A Conjecture about Conserved Symmetric Tensors

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Abstract: We consider T(x), a tensor of arbitrary rank that is symmetric in all of its indices and conserved in the sense that the divergence on any one index vanishes. Our conjecture is that all integral moments of this tensor will vanish if the number of coordinates in that integral moment is less than the rank of the tensor. This result is proved explicitly for a number of particular cases, assuming adequate dimensionality of the Euclidean space of coordinates (x); but a general proof is lacking. Along the way, we find some neat results for certain large matrices generated by permutations.

1.　Introduction

In an n-dimensional real Euclidean space we consider symmetric tensors of any rank that are "conserved" as follows,

$$\sum_{i=1,n} \frac{\partial}{\partial x_i} \, T_{iw}(x) = 0, \tag{1}$$

where w is a string of indices (a "word") of arbitrary length. If $w = abc$, then $iw = iabc$, etc.; and we use the notation $[w]$ to denote the length of the word w.

　　Our interest is in integrals of the form

$$(w_L; w_R) \equiv \int d^n x \; x^{w_L} \, T_{w_R}(x), \tag{2}$$

which we call "integral moments" of the tensor T, and x^w is a product of coordinates identified by the letters in the word w. We denote the null word by 0, so that $x^0 = 1$. We

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assume that the tensors are functions well confined in space, so that all such integrals of interest converge and we can do partial integration ignoring surface terms.

Our conjecture is that all integrals of the type (2) will vanish so long as $[w_L] < [w_R]$; and the identities (1) are necessary for this result to be true.

Usually one speaks about tensors in relation to some group of transformations; but here that plays no role. This is just about algebraic manipulation of the indices.

This topic arose from a recent study of the General Theory of Relativity [1], where we were looking at the asymptotic form of the potential produced by a source represented by such a symmetric tensor. The main conclusion was that there is no long range potential $\sim 1/r$.

2. The System of Equations

The general identity we start with is this,

$$< w_1 | w_2 > \equiv - \sum_i \int d^n x \ x^{w_1} \frac{\partial}{\partial x_i} T_{i w_2}(x) = \sum_i (\partial_i w_1; i w_2) = 0, \tag{3}$$

where $\partial_i w$ means removing any occurrence of the letter i in the word w. Since we are interested in symmetric tensors, the order of letters in any word is unimportant.

We can organize this host of equations by looking at the combined word $W = w_1 w_2 = w_L w_R$; and we see that the equations (3) separate into distinct subsets for each combined word.

3. Examples

We proceed from the simplest examples of (3) to more complicated ones. In what follows, I use the letters $a, b, c, \ldots$ to denote distinct values of the index $i = 1, 2, \ldots, n$. It will be advantageous to separate subsets of letters according to Young Tableaux, such as $a^3 b^2 cd$, for example. The word w, as used below may be arbitrary.

$$[w_L] = 0; W = aw : \tag{4}$$

$$< a | w > = (0; aw) = 0. \tag{5}$$

This is the simplest case, which says, when written out,

$$- \int d^n x \ x_a \sum_{i=1,n} \frac{\partial}{\partial x_i} T_{iw}(x) = \int d^n x \ T_{aw}(x) = 0. \tag{6}$$

We proceed to $[w_L] = 1$.

$$[w_L] = 1; W = a^2 bw : \tag{7}$$

$$< aa | bw > = 2(a; abw) = 0, \tag{8}$$

$$< ab | aw > = (a; abw) + (b; aaw) = 0. \tag{9}$$

So all these are zero.

$$[w_L] = 1; W = abcw : \tag{10}$$
$$< ab|cw >= (a; bcw) + (b; acw) = 0, \tag{11}$$
$$< bc|aw >= (b; acw) + (c; abw) = 0, \tag{12}$$
$$< ca|bw >= (c; abw) + (a; bcw) = 0. \tag{13}$$

The solution of these three simultaneous equations in three unknowns is that all three are zero. We had to introduce the label c explicitly to get this result; which means that the tensors involved here are at least of second rank and also that the dimensionality of the space must be $n \geq 3$.

