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Analytic Functions of a General Matrix Variable

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Abstract

Recent innovations on the differential calculus for functions of non-commuting variables, begun for a quaternionic variable, are now extended to the case of a general matrix over the complex numbers. The expansion of F(X+Delta) is given to first order in Delta for general matrix variables X and Delta that do not commute with each other.

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1 Introduction

In a recent paper [1] I showed how to expand

$$F(x+\delta) = F(x) + F^{(1)}(x) + O(\delta^2) \tag{1.1}$$

when both x and δ were general quaternionic variables, thus did not commute with each other:

$$F^{(1)}(x) = F'(x)\delta_1 + [F(x) - F(x^*)](x - x^*)^{-1}\,\delta_2, \quad \delta = \delta_1 + \delta_2, \tag{1.2}$$

with specific formulas on how to construct the two components of δ.

Now we shall extend that analysis to a more general situation.

Consider the N x N matrices X over the complex numbers and arbitrary analytic functions $F(X)$ with such a matrix as its variable. We seek a general construction for the first order term $F^{(1)}(X)$ when we expand $F(X+\Delta)$ given that Δ is small but still a general N x N matrix that does not commute with X.

The first step, as before, is to represent the function F as a Fourier transform,

$$F(X) = \int dp f(p) e^{ipX} \tag{1.3}$$

where the integral may go along any specified contour in the complex p-plane; and then we also make use of the expansion,

$$e^{(X+\Delta)} = e^X[1 + \int_0^1 ds e^{-sX}\Delta\, e^{sX} + O(\Delta^2)]. \tag{1.4}$$

Another well-known expansion, relevant to what we see in (1.4), is

$$e^A\; B\; e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \ldots, \tag{1.5}$$

involving repeated use of the commutators, $[A,B] = AB - BA$.

2 Diagonalization

The first step is to assume that we can find a matrix S that will diagonalize the matrix X at any given point in the space of such matrices.

$$A = S\; X\; S^{-1}, \qquad A_{i,j} = \delta_{i,j}\;\lambda_i, \qquad i,j = 1,\ldots,N \tag{2.1}$$

and we carry out the same transformation on the matrix Δ:

$$B = S \; \Delta \; S^{-1} \tag{2.2}$$

but, of course, the matrix B will not be diagonal.

Our task is to separate the matrix B into separate parts, each of which will behave simply in the expansion of Eq. (1.5). The first step is to recognize that the diagonal part of the matrix B, call it B_0, commutes with the diagonal matrix A and thus we have

$$e^{tA} \; B_0 \; e^{-tA} = B_0, \tag{2.3}$$

where we have $t = -isp$ from (1.3), (1.4).

Next we look separately at each off-diagonal element of the matrix B: that is, $B_{(i,j)}$ is the matrix that has only that one off-diagonal ($i \neq j$) element of the given matrix B, and all the rest are zeros. The first commutator is simply

$$[A, B_{(i,j)}] = r_{ij} \; B_{(i,j)}, \qquad r_{ij} = \lambda_i - \lambda_j \tag{2.4}$$

and then the whole series can be summed:

$$e^{tA} \; B_{(i,j)} e^{-tA} = e^{tr_{ij}} \; B_{(i,j)}. \tag{2.5}$$

Putting this all together, we have

$$S \; F^{(1)} \; S^{-1} = \int dp f(p) \; ip \; e^{ipA} \; \int_0^1 ds \; [B_0 + \sum_{i \neq j} \; e^{-ispr_{ij}} \; B_{(i,j)}]. \tag{2.6}$$

It is trivial to carry out the integrals over s; and we thus come to the final answer

$$F^{(1)}(X) = F'(X) \; \Delta_0 + \sum_{i \neq j} [F(X) - F(X - r_{ij} I)] \; r_{ij}^{-1} \; \Delta_{(i,j)} \tag{2.7}$$

where

$$\Delta_\mu \equiv S^{-1} \; B_\mu \; S, \qquad \mu = 0, (i,j). \tag{2.8}$$

3 Discussion

The general structure of the result, Eq.(2.7), is similar to what we found in earlier work, Eq.(1.2): the first term $F'(X)$ looks like ordinary differential calculus and goes with that part of Δ that commutes with the local coordinate X; the remaining terms are non-local, involving the function F evaluated at discrete points separated from X by specific multiples of the unit matrix I.

While this final formula appears not as explicit as the previous result found for quaternionic variables (or for variables based upon the algebra of SU(2)), in any practical situation we have computer programs that can calculate the matrix operations referred to above with great efficiency.

The quantities r_{ij}, which may be real or complex numbers, can be called the "roots" following the familiar treatment of Lie algebras. They have some properties, such as $r_{ij} = -r_{ji}$ and sum rules that involve traces of the original matrix X and powers of X.

What happens if the eigenvalues of X are degenerate? Suppose, for example, that $\lambda_1 = \lambda_2$. This means that r_{12} and r_{21} are zero. If we look at Eq. (2.7), we see that the terms $\Delta_{(1,2)}$ and $\Delta_{(2,1)}$ then have the coefficient $F'(x)$. Thus they simply add in with Δ_0. In the extreme case when all the eigenvalues are identical, then the answer is $F(X+\Delta) = F(X) + F'(X)\,\Delta + O(\Delta^2)$, which is old fashioned differential calculus for commuting variables.

Appendix A

A more abstract form of the result Eq. (2.7) is the following.

$$F^{(1)}(X) = F'(X)\,\Delta_0\, + [C, F(X)], \tag{A.1}$$

$$[C, X] = \Delta - \Delta_0, \qquad [\Delta_0, X] = 0. \tag{A.2}$$

Here the matrix C is defined implicitly, through its commutator with X, rather than explicitly; and the matrix Δ_0 is the same as previously discussed. One may readily confirm the correctness of this formula in the case of $F(X) = X^n$.

This alternative formalism may also be applied to the case of a quaternionic variable x, which was studied in reference [1]. In that case we find $\Delta_0 = \delta_1$ and $C = \frac{1}{4r^2}[x, \delta]$.

It is interesting that this new formalism manages to hide the non-locality, which was a prominent feature of the original analysis.

References

[1] C. Schwartz, arXiv:0803.3782 [math.FA]