Extending Differential Calculus for Noncommuting Variables

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Abstract: Recent innovations in the differential calculus for functions of non-commuting variables, begun for a quaternionic variable, are readily extended to the case of a general Clifford algebra and then extended more to the case of a variable that is a general square matrix over the complex numbers. The expansion of F(x+delta) is given to first order in delta for general matrix variables x and delta that do not commute with each other. Further extension leads to the broader study of commutators with functions, [y, f(x)] , expressed in terms of [y, x].

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1. Introduction

In a recent paper [1] I showed how to expand

$$F(x + \delta) = F(x) + F^{(1)}(x) + O(\delta^2) \tag{1..1}$$

when both x and δ were general quaternionic variables, thus did not commute with each other. It starts with the separation of the infinitesimal displacement into two parts, $\delta = \delta_\parallel + \delta_\perp$ as follows.

$$\delta_\parallel = \frac{1}{2}(1 - u_x\delta u_x), \qquad \delta_\perp = \frac{1}{2}(1 + u_x\delta u_x), \tag{1..2}$$

$$x = x_0 + ix_1 + jx_2 + kx_3 = x_0 + ru_x, \qquad r = \sqrt{x_1^2 + x_2^2 + x_3^2}. \tag{1..3}$$

This construction leads to the simple operator identities,

$$\delta_\parallel\, x = x\, \delta_\parallel, \qquad \delta_\perp\, x = x^*\, \delta_\perp, \tag{1..4}$$

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where x^* is the complex conjugate of x, having all imaginary components changed in sign. The final result is,

$$F^{(1)}(x) = F'(x)\delta_{\|} + [F(x) - F(x^*)](x - x^*)^{-1}\,\delta_{\perp}, \tag{1..5}$$

where $F'(x)$ is the ordinary derivative of the function $F(x)$.

It was noted in [1] that this result for quaternionic variables can be written in terms of a formal set of commutators:

$$F(x+\delta) = F(x) + F'(x)\delta_{\|} + [C, F(x)] + O(\delta^2), \qquad [C, x] = \delta_{\perp}. \tag{1..6}$$

It is a trivial matter to extend this result for quaternions to any Clifford algebra.

$$x = x_0 + \sum_{i=1,n} e_i\, x_i, \qquad e_i e_j + e_j e_i = -2\delta_{ij}, \tag{1..7}$$

since all we need to do is extend the definitions of u_x and r:

$$\sum_{i=1,n} e_i\, x_i = r u_x, \qquad r = \sqrt{\sum_{i=1,n} x_i^2} \tag{1..8}$$

and then use the very same formulas as in Eqs. (1..2), (1..5). There is some further material in Appendix D, where the venerable Fueter differential equation (for functions of a quaternionic variable) is extended to a general Clifford algebra.

The purpose of this paper is to extend that earlier analysis of differential calculus to a larger family of non-commuting variables. First we shall consider NxN matrices; then we shall look at more general formulations.

2. Finite Matrices as the Variable

Consider the N x N matrices X over the complex numbers and arbitrary analytic functions $F(X)$ with such a matrix as its variable. We seek a general construction for the first order term $F^{(1)}(X)$ when we expand $F(X+\Delta)$ given that Δ is small but still a general N x N matrix that does not commute with X.

The first step, following earlier work, is to represent the function F as a Fourier transform,

$$F(X) = \int dp f(p) e^{ipX} \tag{2..1}$$

where the integral may go along any specified contour in the complex p-plane; and then we also make use of the expansion,

$$e^{(X+\Delta)} = e^X[1 + \int_0^1 ds e^{-sX}\Delta\, e^{sX} + O(\Delta^2)]. \tag{2..2}$$

Another well-known expansion, relevant to what we see in (2..2), is

$$e^A\, B\, e^{-A} = B + [A, B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \dots, \tag{2..3}$$

involving repeated use of the commutators, $[A, B] = AB - BA$.

The next step is to assume that we can find a matrix S that will diagonalize the matrix X at any given point in the space of such matrices.

$$A = S\ X\ S^{-1}, \qquad A_{i,j} = \delta_{i,j}\ \lambda_i, \qquad i, j = 1, \ldots, N \tag{2..4}$$

and we carry out the same transformation on the matrix Δ:

$$B = S\ \Delta\ S^{-1} \tag{2..5}$$

but, of course, the matrix B will not be diagonal.