Now we go to $[w_L] = 2$; and here we need $[W] \geq 5$. The case $W = a^5 w$ is trivial and $W = a^4 bw$ is similar to what is above in (7).

$$[w_L] = 2; W = a^3 b^2 w \tag{14}$$
$$< aaa|bbw >= 3(aa; abbw) = 0, \tag{15}$$
$$< aab|abw >= 2(ab; aabw) + (aa; abbw) = 0, \tag{16}$$
$$< abb|aaw >= (bb; aaaw) + 2(ab|aabw) = 0. \tag{17}$$

The solution is that all three unknowns are zero.

$$[w_L] = 2; W = a^3 bcw \tag{18}$$
$$< aaa|bcw >= 3(aa; abcw) = 0, \tag{19}$$
$$< aab|acw >= 2(ab; aacw) + (aa; abcw) = 0, \tag{20}$$
$$< aac|abw >= 2(ac; aabw) + (aa; abcw) = 0, \tag{21}$$
$$< abc|aaw >= (ab; aacw) + (ac; aabw) + (bc; aaaw) = 0. \tag{22}$$
$$\tag{23}$$

The solution is that all four unknowns are zero.

$$[w_L] = 2; W = a^2 b^2 cw : \tag{24}$$
$$< aab|bcw >= 2(ab; abcw) + (aa; bbcw) = 0, \tag{25}$$
$$< abb|acw >= 2(ab; abcw) + (bb; aacw) = 0, \tag{26}$$
$$< bbc|aaw >= 2(bc; aabw) + (bb; aacw) = 0, \tag{27}$$
$$< abc|abw >= (ab; abcw) + (ac; abbw) + (bc; aabw) = 0, \tag{28}$$
$$< aac|bbw >= 2(ac; abbw) + (aa; bbcw) = 0. \tag{29}$$

These 5 equations in 5 unknowns have the solution that all unknowns are zero.

$$[w_L] = 2; W = a^2bcdw : \tag{30}$$

$$< aab|cdw >= 2(ab; acdw) + (aa; bcdw) = 0, \tag{31}$$

$$< aac|bdw >= 2(ac; abdw) + (aa; bcdw) = 0, \tag{32}$$

$$< aad|bcw >= 2(ad; abcw) + (aa; bcdw) = 0, \tag{33}$$

$$< abc|adw >= (ab; acdw) + (ac; abdw) + (bc; aadw) = 0, \tag{34}$$

$$< acd|abw >= (ac; abdw) + (ad; abcw) + (cd; aabw) = 0, \tag{35}$$

$$< abd|acw >= (ab; acdw) + (ad; abcw) + (bd; aacw) = 0, \tag{36}$$

$$< bcd|aaw >= (bc; aadw) + (bd; aacw) + (cd; aabw) = 0. \tag{37}$$

$$\tag{38}$$

These 7 equations in 7 unknowns have the solution that all unknowns are zero.

$$[w_L] = 2; W = abcdew : \tag{39}$$

$$< abc|dew >= (ab; cdew) + (bc; adew) + (ac; bdew) = 0, \tag{40}$$

$$\textit{nine more equations by permutations.} \tag{41}$$

The solution of these ten simultaneous equations in ten unknowns is that all ten are zero. We had to introduce the labels *de* explicitly to get this result; which means that the tensors involved here are at least of third rank and also $n \geq 5$.

4. First Steps Toward a General Proof

Let's start with the following set of cases, where the explicit part of W contains at most two distinct labels.

$$[w_L] = k; \quad [w_R] \geq k + 1; \quad W = a^{2k+1-m}b^m w; \quad 0 \leq m \leq k \tag{42}$$

$$< a^{k+1-r}b^r|a^{k-m+r}b^{m-r}w >= 0 = \tag{43}$$

$$(k + 1 - r)(a^{k-r}b^r; a^{k+1-m+r}b^{m-r}w) + r(a^{k+1-r}b^{r-1}; a^{k-m+r}b^{m+1-r}w), \tag{44}$$

for $0 \leq r \leq m$. For each set of values for k and m, this is a series of equations,

$$(k + 1 - r)Q(r) + rQ(r - 1) = 0, \tag{45}$$

which leads to $Q(r) = 0$ for all allowed values of r.

Next, let's consider this set of cases:

$$[w_L] = k; \quad W = a^{k+1} x_k w, \tag{46}$$

where x_k is some given word of length k. I also introduce the notation $x_{k,r,\alpha}$ to stand for the word that is made from some subset of r letters in the word x_k; there are many such

subsets and so the label α is meant to distinguish them from one another.Then we see the series of equations,

$$< a^{k+1}|x_k w >= 0 = (k+1)(a^k; a \, x_k w), \tag{47}$$

$$< a^k \, x_{k,1,\alpha}|a(x_k/x_{k,1,\alpha})w >= 0 = \tag{48}$$

$$k(a^{k-1} \, x_{k,1,\alpha}; a^2 \, (x_k/x_{k,1,\alpha})w) + (a^k; a \, x_k w). \tag{49}$$

The quotient $x_k/x_{k,r,\alpha}$ stands for that word which results when those r letters are removed from x_k. Eq. (47) involves $r = 0$; and Eqs. (48, 49) involve $r = 1$ as well as $r = 0$.

