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Calculus with a quaternionic variable

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Most of theoretical physics is based on the mathematics of functions of a real or a complex variable; yet we frequently are drawn in trying to extend our reach to include quaternions. The noncommutativity of the quaternion algebra poses obstacles for the usual manipulations of calculus, but we show in this paper how many of those obstacles can be overcome. The surprising result is that the first order term in the expansion of $F(x+\delta)$ is a compact formula involving both $F'(x)$ and $[F(x)-F(x^*)]/(x-x^*)$. This advance in the differential calculus for quaternionic variables also leads us to some progress in studying integration. © *2009 American Institute of Physics.* [DOI: 10.1063/1.3058642]

I. INTRODUCTION

We are very familiar with functions of a real or complex variable x which we can expand, in the mode of differential calculus, as

$$F(x+\delta) = F(x) + F'(x)\delta + \tfrac{1}{2}F''(x)\delta^2 + \cdots . \tag{1.1}$$

However, what if we consider a quaternionic variable

$$x = x_0 + ix_1 + jx_2 + kx_3 \tag{1.2}$$

involving four real variables x_μ, $\mu=0, 1, 2, 3$, along with those quaternions i, j, k which do not commute with one another,

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \ldots . \tag{1.3}$$

The small quantity δ will also involve all those quaternions. How then can we expect anything as neatly packaged as Eq. (1.1)?

This is a long-standing challenge to mathematicians. The earliest attempt to extend the usual concept of the derivative dF/dx with a quaternionic dx failed dramatically. The subsequent approach focused on the four real variables,

$$dF(x) = \sum_\mu \frac{\partial F(x)}{\partial x_\mu} dx_\mu . \tag{1.4}$$

That approach, often called quaternionic analyticity, springs from the work in the 1930s by Fueter[1] and his school, with more accessible articles reviewing that subject available in Refs. 2 and 3. Some more recent attempts to advance that work may be found in Refs. 4–6. In Appendix C I provide a rough summary of the Fueter approach.

The first new result presented in this paper is an alternative approach to the differential calculus—something between relying on the whole quaternionic variable, dF/dx, and resorting to the four-component real variables, as in Eq. (1.4). This starts, in Sec. II, with the separation of the

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quaternionic displacement δ into two parts, one "parallel" and the other "perpendicular" to the quaternionic variable x as it may be envisioned in that four-dimensional space.

The subsequent sections show how this leads to a surprisingly compact and general formula for the quaternionic version of the expansion (1.1):

$$F(x+\delta) = F(x) + F'(x)\delta_{\parallel} + (F(x) - F(x^*))/(x - x^*)\delta_{\perp} + O(\delta^2). \tag{1.5}$$

In the second part of this paper we look at integration and find that the new form of the quaternionic differential leads to new results in this other realm of calculus.

II. LOCAL COORDINATES

The standard approach to quaternionic variables starts with a global set of imaginary coordinates,

$$x = x_0 + ix_1 + jx_2 + kx_3, \quad \delta = \delta_0 + i\delta_1 + j\delta_2 + k\delta_3. \tag{2.1}$$

We now want to write x in a different way:

$$x = x_0 + u_x r, \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad u_x^2 = -1, \tag{2.2}$$

where u_x is a unit imaginary that varies in its i, j, k compositions as x moves from one point to another in that four-dimensional space. This is analogous to the use of polar coordinates in two-dimensional Euclidean space.

Now we want to decompose the quaternionic quantity δ in a particular way that refers to this local coordinate system,

$$\delta = \delta_{\parallel} + \delta_{\perp}, \quad \delta_{\parallel} = \tfrac{1}{2}(\delta - u_x \delta u_x), \quad \delta_{\perp} = \tfrac{1}{2}(\delta + u_x \delta u_x), \tag{2.3}$$

which leads to the algebraic relations

$$\delta_{\parallel} u_x = u_x \delta_{\parallel}, \quad \delta_{\perp} u_x = -u_x \delta_{\perp}. \tag{2.4}$$

The essence of this approach is expressed in the nomenclatures parallel and perpendicular for these two components of δ as they relate to the local quaternion x. The most useful way to write these relations is

$$\delta_{\parallel} x = x\delta_{\parallel}, \quad \delta_{\perp} x = x^* \delta_{\perp}, \tag{2.5}$$

where * is the complex conjugation operator that changes the sign of all imaginaries.[1] Now we shall give three examples of how to expand $F(x+\delta)$ with this simple machinery.