Our task is to separate the matrix B into various parts, each of which will behave simply in the expansion of Eq. (2..3). The first step is to recognize that the diagonal part of the matrix B, call it B_0, commutes with the diagonal matrix A and thus we have

$$e^{tA}\ B_0\ e^{-tA} = B_0, \tag{2..6}$$

where we have $t = -isp$ from (2..1), (2..2).

Next we look separately at each off-diagonal element of the matrix B: that is, $B_{(i,j)}$ is the matrix that has only that one off-diagonal ($i \neq j$) element of the given matrix B, and all the rest are zeros. The first commutator is simply

$$[A, B_{(i,j)}] = r_{ij}\ B_{(i,j)}, \qquad r_{ij} = \lambda_i - \lambda_j \tag{2..7}$$

and then the whole series can be summed:

$$e^{tA}\ B_{(i,j)} e^{-tA} = e^{tr_{ij}}\ B_{(i,j)}. \tag{2..8}$$

Putting this all together, we have

$$S\ F^{(1)}\ S^{-1} = \int dp f(p)\ ip\ e^{ipA} \int_0^1 ds\ [B_0 + \sum_{i \neq j} e^{-ispr_{ij}}\ B_{(i,j)}]. \tag{2..9}$$

It is trivial to carry out the integrals over s; and we thus come to the final answer

$$F^{(1)}(X) = F'(X)\ \Delta_0 + \sum_{i \neq j} [F(X) - F(X - r_{ij} I)]\ r_{ij}^{-1}\ \Delta_{(i,j)} \tag{2..10}$$

where

$$\Delta_\mu \equiv S^{-1}\ B_\mu\ S, \qquad \mu = 0, (i, j). \tag{2..11}$$

The general structure of the result, Eq. (2..10), is similar to what we found in earlier work, Eq. (1..5): the first term $F'(X)$ looks like ordinary differential calculus and goes with that part of Δ that commutes with the local coordinate X; the remaining terms are non-local, involving the function F evaluated at discrete points separated from X by specific multiples of the unit matrix I.

While this final formula appears not as explicit as the previous result found for quaternionic variables, in any practical situation we have computer programs that can calculate the matrix operations referred to above with great efficiency.

A particular example, which was drawn out in [1], is the case of a variable built upon the matrices of any rank which generate the Lie algebra SU(2).

The quantities r_{ij}, which may be real or complex numbers, can be called the "roots" following the familiar treatment of Lie algebras. They have some properties, such as $r_{ij} = -r_{ji}$ and sum rules that involve traces of the original matrix X and powers of X.

What happens if the eigenvalues of X are degenerate? Suppose, for example, that $\lambda_1 = \lambda_2$. This means that r_{12} and r_{21} are zero. If we look at Eq. (2..10), we see that the terms $\Delta_{(1,2)}$ and $\Delta_{(2,1)}$ then have the coefficient $F'(x)$. Thus they simply add in with Δ_0. In the extreme case when all the eigenvalues are identical, then the answer is $F(X+\Delta) = F(X) + F'(X)\ \Delta + O(\Delta^2)$, which is old fashioned differential calculus for commuting variables.

Again, it is noted that the result Eq. (2..10) can be written as,

$$F(X+\Delta) = F(X) + F'(X)\Delta_0 + [C, F(X)] + O(\Delta^2), \qquad [C, X] = \Delta - \Delta_0. \tag{2..12}$$

3. General Commutator Formulation

Now we shall explore the more abstract form of the results reported above which involves commutator equations for any noncommutative system.

$$F^{(1)}(x) = F'(x)\ \delta_0\ + [C, F(x)], \tag{3..1}$$

$$[C, x] = \delta - \delta_0, \qquad [\delta_0, x] = 0. \tag{3..2}$$

Here the operator C is defined implicitly, through its commutator with x, rather than explicitly; and δ_0 is that portion of δ that commutes with x (called $\delta_{\|}$ or Δ_0 above). One may readily confirm the correctness of this formulation in the case of $F(x) = x^n$.

Here is a more formal-looking derivation of these equations. [2] Start with the following identity for suitably smooth functions $F(x)$.

$$e^C F(x) e^{-C} = F(e^C x e^{-C}), \tag{3..3}$$

$$F(x) + [C, F(x)] + \ldots = F(x + [C, x] + \ldots), \tag{3..4}$$

where the dots indicate terms of second order or higher in the quantity C, which is assumed to be small. If we now define the operator C by $[C, x] = \delta - \delta_0$, with $[\delta_0, x] = 0$, this yields,

$$F(x + \delta - \delta_0) = F(x+\delta) - F'(x)\ \delta_0 + \ldots = F(x) + [C, F(x)] + \ldots. \tag{3..5}$$

which connects Eq. (3..1) to the original statement of the differential calculus in Eq. (1..1).