When we use the result of Eq. (47) in Eq. (49) we see that all those integral moments formed with $r = 0$ and $r = 1$ vanish. We then go on to look at $r = 2$ and find that this is an inductive series of equations. We conclude that all integral moments built from the ansatz (46) vanish.

How does one go on to extend this proof? If we look at some of the earlier examples, for example $[w_L] = 1$, $W = abcw$, we see that this nice inductive situation does not apply in all cases.

Here is one more set of cases that we can solve analytically.

$$[w_L] = k; \quad W = a^k b^k cw \tag{50}$$

$$< a^{k-r}b^{r+1}|a^r b^{k-r-1}cw >= (k-r)P_{r+1} + (r+1)P_r = 0, \tag{51}$$

$$P_r \equiv (a^{k-r}b^r; a^r b^{k-r}cw); \tag{52}$$

$$< a^{k-r}b^r c|a^r b^{k-r}w >= (k-r)Q_r + rQ_{r-1} + P_r = 0, \tag{53}$$

$$Q_r \equiv (a^{k-r-1}b^r c; a^{r+1}b^{k-r}w). \tag{54}$$

We can solve the Eqs. (51) to yield,

$$P_r = (-1)^r \, \frac{r! \, (k-r)!}{k!} \, P_0, \quad r = 0, k; \tag{55}$$

and also Eqs. (53) yield,

$$Q_r = (-1)^r \, \frac{r! \, (k-r-1)!}{(k-1)!} \, (-rP_0/k + Q_0), \quad r = 0, k-1. \tag{56}$$

Now, if we look at the two extreme cases for Eqs. (53), namely $r = 0$ and $r = k$, we find

$$kQ_0 + P_0 = 0, \tag{57}$$

$$kQ_{k-1} + P_k = (-1)^k \, (kP_0 - kQ_0) = 0. \tag{58}$$

The solution of this is $P_0 = Q_0 = 0$, which makes all of the solutions equal to zero.

This suggests how we might solve the general problem involving at most three distinct labels; but it gets rather tedious.

5.   Another Special Case

Consider now the special case of W, of length (2k+1), consisting of all different letters. We saw examples of this in (10) and (39).

Let x, y, z represent any of the (k+1) length words contained in W; we want to use these words to label the rows and columns of the simultaneous linear equations we are studying. There are N of them, where N = (2k+1)!/k!(k+1)!.

For each chosen word x, there is its complement, $\bar{x} = W/x$, a word of length k. There are also N of them.

The basic equations (3) are,

$$< x|\bar{x} >= 0 = \sum_a (x/a; y = a\bar{x}), \tag{59}$$

where a is any one of the letters contained in x and y is formed by adding this letter a to $\bar{x}$. We can write this as the NxN system of simultaneous linear equations with the matrix $A_{x,y}$ whose entries are all +1 or zero.

We can see that this matrix A is symmetric. Consider the words y, which label the columns of $A_{x,y}$. We saw how y is derived from $\bar{x} = W/x$ for each nonzero element in the row of A labeled by x. Consider now the row labeled by one of those y words: $A_{y,z}$. We have the nonzero elements given by $z = b\bar{y}$ for some letter b contained in y. There will be one case, $b = a$, that will yield $z = x$, exactly the word that y was derived from. So we have shown that $A_{x,y} = A_{y,x}$.

We also see that A has only zeroes on the diagonal. So we have Tr(A) = 0.

If we look at the matrix A^2 , we see that on its diagonal will be the number (k+1), which is just how many 1s there are in each row (and each column) of A. So we conclude that $Tr(A^2) = N(k + 1)$.

We are interested in exploring the eigenvalues, E_i, of the matrix A. There are N of them and they are real numbers.

It is easy to find one eigenvector of A. It has all entries +1 and its eigenvalue is (k+1).

We can also calculate (with a computer) the determinant of A, and this is equal to the product of all its eigenvalues.

Again, using the computer, we can search out the eigenvalues by calculating det(A-EI) and seeing where (and how) it goes to zero as a function of E. In the results shown in the table below, the superscript m in $(E)^m$ indicates the multiplicity of any eigenvalue, shown by the behavior $(E - E_i)^m$ of the calculated determinant in the neighborhood of a zero.