III. THE FUNCTION $F(x) = x^n$

We calculate directly

$$(x+\delta)^n = x^n + \sum_{m=0}^{n-1} x^{n-m-1}\delta x^m + O(\delta^2). \tag{3.1}$$

Putting in the separation $\delta = \delta_{\parallel} + \delta_{\perp}$ and using the properties of Eq. (2.5), the sum becomes

$$\sum = \sum_{m=0}^{n-1} (x^{n-1}\delta_{\parallel} + x^{n-m-1}x^{*m}\delta_{\perp}) = nx^{n-1}\delta_{\parallel} + (x^n - x^{*n})(x - x^*)^{-1}\delta_{\perp}, \tag{3.2}$$

where we evaluated a finite geometric series.

[1]The efficacy of this technique was discovered as the result of a more long-winded calculation, which may be seen in Ref. 7.

IV. THE EXPONENTIAL FUNCTION

For a general quaternionic variable x, we can define the exponential function in the usual way:

$$e^x = \lim_{N\to\infty}\left(1+\frac{x}{N}\right)^N, \tag{4.1}$$

and this leads us to the expansion

$$e^{(x+\delta)} = e^x\left[1+\int_0^1 dse^{-sx}\delta e^{sx} + O(\delta^2)\right], \tag{4.2}$$

which is correct for the situation where x and δ do not commute. For a derivation of this formula, see Appendix A.

Putting in a real parameter p and following the course set above, we get the expansion

$$e^{p(x+\delta)} - e^{px} = \int_0^1 dspe^{(1-s)px}(\delta_\parallel + \delta_\perp)e^{spx} \tag{4.3}$$

$$= \int_0^1 dspe^{px}\delta_\parallel + \int_0^1 dspe^{(1-s)px}e^{spx^*}\delta_\perp \tag{4.4}$$

$$= pe^{px}\delta_\parallel + (e^{px} - e^{px^*})(x-x^*)^{-1}\delta_\perp \tag{4.5}$$

to first order in δ.

V. GENERAL ANALYTIC FUNCTION $F(x)$

For a general analytic function $F(x)$ of a quaternionic variable x, we start by assuming a representation as a Laplace transform:

$$F(x) = \int dpf(p)e^{px}, \tag{5.1}$$

where p is a real variable. We then use the result of the previous section to obtain

$$F(x+\delta) - F(x) = F'(x)\delta_\parallel + (F(x) - F(x^*))(x-x^*)^{-1}\delta_\perp + O(\delta^2), \tag{5.2}$$

where $F'(x)$ is the derivative of the function $F(x)$ calculated as if x were a real variable. This is our general result. The particular result of Sec. III, for $F(x)=x^n$, also fits this general formula, and thus it also works for any power series $F(x)=\Sigma_n c_n x^n$.

The authors of Ref. 4 have taken an approach somewhat similar to what is done here. They introduced a local unit imaginary (which they call *iota*) that is the same as what we have defined as u_x. However, they limit their differentiations to displacements that are restricted to the two-dimensional space of what we call $\delta_\parallel$ without allowing any of $\delta_\perp$. In that way they merely reproduce what is known about ordinary complex variables.

VI. FURTHER EXERCISES

Let us define the first order differential operator $\mathcal{D}$, from Eq. (5.2), as

$$F(x+\delta) = F(x) + \mathcal{D}F(x) + O(\delta^2), \tag{6.1}$$

with

$$\mathcal{D}F(x) = F'(x)\delta_{\parallel} + (F(x) - F(x^*))(x - x^*)^{-1}\delta_{\perp}. \tag{6.2}$$

Several interesting exercises are now suggested.

Calculate $\mathcal{D}(F(x)G(x))$ and verify the applicability of Leibnitz' rule. Calculate $\mathcal{D}(1/G(x))$ and also $\mathcal{D}F(G(x))$.