It is interesting that this new formalism manages to hide the non-locality, which was a prominent feature of the previous examples.

In the rest of this paper I want to explore this commutator formalism as a general mathematical problem: how to express a commutator with a function $[y, f(x)]$ in terms of $[y, x]$.

In any non-commutative (but associative) algebra, we have the basic identity,

$$[x, yz] = y[x, z] + [x, y]z; \tag{3..6}$$

and repetition of this identity leads to the familiar formula,

$$[y, x^n] = \sum_{m=0}^{n-1} x^m \, [y, x] \, x^{n-m-1}. \tag{3..7}$$

We want to seek similar expressions for some familiar functions of x:

$$[y, f(x)] = \textit{ something built around } [y, x]. \tag{3..8}$$

Some earlier studies [3] of this general problem assumed that the commutator $[y, x]$ is a constant; but here we want to go beyond that assumption.

4. Non-Integral Power

Let's start with $f(x) = x^\nu$ for an arbitrary number ν. We introduce a parameter p, which commutes with x and y, and consider the function:

$$Q_\nu(p) = [y, (x + p)^\nu]. \tag{4..1}$$

We have the algebraic identity

$$Q_\nu(p) = [y, (x + p)(x + p)^{\nu-1}] = (x + p)Q_{\nu-1}(p) + [y, x](x + p)^{\nu-1}; \tag{4..2}$$

and we can also take the derivative,

$$\frac{dQ_\nu(p)}{dp} = \nu \, Q_{\nu-1}(p). \tag{4..3}$$

Combining these two equations we get a simple differential equation for $Q_\nu(p)$, which is easily solved.

$$Q_\nu(p) = \nu \, (x + p)^\nu \int_p^\infty ds \, (x + s)^{-\nu-1} \, [y, x] \, (x + s)^{\nu-1}. \tag{4..4}$$

Now, setting $p = 0$, we have the desired result,

$$[y, x^\nu] = \nu \, x^\nu \int_0^\infty ds \, (x + s)^{-\nu-1} \, [y, x] \, (x + s)^{\nu-1}. \tag{4..5}$$

An alternative version of this formula, starting with a rewrite of Eq. (4..2), is

$$[y, x^\nu] = \nu \int_0^\infty ds \, (x + s)^{\nu-1} \, [y, x] \, (x + s)^{-\nu-1} \, x^\nu. \tag{4..6}$$

One special case of this general formula is the familiar result,

$$[y, x^{-1}] = -x^{-1} \, [y, x] \, x^{-1}; \tag{4..7}$$

and another, unfamiliar, special case comes from taking the limit as ν goes to zero:

$$[y, ln(x)] = \int_0^\infty ds \, \frac{1}{x+s} \, [y, x] \, \frac{1}{x+s}. \tag{4..8}$$

From this we can derive the following.

$$[y, tan^{-1}(x)] = \int_0^1 ds \, \frac{1}{1+s^2x^2} \Big([y, x] - s^2x[y, x]x\Big) \frac{1}{1+s^2x^2}. \tag{4..9}$$

An alternative representation can be found, using the formula (A.1) in Appendix A, for the case $-1 < \nu < 0$:

$$[y, x^\nu] = \frac{sin \; \pi\nu}{\pi} \int_0^\infty ds \; s^\nu \, \frac{1}{x+s} \, [y, x] \, \frac{1}{x+s}. \tag{4..10}$$

This can also be written as,

$$[y, x^\nu] = \mu \frac{sin \; \pi\nu}{\pi} \int_0^\infty ds \, \frac{1}{x+s^\mu} \, [y, x] \, \frac{1}{x+s^\mu} \; , \qquad \mu = 1/(\nu+1). \tag{4..11}$$

