Math 1B midterm review, Spring 2002

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Series solutions

Important concepts

1. Sequences. Convergence and divergence. Monotonic sequences.
2. Series tests

   Single series tests

   Integral test. The function must be positive and decreasing.
   Divergence test.
   Alternating series.
   Absolute convergence testing. Note that absolute convergence does not imply conditional convergence.
   Root and ratio test.

   Comparison tests

   Sums and multiples of series. Be careful - the difference of two diverging series may be convergent.
   Direct comparison test. All terms must be positive.
   Limit comparison test. All terms must be positive.

   Tests with a known remainder

   Integral test.
   Direct comparison test.
   Alternating series.

   Useful series to know

   Arithmetic series.
   Geometric series.
   Telescoping series.
   Harmonic series.
   Alternating harmonic series.
   p-series.
   Binomial series.

3. Series heuristics

   (a) **Is it one of the above series?** See if the series is one of the above series, or if it can be manipulated (using some algebra) into one of the above forms. If this is possible, then simply apply the corresponding formula for convergence.

(b) **Divergence test.** The divergence is usually easy to apply. It will tell immediately if a series diverges, and can usually be done in less than a minute.

(c) **Comparison with p-series.** If the terms of the series are a ratio of polynomials, then often a comparison with p-series will be effective. Try either the limit comparison or the direct comparison. A good rule of thumb is to use the difference of the highest powers in the numerator and denominator as the value for p).

(d) **Root and ratio test.** Any series with an expression to an n-th power may be a candidate for the root test. Such series are also candidates for the ratio test, as well as any series involving factorials and products. (Using the ratio test will often eliminate factorials in the resulting expression.)

(e) **Comparison tests.** If a series is close to, but not exactly the same as, a standard series, a comparison test is a good one to try. Remember the following two standard limits; they may be helpful in a limit comparison test:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$$

(f) **Absolute convergence testing.** If a series has both positive and negative terms, but the signs switch arbitrarily, then the test for absolute convergence must be used. Test to see if the corresponding series (of absolute values of the original series) converges or diverges.

(g) **Alternating series test.** A series with alternating positive and negative is a candidate for the alternating series test. Be sure to verify that the series satisfies the hypotheses for the alternating series test. Don't be fooled by series that don't look like alternating series, but really are. For example, does

$$\sum_{n=1}^{\infty} \cos(n\pi)(\frac{1}{n})$$

converge or diverge?

(h) **Integral test.** If it looks like the infinite integral of the series can be evaluated, then the integral test is a possible choice. Be sure the series satisfies the hypotheses for the integral test. Usually a last choice, as integration is tedious in general, as is verifying the integral test hypotheses.

4. Power series. Taylor and Maclaurin series.

   Power series and radius and interval of convergence

   Taylor and Maclaurin series for arbitrary functions

   Finding the sum of a power series

Basic Problems

Determine the convergence or divergence of the series.

1. $\sum_{n=1}^{\infty} \frac{1}{e^{2n}}$ 2. $\sum_{n=1}^{\infty} ne^{-n^2}$ 3. $\sum_{n=1}^{\infty} \frac{3}{n2^n}$ 4. $\sum_{n=1}^{\infty} \frac{n^2+1}{n^4+1}$

5. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ 6. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{2^n}$ 7. $\sum_{n=1}^{\infty} \frac{(-1)^n}{5+n}$ 8. $\sum_{n=1}^{\infty} \frac{1}{(2n)!}$

Find the radius of convergence and the interval of convergence.

9. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ 10. $\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$

Find a series expansion (centered at 0) for the following functions.

11. $f(x) = \frac{x}{1-x}$ 12. $f(x) = \cos \pi x$ 13. $f(x) = x^2 e^{-x}$

14. $f(x) = \int \frac{1}{1+x^4}\, dx$ 15. $f(x) = \int \frac{\sin x}{x}\, dx$

Advanced Problems

1. $\sum_{n=1}^{\infty} sin(\frac{1}{n})$ 2. $\sum_{n=1}^{\infty} e^{-n} n!$ 3. $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ 4. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

5. $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n\sqrt{n}}$ 6. $\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{n!}$ 7. $\sum_{n=1}^{\infty} \frac{n+5}{\sqrt[3]{n^7+n^2}}$ 8. $\sum_{n=1}^{\infty} (-1)^n \sin(\frac{\pi}{n})$

9. $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$

Find series expansions for the following functions (centered at 0).

10. $f(x) = x \ln(1+x)$ 11. $f(x) = x \arctan(\frac{x}{3})$

12. $f(x) = \ln(9+x^2)$ 13. $f(x) = \sin^2 x$

14. $f(x) = \sin(x^4)$ 15. $f(x) = \int_0^x \sin(t^2)\, dt$

Find a closed formula for the sum.

16. $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!}$ 17. $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{2n} n!}$

18. $\sum_{n=0}^{\infty} \frac{(x+2)^n}{(n+3)!}$

Differential Equations

Important concepts

1. Techniques for solving differential equations

   Separation of variables

   Integrating factors (Linear equations)

Basic Problems

Solve the differential equation.

1. $\frac{du}{dt} = 2 + 2u + t + tu$
2. $y' + 2y = 2e^x$
3. $1 + xy = xy'$
4. $y' = x^2y$
5. $(1 + t)\frac{du}{dt} + u = 1 + t$
6. $xyy' = \ln x$, $y(1) = 2$

Miscellaneous

Showing a sequence is increasing or decreasing

There are three ways to show that a sequence is increasing. Some of them are easier to apply in certain situations than others. The first test assumes that all elements are positive:

Division If $\frac{a_{n+1}}{a_n} \geq 1$ then the sequence is increasing. This follows from dividing both sides of $a_{n+1} \geq a_n$.

Subtraction If $a_{n+1} - a_n \geq 0$ then the sequence is increasing. This follows from subtracting both sides of $a_{n+1} \geq a_n$.

Differentiation If $f'(n) \geq 0$ then the sequence is increasing. This follows from the definition of derivative.

There are similar formulas to show that a sequence is decreasing.