VII. ALTERNATIVE ARRANGEMENTS

Still another way to represent our result for the first order differential is in terms of some *partial* derivatives, defined as follows:

$$\mathcal{D}F(x) = \frac{\partial F(x)}{\partial x_{\parallel}}dx_{\parallel} + \frac{\partial F(x)}{\partial x_{\perp}}dx_{\perp}, \tag{7.1}$$

$$\frac{\partial F(x)}{\partial x_{\parallel}} \equiv F'(x), \quad \frac{\partial F(x)}{\partial x_{\perp}} \equiv (F(x) - F(x^*))(x - x^*)^{-1}. \tag{7.2}$$

Suppose we restrict the functions $F(x)$ to be real: that is, the coefficients c_n in $F=\Sigma_n c_n x^n$ or the amplitudes $f(p)$ in the Laplace transform are real numbers. Then it is noted that the terms in Eq. (5.2) can be written with the displacement quaternions, $\delta_{\parallel}$ and $\delta_{\perp}$, written either to the right or to the left of their accompanying factors. This is obvious in the case of $F'(x)$, since $\delta_{\parallel}$ commutes with x. For the second term, we know that $\delta_{\perp}$ does not commute with x; it takes the complex conjugate. However, we note that the whole expression $(F-F^*)/(x-x^*)$ is real; therefore this rearrangement is possible. The same rearrangement can be done with Eq. (7.1).

These considerations lead us to note that the second term in the equation for $\mathcal{D}F(x)$ can be written in terms of commutators as

$$(F(x) - F(x^*))(x - x^*)^{-1}\delta_{\perp} = [C, F(x)], \tag{7.3}$$

where C is defined by

$$[C, x] = \delta_{\perp}, \quad C = \frac{1}{x^* - x}\delta_{\perp}. \tag{7.4}$$

What is somewhat surprising about this alternative arrangement, Eq. (7.3), is that the expression on the left hand side is manifestly nonlocal, involving things evaluated at the point x and also at the remote point x^*; however, the right hand side appears to be local, involving only x. This confusion is removed when one recognizes that C is a nonlocal operator, involving $\delta_{\perp}$, which changes x to x^*.

VIII. SECOND ORDER TERMS

Let us return to the exponential function (4.1) and proceed with the expansion

$$e^{(x+\delta)} = e^x\left[1 + \int_0^1 ds e^{-sx}\delta e^{sx} + \int_0^1 dt \int_0^{1-t} ds e^{-(s+t)x}\delta e^{tx}\delta e^{sx} + O(\delta^3)\right]. \tag{8.1}$$

The best approach is to combine the exponential function and the Laplace transform from the beginning. Writing $F(x+\delta)=F(x)+F^{(1)}+F^{(2)}+\cdots$, we now look at

$$F^{(2)} = \int dp f(p) e^{px} p^2 \int_0^1 dt \int_0^{1-t} ds e^{-(s+t)px}\delta e^{tpx}\delta e^{spx}. \tag{8.2}$$

Again, we decompose δ and after a bit more work arrive at the result for the second order term,

$$F^{(2)} = \tfrac{1}{2}F''(x)\delta_{\parallel}^{2} + (F(x) - F(x^{*}))(x - x^{*})^{-2}(\delta_{\perp}\delta_{\parallel} - \delta\delta_{\perp}) + F'(x)(x - x^{*})^{-1}\delta\delta_{\perp} + F'(x^{*})(x^{*} - x)^{-1}\delta_{\perp}\delta_{\parallel}. \tag{8.3}$$

It is also true that, with the first order term given as $F^{(1)}(x)=\mathcal{D}F(x)$, the second order result can be written as

$$F^{(2)}(x) = \tfrac{1}{2}\mathcal{D}\mathcal{D}F(x). \tag{8.4}$$

To verify this one needs the preliminary formulas,

$$\mathcal{D}x = \delta, \quad \mathcal{D}x^{*} = \delta^{*}, \quad \mathcal{D}\delta = 0, \quad \mathcal{D}u_x = \frac{1}{r}\delta_{\perp}, \tag{8.5}$$

along with $\delta_{\parallel}^{*}\delta_{\perp} = \delta_{\perp}\delta_{\parallel}$ and $\delta_{\perp}^{*} = -\delta_{\perp}$. See further in Appendix D.

IX. DISCUSSION ON DIFFERENTIALS

It is surprising how simple and how general the new results obtained here are. It is also noteworthy that our differential operators are no longer local: they involve $F(x^{*})$ along with $F(x)$.