TABLE: Computed properties of the matrices A

k	N	Det	Eigenvalues $\sim (E)^m$
1	3	2	$(+2)^1, (-1)^2$
2	10	48	$(+3)^1, (-2)^4, (+1)^5$
3	35	47,775,744	$(+4)^1, (-3)^6, (+2)^{14}, (-1)^{14}$
4	126	$10^{32.8}$	$(+5)^1, (-4)^8, (+3)^{27}, (-2)^{48}, (+1)^{42}$
5	462	$10^{136.4}$	$(+6)^1, (-5)^{10}, (+4)^{44}, (-3)^{110}, (+2)^{165}, (-1)^{132}$
6	1716	$10^{557.7}$	$(+7) \dots$
7	6435	$10^{2259.5}$	$(+8) \dots$

It is surprising how these results look. There are few eigenvalues; they are all whole numbers; and they form a neat pattern as we go up in k. Even the multiplicities, shown as exponents on the eigenvalues, may be represented by simple formulas: in the second column we see $m = 2k$ and in the third column $m = (2k+1)(k-1)$.

We shall look for more sum rules. From any trace formula we have a sum rule for the eigenvalues.

$$Trace(A^r) = \sum_{i=1,N} (E_i)^r \equiv \Sigma_r. \tag{60}$$

Let's return to the matrix A^2 and write

$$(A^2)_{x,z} = \sum_y A_{x,y} A_{y,z}, \tag{61}$$

$$y = a\bar{x}, \ \forall \, a \in x; \quad z = b\bar{y}, \ \forall \, b \in y. \tag{62}$$

There are two distinct cases. One is where $b = a$ and this is just the diagonal part of A^2 as earlier noted. The other is where $b \in \bar{x}$. This leads us to the following construction.

$$A^2 = (k+1)\, I + \Delta_2, \tag{63}$$

$$(\Delta_2)_{x,z} = \delta_{z,bx/a}, \quad a \in x, \ \ b \in \bar{x}, \tag{64}$$

and I is the unit matrix. Reading this, it says that Δ_2 connects to a new word z that has one letter removed from the original word x and replaced by a letter from the complement $\bar{x}$.

We shall use this formula (64) to calculate some higher power traces. First, however, we will need the formula,

$$(\Delta_2)^2 = C_0\, I + C_2\, \Delta_2 + C_4\, \Delta_4. \tag{65}$$

From (64), we count $C_0 = k(k+1)$; and with some care we count $C_2 = (2k-1)$. Δ_4 is a matrix that connects from x to a word with two letters removed and replaced with two letters from $\bar{x}$. We count $C_4 = 4$ because there are 4 such paths; and the number of

such paired sets is $[(k+1)k/2][k(k-1)/2]$. We also note that there is zero Trace for Δ_2, Δ_4 and also $\Delta_2 \Delta_4$.

With this, we now calculate,

$$A^4 = (k+1)(2k+1) \, I + (4k+1) \, \Delta_2 + 4 \, \Delta_4; \tag{66}$$

and this leads to,

$$\Sigma_4 = Trace(A^4) = N(k+1)(2k+1), \tag{67}$$
$$\Sigma_6 = Trace(A^6) = N[(k+1)^2(2k+1) + (4k+1)k(k+1)] \tag{68}$$
$$\Sigma_8 = Trace(A^8) = N[(k+1)^2(2k+1)^2 + (4k+1)^2 k(k+1) \tag{69}$$
$$+4(k+1)k^2(k-1)]. \tag{70}$$

We have verified that all the eigenvalues given in the table above do satisfy these summation formulas.

We can guess that the sum rules for the trace of the odd powers of A will be zero; but this is verified only within the limitations that $r < (2k+1)$.

All we really wanted here was to see that there were no eigenvalues equal to zero; however, what we have uncovered is quite suggestive of a larger mathematical reservoir hiding behind these elementary investigations.

6.   Discussion

Well, this looks like there should be a general theorem and it might involve something about irreducible representations of the permutation group. But I don't see how to prove it.

As an example of the boundaries of this conjecture, suppose we review the above calculations for $[w_L] = 2$ but limit ourselves to tensors of rank two. Then we find, for instance at $W = a^2 b^2$,

$$(aa; bb) + 2(ab; ab) = 0, \tag{71}$$

which tells us a relation between two integrals; but neither of them must be zero.

Another example: suppose the tensor is not symmetric in its indices. Consider $[w_L] = 1, W = abc$ and say that the second rank tensor is anti-symmetric in its indices. Then one finds three equations, which have the solution,

$$(a; bc) = (b; ca) = (c; ab), \tag{72}$$

but they need not vanish.

References

[1] C. Schwartz, "Tachyons in General Relativity," *J. Math. Phys.* **52**, 052501 (2011); arXiv:1011.4847