5. Exponential

To study the exponential function, we again introduce a real parameter,

$$R(p) = [y, e^{px}]. \tag{5..1}$$

We take the derivative with respect to p,

$$\frac{dR(p)}{dp} = [y, xe^{px}] = xR(p) + [y, x]e^{px}, \tag{5..2}$$

and solve this simple differential equation to get the result:

$$[y, e^x] = \int_0^1 ds \; e^{(1-s)x} \; [y, x] \; e^{sx}. \tag{5..3}$$

I would not be surprised if this formula has been discovered before. A very similar looking formula, which is well-known, is

$$\frac{d}{d\lambda} \, e^{x(\lambda)} = \int_0^1 ds \; e^{(1-s)x} \, \frac{dx(\lambda)}{d\lambda} \, e^{sx}. \tag{5..4}$$

This result, Eq. (5..3), can also be derived from Eq. (4..5) if we make the substitution, $x \to (1 + x/\nu)$ and take the limit $\nu \to \infty$.

6. Some Other Familiar Functions

The Chebyshev polynonials of the second kind $U_n(x)$ are defined by a simple generating function.[4]

$$\frac{1}{1-2xt+t^2} = \sum_{n=0}^\infty t^n \; U_n(x). \tag{6..1}$$

If we take the commutator of this equation with y, we find

$$\sum_n t^n\ [y, U_n(x)] = 2t \frac{1}{1-2xt+t^2}\ [y,x]\ \frac{1}{1-2xt+t^2}. \qquad (6..2)$$

Now using the expansion (6..1) again; and collecting terms with the same power of t, we find,

$$[y, U_n(x)] = 2 \sum_{m=0}^{n-1} U_m(x)\ [y,x]\ U_{n-m-1}(x). \qquad (6..3)$$

This is a remarkably simple formula; it looks similar to the earliest Eq. (3..7).

One can get a very similar-looking formula for the set of polynomials generated by $1/(1 - xt + \chi(t))$, where $\chi(t)$ is any power series starting off with t^2. The special case $\chi = 0$ gives us Eq. (3..7). An application of this sort of generating function, with multiple noncommuting variables, may be seen in an earlier study of infinite dimensional free algebra in large N matrix theory.[5]

The Chebyshev polynomials of the first kind $T_n(x)$ come from a slightly different generating function and they yield the following formula.

$$[y, T_n(x)] = -[y,x]\ U_{n-1}(x) + 2 \sum_{m=0}^{n-1} T_m(x)[y,x]\ U_{n-m-1}(x). \qquad (6..4)$$

Gegenbauer polynomials are given by this generating function,

$$(1-2xt+t^2)^{-\nu} = \sum_{n=0}^{\infty} t^n\ C_n^{\nu}(x). \qquad (6..5)$$

For $\nu = 1$ these are the Chebyshev and for $\nu = 1/2$ these are the Legendre polynomials. If I try to use the formulas above directly, then it results in a very complicated mess. In Appendix A is a formula that relates this generating function Eq. (6..5) to an integral over the simpler one of Eq. (6..1),

$$C_n^{\nu}(x) = \frac{sin\pi\nu}{\pi}\ 2\int_0^1 du\ u^{n-1}\ (1/u^2-1)^{-\nu}\ U_n(ux), \qquad 0 < \nu < 1; \qquad (6..6)$$

and this lets us calculate $[y, C_n^{\nu}(x)]$ using results above.

The Laguerre polynomials are given by another generating function,

$$(1-t)^{-\alpha-1} e^{-xt/(1-t)} = \sum_{n=0}^{\infty} t^n\ L_n^{(\alpha)}(x). \qquad (6..7)$$

Following procedures given above, we find the commutator equation,

$$[y, L_n^{(\alpha)}(x)] = -\int_0^1 ds \sum_{m=0}^{n-1} L_m^{(\alpha)}((1-s)x)\ [y,x]\ L_{n-m-1}^{(0)}(sx), \qquad (6..8)$$

which involves an integral and a finite sum.

Bessel functions of integral order are given from this generating function:

$$e^{x(t-t^{-1})/2} = J_0(x) + \sum_{n=1}^{\infty} [t^n + (-t)^{-n}] J_n(x). \qquad (6..9)$$

If we take the commutator with y, use Eq. (5..3) and again compare the coefficients of each power of t, we get a rather complicated formula, involving an integral and an infinite sum. Here is the result for J_0.

$$[y, J_0(x)] = \int_0^1 ds \sum_{m=0}^{\infty} (-1)^{m+1}[J_m((1-s)x)\ [y,x]\ J_{m+1}(sx) + \tag{6..10}$$

$$J_{m+1}((1-s)x)\ [y,x]\ J_m(sx)]. \tag{6..11}$$

Another representation for some Bessel functions is the following [6].

$$x^\mu\ H_\mu^{(1)}(x) = \frac{2}{\pi i}\int_0^\infty ds\ s^{\mu+1}\ J_\mu(s)\ \frac{1}{s^2 - x^2}, \tag{6..12}$$

where the variable x lies in the first quadrant of the complex plane. Applying the commutator $[y,\ \ ]$ to this equation will give a simpler-looking formula.