One may ask what restrictions there are on the functions $F(x)$ considered above. At first, one would say that they should be real analytic functions; having terms such as xax where a is a general quaternion would certainly cause trouble.[2] One can extend this condition slightly by allowing $F(x)$ [but not the function $G(x)$ in Sec. VI] to be a real function with arbitrary quaternions multiplying from the left. That is, the power series form $F=\Sigma_n c_n x^n$ could have arbitrary numbers c_n. This bias to the left hand side can be reversed if we change the original steps (4.2), setting $s \rightarrow 1-s$, and (5.1), putting $f(p)$ on the right hand side.

The Taylor series we have discussed above are expansions about the origin $x=0$. In the usual complex analysis such power series may be about any fixed point $x=x_f$, but such a quaternion constant put in the middle of our expressions would appear to cause trouble. That trouble could be avoided by limiting x_f to be real, but there is a better way. If we define a new quaternionic variable $y=x-x_f$ then we may proceed as done above only using the appropriate unit imaginary u_y instead of the original u_x in order to separate the displacement δ into parallel and perpendicular components.

One may also ask if this general method may be applied to some other kind of noncommuting algebra beyond the quaternions. I believe that something very similar can be done starting with a Clifford algebra. Other examples are given in Appendix B and in Ref. 8.

X. INTRODUCTION TO INTEGRATION

When the conventional approach to analyticity of quaternionic functions failed in the differential calculus, the main push was then in the realm of integral calculus. The key result of the Fueter school was a third order differential equation that could define a "regular" function of a quaternionic variable, just as the Cauchy–Riemann equation was a first order constraint on functions of a complex variable $z=x+iy$. That approach is described roughly in Appendix C. Their result is a focus on integrals over a three-dimensional surface in the four-dimensional space. With the construction of the quaternionic differential operator $\mathcal{D}$ we can do something quite different about integration, as is shown in the following two sections.

XI. THE LINE INTEGRAL

In ordinary calculus of functions of the real variable t, we know what is meant by an integral, such as $\int f(t)dt$. However, when we first consider quaternionic (or other noncommuting) variables it is unclear even how to write such an expression. We shall pursue that path in Sec. XII.

[2]This use of the term "real analytic" differs from that found in Ref. 3.

Alternatively, we can start with the defining relation between the integral and the differential:

$$\int_a^b df(t) = \int_a^b \frac{df(t)}{dt} dt = f(b) - f(a), \tag{11.1}$$

and this is what we shall generalize for our noncommuting quaternionic variable x as

$$\int_a^b \mathcal{D}F(x) = F(x_b) - F(x_a). \tag{11.2}$$

We define this integral as an additive operation along a path in that four-dimensional space of the real variables x_μ,

$$x = x_{\text{path}}(s), \quad x_{\text{path}}(0) = x_a, \quad x_{\text{path}}(1) = x_b, \tag{11.3}$$

where s is a real continuous parameter.

Next, we subdivide that path, whatever it may be, into a large number of infinitesimal increments,

$$\int_a^b = \sum_{n=1}^{n=N} \int^{(n)}, \quad \int^{(n)} = \int_{x_{n-1}}^{x_n}, \quad n = 1, \ldots, N, \tag{11.4}$$

where $x_0 = x_a$ and $x_N = x_b$.

In any segment of this path we choose the line of integration, with the integrand $\mathcal{D}F(x)$, to be the sum of two infinitesimal parts:

$$x_n - x_{n-1} = \delta = \delta_\parallel + \delta_\perp. \tag{11.5}$$

The first part is parallel to the direction of x at that point, giving the contribution

$$\int_\parallel \mathcal{D}F(x) = F'(x)\delta_\parallel. \tag{11.6}$$

Then the second part is perpendicular, giving the contribution

$$\int_\perp \mathcal{D}F(x) = [F(x) - F(x^*)](x - x^*)^{-1}\delta_\perp. \tag{11.7}$$

The sum of these two parts is thus nothing other than

$$F(x_n) - F(x_{n-1}) \tag{11.8}$$

to first order in the interval δ. The entire sum then results in Eq. (11.2).

Another general proof can proceed as follows. If we start with the coordinate along the path $x(s) = x_{\text{path}}(s)$, then we can simply write

$$\mathcal{D}x(s) = ds \frac{dx(s)}{ds} \tag{11.9}$$

since there is no commutativity problem in this representation. It is also true that we can express any function composed of powers of x as

$$F(x(s)) = A(s) + B(s)x(s), \tag{11.10}$$

where A and B are real functions, the only quaternions being in the single factor $x(s)$. We then see that the integral becomes quite ordinary:

$$\int_a^b \mathcal{D}F(x(s)) = \int_0^1 ds\frac{dF(x(s))}{ds} = F(x(s))|_0^1 = F(x_b) - F(x_a). \tag{11.11}$$

Since this differential operator $\mathcal{D}$ obeys the Leibnitz rule, we get the identity, usually called "integration by parts,"

$$\int_a^b F(x)\mathcal{D}G(x) = F(x_b)G(x_b) - F(x_a)G(x_a) - \int_a^b (\mathcal{D}F(x))G(x). \tag{11.12}$$

Loosely speaking, integration is the inverse of differentiation. What we see in Eqs. (11.1) and (11.2) is one statement of that relationship. However, there is also the other form, which is stated for real variables as

$$\frac{d}{dt}\int^t f(t')dt' = f(t). \tag{11.13}$$

For our quaternionic variables we start by looking at

$$\mathcal{D}_x\int^x \mathcal{D}_{x'}F(x') \tag{11.14}$$

and then apply the first differential operator to the coordinate x in two parts: first the $\delta_\parallel$ part and then the $\delta_\perp$ part. The result is just the integrand evaluated at the point x:

$$= \mathcal{D}_x F(x), \tag{11.15}$$

and this is just what we should expect from the right hand side of Eq. (11.2), with x_b replaced by x.

XII. THE OTHER LINE INTEGRAL

If we look at the common real integral and try to guess how to generalize it to the noncommutative quaternions, we might start with

$$\int f(t)dt \overset{?}{\rightarrow} \frac{1}{2}\int (dxF(x) + F(x)dx). \tag{12.1}$$

But why should dx only appear on the outside; why not also in the middle of the function $F(x)$?

Let us try a most symmetrical arrangement with the function $F(x)=x^n$:

$$\int t^n dt \overset{?}{\rightarrow} \frac{1}{n+1}\int (dxx^n + xdxx^{n-1} + x^2dxx^{n-2} + \cdots + x^n dx). \tag{12.2}$$

However, we can recognize that the long expression in parentheses on the right hand side of this is nothing other than $\mathcal{D}x^{n+1}$:

$$\mathcal{D}F(x) \equiv F(x+dx) - F(x) \quad \text{to first order in } dx. \tag{12.3}$$

So we would then write

$$\int t^n dt \rightarrow \frac{1}{n+1}\int \mathcal{D}x^{n+1} = \frac{x^{n+1}}{n+1} \tag{12.4}$$

using our defining equation (11.2). Now, this looks quite familiar.

We can extend this to any power series and thus offer the following rule. For any analytic function of a real variable $f(t)$, for which we know the integral

$$\int f(t)dt = h(t), \tag{12.5}$$

we can make the correspondence to quaternionic integration as follows:

$$\int f(t)dt \to \int \mathcal{D}h(x) = h(x). \tag{12.6}$$

While this may look trivial for real and complex variables, it is something new for noncommuting variables. This arises because we have carefully defined and studied the operator $\mathcal{D}$.

XIII. DISCUSSION ON INTEGRATION

Following what was stated earlier, we do require the functions $F(x)$ to be real analytic functions along the path of integration. Terms such as xax would be allowed only for real constants a.

Our first new result Eq. (11.2) implies that the result of the integration depends only on the end points and is independent of the path. This is true if we also require that the function $F(x)$ be single valued. Then, we have the result that the integral over any closed path, ending up at the same point where it started, is zero. This is a significant new result, carrying the world of contour integration over from the complex domain to the quaternionic. Our second new result, Eqs. (12.5) and (12.6), opens up considerable possibilities for integration of quaternionic functions.