For more in this vein, see Appendix B.

One more famous function is the Gamma function $\Gamma(x)$ and we find the following formula for its logarithmic derivative.

$$\psi(x) = \frac{1}{\Gamma(x)}\frac{d\Gamma(x)}{dx}, \qquad [y,\psi(x)] = \sum_{n=0}^{\infty}\frac{1}{n+x}[y,x]\frac{1}{n+x}, \tag{6..13}$$

which looks like the discrete analog of Eq. (4..8); and from this we also get,

$$[y, cot(x)] = -\sum_{n=-\infty}^{\infty}\frac{1}{n\pi + x}\ [y,x]\ \frac{1}{n\pi+x}. \tag{6..14}$$

7. Discussion

The results given in this paper are not only new, they are mostly of an unfamiliar form. The formulas in Sections 4 and 5 involve definite integrals (over a parameter s) that would be easy to evaluate if it were not for the operator $[y,x]$ that stands in the midst of simple functions of s and x. I suggest an interesting exercise for readers is to take the special case $y = d^2/dx^2$ and see how those integrals work out. But also see Appendix C.

Are there any restrictions on the formulas derived above? Within Section 4, I should specify that the operator x has eigenvalues with positive real parts. This is necessary to give meaning to the nonintegral power of x and it will avoid singularities in the integrals over the parameter s.

In Sections 4 and 5 there are formulas for the commutator of y with x^ν, $ln(x)$, and e^{px}; and one should be able to relate those formulas. For example,

$$[y, x^\nu] = [y, e^{\nu ln x}] = \int_0^1 ds\ e^{(1-s)\nu ln(x)}\ [y, \nu ln(x)]\ e^{s\nu ln(x)} = \tag{7..1}$$

$$\nu\ x^\nu \int_0^1 ds\ x^{-\nu s}\int_0^\infty dt\ \frac{1}{t+x}\ [y,x]\ \frac{1}{t+x}\ x^{\nu s}. \tag{7..2}$$

How do we get this to look like Eq. (4..5) ?

Appendix A: Relating Generators

We want to relate the generating function in Eq. (6..5) to that in Eq. (6..1). We start with a simple integral identity,

$$\int_0^\infty \frac{d\zeta}{\zeta + M}\,\zeta^{-\nu} = \frac{\pi}{\sin(\pi\nu)} M^{-\nu}, \qquad M > 0, \qquad 0 < \nu < 1. \tag{A.1}$$

For the problem at hand we have $M = 1 - 2xt + t^2$. Now we can write the generating function for the Gegenbauer polynomials as,

$$\sum_{n=0}^{\infty} t^n\, C_n^\nu(x) = \frac{sin(\pi\nu)}{\pi} \int_0^\infty d\zeta\, \zeta^{-\nu}\, \frac{1}{\zeta + 1 - 2xt + t^2}. \tag{A.2}$$

Now we just need to scale the variables t and x to get,

$$C_n^\nu(x) = \frac{sin(\pi\nu)}{\pi} \int_0^\infty d\zeta\, \zeta^{-\nu} \frac{1}{\zeta+1}\, \frac{1}{q^n}\, U_n(x/q), \qquad q = \sqrt{\zeta + 1}. \tag{A.3}$$

Appendix B: A More Formal Approach

Given any analytic function $f(x)$ we can write the equation,

$$f(x) = \frac{1}{2\pi i} \oint ds\, f(s)\, \frac{1}{s - x}. \tag{B.1}$$

where the countour of the integral in the complex s-plane encloses the point x. Then, taking the commutator, we have,

$$[y, f(x)] = \frac{1}{2\pi i} \oint ds\, f(s) \frac{1}{s - x}\, [y, x]\, \frac{1}{s - x}. \tag{B.2}$$

How one might make practical use this equation is unclear.