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APPENDIX A: EXPANSION OF THE EXPONENTIAL

Here we give a derivation of the formula (4.2) for any noncommuting quantities x and δ,

$$e^{(x+\delta)} = \lim_{N\to\infty}\left[1 + \frac{x}{N} + \frac{\delta}{N}\right]^N \tag{A1}$$

$$= \lim_{N\to\infty}\left\{\left[1 + \frac{x}{N}\right]^N + \sum_{m=0}^{N-1}\left[1 + \frac{x}{N}\right]^{N-m-1}\frac{\delta}{N}\left[1 + \frac{x}{N}\right]^m + O(\delta^2)\right\}. \tag{A2}$$

In taking the limit $N\to\infty$, we convert the sum over m to an integral over $s=m/N$ and this yields

$$e^{(x+\delta)} = e^x + \int_0^1 ds e^{(1-s)x}\delta e^{sx} + O(\delta^2). \tag{A3}$$

APPENDIX B: SU(2,C)

Here we shall extend the general method used above for a quaternionic variable to something built on a Lie Algebra—specifically SU(2). Here is the Lie algebra:

$$[J_1,J_2] = J_3, \quad [J_2,J_3] = J_1, \quad [J_3,J_1] = J_2, \tag{B1}$$

where the three J are understood to be matrices over the complex numbers. In particular, we shall use the relations

$$e^{\theta J_3}J_1e^{-\theta J_3} = J_1 \cos\theta + J_2 \sin\theta, \quad e^{\theta J_3}J_2e^{-\theta J_3} = J_2 \cos\theta - J_1 \sin\theta, \tag{B2}$$

which follow from ((B1).

The new variable x is to be constructed with four real parameters as

$$x = x_0 I + x_1 J_1 + x_2 J_2 + x_3 J_3 \tag{B3}$$

and we want to expand $F(x+\delta)=F(x)+F^{(1)}+O(\delta^2)$, where δ is a small quantity in that same space of matrices as x. Our first step is to define a local coordinate system at the given point x. By a suitable linear transformation (rotation) of the Lie algebra we make the coordinate x appear as

$$x = x_0 I + r J_3, \tag{B4}$$

where we recognize that $r^2=x_1^2+x_2^2+x_3^2$.

We can now separate the displacement $\delta=\delta_\parallel+\delta_\perp$ as follows:

$$\delta_\parallel = \delta_0 I + \delta_3 J_3, \quad \delta_\perp = \delta_1 J_1 + \delta_2 J_2. \tag{B5}$$

Now we are ready to study the first order term in the expansion, again using the representation of $F(x)$ in terms of the exponential function,

$$F^{(1)} = \int dp f(p) p e^{px} \int_0^1 ds e^{-spx} \delta e^{spx}. \tag{B6}$$

Since $\delta_\parallel$ commutes with x, the first part of this is simply $F'(x)\delta_\parallel$. For the part with $\delta_\perp$ we use the formulas (B2), where θ is replaced by $-spr$. The integrals over s are trivial and we merely write $\sin(pr)$ and $\cos(pr)$ in terms of $e^{\pm ipr}$ to get our final result,

$$F(x+\delta) = F(x) + F'(x)\delta_\parallel + \{F(x+ir) - F(x-ir)\}\frac{1}{2ir}\delta_\perp$$
$$+ \{F(x+ir) + F(x-ir) - 2F(x)\}\frac{1}{2r}[J_3,\delta_\perp] + O(\delta^2). \tag{B7}$$

It should be noted that the δ-related factors in Eq. (B7) can be written in the following way:

$$[J_3,\delta_\perp] = \frac{1}{r}[x,\delta], \tag{B8}$$

$$\delta_\perp = -\frac{1}{r^2}[x,[x,\delta]], \tag{B9}$$

$$\delta_\parallel = \delta - \delta_\perp. \tag{B10}$$

This means that we do not have to carry out the "rotation" that gave us Eq. (B4) explicitly; the talk about choosing a local coordinate system is merely rhetorical.

I expect that this method can be extended to other Lie algebras, with the quantity $\delta_\perp$ subdivided into distinct portions according to the roots of the particular algebra. The system of Eqs. (B9) and (B10) would be adapted to make those separations using the known values of the roots, and those root values would also appear in the final generalization of Eq. (B7). Extension of this method to general matrix variables, over the complex numbers, is given in Ref. 8.

APPENDIX C: FUETER'S DIFFERENTIAL EQUATION

The literature on Fueter's analysis of quaternionic functions points to a third order differential equation as his key result, extending the familiar Cauchy–Riemann (first order) equation for functions of a complex variable,

$$\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)f(z=x+iy)=0. \tag{C1}$$

Here I wish to present a rather simple derivation of that result, starting with the exponential function of our quaternionic variable,

$$x=x_0+ix_1+jx_2+kx_3=x_0+\mathbf{u}\cdot\mathbf{x}, \quad \mathbf{u}=(i,j,k), \quad r^2=\mathbf{x}\cdot\mathbf{x}, \tag{C2}$$

$$e^{px}=e^{px_0}\left(\cos pr+\mathbf{u}\cdot\mathbf{x}\frac{\sin pr}{r}\right). \tag{C3}$$

Here is Fueter's first order differential operator,

$$\Box=\frac{\partial}{\partial x_0}+i\frac{\partial}{\partial x_1}+j\frac{\partial}{\partial x_2}+k\frac{\partial}{\partial x_3}=\frac{\partial}{\partial x_0}+\mathbf{u}\cdot\nabla, \tag{C4}$$

and we calculate its action on the exponential function and find the result

$$\Box e^{px}=-2e^{px_0}\frac{\sin pr}{r}. \tag{C5}$$

This is not zero [as in the complex case, Eq. (C1)] but it is rather simple (and real). Moreover, we recognize this function as a solution of the four-dimensional Laplace equation,

$$\triangle_4=\Box\Box^*=\frac{\partial^2}{\partial x_0^2}+\triangle_3. \tag{C6}$$

So here are two forms of Fueter's third order differential equation,

$$\triangle_4\Box e^{px}=0, \quad \triangle_4\Box x^n=0, \tag{C7}$$

where the second result comes from expanding the first result in a power series in the parameter p. Thus any superposition (with real coefficients) of the powers or the exponential will satisfy this condition, and this is the basis for what they define as holomorphic functions of a quaternionic variable. They also exclude functions with terms such as xax with arbitrary quaternion constants a—just as we have done in the present paper.

From those differential equations (C7) (Cauchy–Riemann–Fueter), some integral theorems follow. In the case of complex variables (C1), we get the familiar result that any integral around a closed path in the complex plane will be zero [with suitable analytic and single-valued behavior of the function $f(z)$]. In the quaternionic case, the relevant integral is over a closed three-dimensional surface in the four-dimensional space of the x_μ.

Here is a surprise! Look at the result of our differential operator $\mathcal{D}$ acting on the exponential function e^{px}, Eq. (4.5). The coefficient of $\delta_\perp$ is the same function that we see on the right hand side of Eq. (C5). So, here is a new identification:

$$\frac{\partial F(x)}{\partial x_\perp}=-\frac{1}{2}\Box F(x). \tag{C8}$$

APPENDIX D: CACULATING $\mathcal{D}u_x$

Here is a derivation of the last item in Eq. (8.5), which is made easy if we take a geometric perspective as we write the coordinate in four dimensions as $x=x_0+ru_x$. Start by writing

$$\mathcal{D}u_x = \alpha\delta_{\parallel} + \beta\delta_{\perp}, \tag{D1}$$

where α and β are to be determined. First, consider a displacement that has only $\delta_{\parallel}$: this should not change u_x at all, since such a displacement can only change x_0 and r. Thus, we see that $\alpha=0$.

Next, consider a displacement that has only $\delta_{\perp}$: this should not change x_0 or r. So we write

$$\mathcal{D}u_x = \mathcal{D}_{\perp}u_x = \mathcal{D}_{\perp}(x-x_0)/r = \frac{1}{r}\mathcal{D}_{\perp}x = \frac{1}{r}\delta_{\perp}. \tag{D2}$$

Following the result for $F^{(2)}$ in Sec. VIII, one may wonder whether the entire Taylor series might be written as

$$F(x+\delta) = \sum_{k=0}^{\infty}\frac{1}{k!}\mathcal{D}^k F(x) = e^{\mathcal{D}}F(x). \tag{D3}$$

This may be readily verified for the functions $F(x)=x^n$, starting with the equation

$$e^{\mathcal{D}}xe^{-\mathcal{D}} = x+\delta. \tag{D4}$$

For the exponential function, define

$$Q(p) = e^{\mathcal{D}}e^{px}, \tag{D5}$$

then calculate dQ/dp and use Eq. (D4).

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