Appendix C: Troublemaking

To illustrate the unfamiliarity of the "integrals-with-operators" seen in this paper, lets start with the earliest result, Eq. (4..5).

$$[y, x^\nu] = \nu\, x^\nu \int_0^\infty ds\, (x + s)^{-\nu-1}\, [y, x]\, (x + s)^{\nu-1}. \tag{C.1}$$

Let me now scale the integration parameter as $s = xt$. This should be ok since we also noted that we would want x to have positive real values. Then this formula would read,

$$[y, x^\nu] = \nu x^\nu \int_0^\infty xdt\, x^{-\nu-1}\, (1 + t)^{-\nu-1}\, [y, x]\, x^{\nu-1}\, (1 + t)^{\nu-1}. \tag{C.2}$$

It would now appear that we can move the factors $(1 + t)$ past the operators y and x and then simply evaluate the t-integral, leaving

$$[y, x^\nu] = \nu\, [y, x]\, x^{\nu-1}, \tag{C.3}$$

which is clearly wrong! The error may be ascribed to the step where we moved $(1+t)$ past the operator y, since this new variable t was constructed to involve x, which does not commute with y.

Appendix D: Fueter Equation for a General Clifford Variable

Fueter [7], long ago, studying calculus with a quaternionic variable, found a third order differential equation that any "holomorphic" function of such a variable would satisfy. We generalize that result for noncommuting variables based on a general Clifford algebra.

For reference, we start with the familiar Cauchy-Riemann equation for an analytic function of a complex variable,

$$(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y})f(z=x+iy)=0. \tag{D.1}$$

Now we want to consider the non-commuting variables x that are based on an n-dimensional Clifford algebra,

$$x=x_0+\sum_{i=1,n} e_i\, x_i, \qquad e_ie_j+e_je_i=-2\delta_{i,j} \quad , \tag{D.2}$$

and the quantities x_0, x_i are real numbers. The procedure we follow to get the generalized Fueter differential equation is that given in [1] for quaternions (n=3).

Define the first order differential operator,

$$\Box_{n+1}=\frac{\partial}{\partial x_0}+\sum_{i=1,n} e_i\,\frac{\partial}{\partial x_i}, \tag{D.3}$$

and see how that acts upon the particular function $F(x)=e^{px}$ for some real parameter p.

$$e^{px}=e^{px_0}(cos\ pr+\Sigma\ \frac{sin\ pr}{r}), \tag{D.4}$$

$$r=\sqrt{\sum_{i=1,n} x_i^2}, \qquad \Sigma=\sum_{i=1,n} e_ix_i, \qquad \Sigma^2=-r^2 \tag{D.5}$$

$$\Box_{n+1}\, e^{px}=(1-n)\; e^{px_0}\frac{sin\ pr}{r}. \tag{D.6}$$

For $n=1$ we have the Cauchy-Riemann formula, Eq. (D.1). For $n=3$ we see that the right hand side of Eq. (D.6) vanishes under action of the Laplacian operator in 4-dimensions.

$$\triangle_4=\Box_4^*\,\Box_4=\frac{\partial^2}{\partial x_0^2}+\frac{1}{r}\frac{\partial^2}{\partial r^2}r. \tag{D.7}$$

This gives us Fueter's third order differential equation for functions of a quaternionic variable.

$$\triangle_4\,\Box_4\,F(x)=0, \tag{D.8}$$

assuming that any function $F(x)$ can be written as a real superposition of the exponentials e^{px}.

Now, how do we go for $n > 3$? It appears that we can still write,

$$\text{“}\triangle_4\text{”}\ \square_{n+1}\ F(x) = 0, \tag{D.9}$$

where I have put “quotation marks” around that operator $\triangle_4$ to mean that it is literally what is written in Eq. (D.7) and the quantity r is meant to be the n-dimensional radial distance defined in Eq. (D.5). So, this is not really the Laplacian operator in n+1 dimensions; but it is a third order differential equation that is satisfied.

Alternatively, it can be shown that one can introduce the correct Laplacian operator in n+1 dimensions,

$$\triangle_{n+1} = \square^*_{n+1}\ \square_{n+1} = \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}, \tag{D.10}$$

and find the following formula, an n-th order differential equation, which appears to be valid only for odd values of n.

$$(\triangle_{n+1})^{(n-1)/2}\ \square_{n+1}\ F(x) = 0. \tag{D.11}$$

This last result has been found earlier by researchers [8] who followed the classical line of Fueter’s analysis. But the earlier formula Eq. (D.9) appears to be new.

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