Relativistic quaternionic wave equation. II

Charles Schwartz[a]

Department of Physics, University of California, Berkeley, California 94720

Further results are reported for the one-component quaternionic wave equation recently introduced. A Lagrangian is found for the momentum-space version of the free equation, and another, nonlocal in time, is found for the complete equation. Further study of multiparticle systems has us looking into the mathematics of tensor products of Hilbert spaces. The principles of linearity and superposition are also clarified to good effect in advancing the quaternionic theory. © *2007 American Institute of Physics.* [DOI: 10.1063/1.2735441]

I. INTRODUCTION

In a recent paper,[1] we introduced the relativistic quaternionic wave equation, with $\psi = \psi(\mathbf{x}, t)$,

$$\frac{\partial}{\partial t} \psi i = \mathbf{u} \cdot \nabla \psi + m\psi j = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right)\psi + m\psi j, \tag{1.1}$$

which we call the free wave equation, and an extended version,

$$\frac{\partial}{\partial t} \psi i = \mathbf{u} \cdot \nabla \psi + e\varphi\psi - e\mathbf{u} \cdot \mathbf{A}\psi i + m\psi e^{ieW} j, \tag{1.2}$$

with external real potentials φ, $\mathbf{A}$, W. For either equation we found a conservation law,

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \mathbf{j}, \tag{1.3}$$

where

$$\rho = \psi^* \psi, \quad \mathbf{j} = \left\{ \frac{-i}{2}, \psi^* \mathbf{u}\psi \right\} \tag{1.4}$$

and $\{X, Y\} = XY + YX$.

A central detail of this study is the recognition that we must allow numbers (quaternions) to multiply either on the left or on the right, so we use the notation $(a \| b)\psi = a\psi b$.

Here we report some further results from this continuing study, including two versions of a Lagrangian, further exploration of many-particle states, and clarification of the principles of linearity and superposition.

II. A LAGRANGIAN

Formerly, we stated that we were unable to find a suitable Lagrangian for the quaternionic wave equation. Here are some new results.

For the free wave equation, we go to momentum space,

---

[a]Electronic mail: schwartz@physics.berkeley.edu

48, 052303-1

$$\psi(\mathbf{x},t) = \sum_{\mathbf{p},\eta} \frac{1}{(2\pi)^{3/2}} \exp(\eta \mathbf{u} \cdot \hat{p}\mathbf{p} \cdot \mathbf{x}) \phi_{\mathbf{p},\eta}(t), \tag{2.1}$$

where $\hat{p} = \mathbf{p}/p$, $p = |\mathbf{p}|$, and $\eta = \pm 1$. Each component $\phi(t)$ (dropping the labels $\mathbf{p}$, η) satisfies the equation

$$\frac{d\phi}{dt} = \phi\Omega, \tag{2.2}$$

where $\Omega = i\eta p + km = -\Omega^*$, $|\Omega| = \sqrt{p^2 + m^2}$, and we find that the following Lagrangian:

$$L = (\dot{\phi} - \phi\Omega)\phi^*, \tag{2.3}$$

when integrated over time, does provide a suitable action. That is, variation of each of the four real components of ϕ does lead unambiguously to the specified equation of motion [Eq. (2.2)].

This Lagrangian leads us to identify a new constant of the motion, in addition to $\phi\phi^*$, as follows:

$$\frac{d}{dt}\phi\Omega\phi^* = 0, \tag{2.4}$$

and this looks like the quaternionic version (imaginary) of the energy.

This is unconventional in that we have multiplied the equation (in ϕ) by its conjugate ϕ^* on the *right* rather than on the left. (Alternatively, we can say that we should reidentify which is the original wave function and which is its complex conjugate.) We also note that this action is imaginary, not real. Thus, there are actually 12 real equations resulting from the variational principle, and they are all consistent, implying the four real equations originally given.

In fact, if we write this Lagrangian as $L = iL_1 + jL_2 + kL_3$, any one of the real quantities L_1 or L_2 or L_3 taken by itself is an adequate Lagrangian to recover the complete equation of motion [Eq. (2.2)] for the quaternionic amplitude ϕ.

Trying to find a coordinate space version of this Lagrangian, we are led to introduce the helicity projection operators,

$$\pi_\eta = \frac{1}{2}[1 - \eta\mathbf{u} \cdot \nabla/\sqrt{-\nabla^2}], \quad \eta = \pm 1, \tag{2.5}$$

$$\pi_\eta \pi_{\eta'} = \delta_{\eta,\eta'} \pi_\eta, \ \mathbf{u} \cdot \nabla \pi_\eta = -\eta\sqrt{-\nabla^2}\pi_\eta, \tag{2.6}$$

and the helicity-projected wave functions, $\psi_\eta = \pi_\eta \psi(\mathbf{x},t)$. Then we can write the above Lagrangian as

$$L = \int d^3x \sum_\eta \left[\frac{\partial \psi_\eta}{\partial t} + \mathbf{u} \cdot \nabla \psi_\eta i - m\psi_\eta k \right] \psi_\eta^*. \tag{2.7}$$

This looks nice, but, of course, it is nonlocal in $\mathbf{x}$, and I cannot extend this to include external potentials. See Sec. IV for another attempt.

III. SOME QUATERNION IDENTITIES

Given an arbitrary quaternion ψ, there is the familiar identity $\psi^*\psi = \psi\psi^*$. Here is something else: for any imaginary quaternion numbers α and β, we have the identity

$$\{\alpha, \psi^* \beta \psi\} = \{\beta, \psi \alpha \psi^*\}, \tag{3.1}$$

involving anticommutators $\{,\}$. The easiest way to prove this is to multiply the left hand side by $\psi \cdots \psi^* / |\psi|^2$. The result is the right hand side, but the left hand side is real, so it is unaffected by this operation.

For example, the conserved current $\mathbf{j}$ which was previously written as $\{-i/2, \psi^* \mathbf{u} \psi\}$ can also be written as $\{-\mathbf{u}/2, \psi i \psi^*\}$.

Writing $\psi = \psi_1 + \psi_2$, we have two further identities,

$$\psi_1^* \psi_2 + \psi_2^* \psi_1 = \psi_2 \psi_1^* + \psi_1 \psi_2^*, \tag{3.2}$$

$$\{\alpha, \psi_1^* \beta \psi_2 + \psi_2^* \beta \psi_1\} = \{\beta, \psi_2 \alpha \psi_1^* + \psi_1 \alpha \psi_2^*\}. \tag{3.3}$$

In a similar vein, we can generalize the conservation law [Eq. (1.3)] to include a mixed density $\rho_{1,2} = \psi_1^* \psi_2 + \psi_2^* \psi_1$, with a similar expression for the current $\mathbf{j}_{1,2}$.

A special case of the result [Eq. (3.3)], following the substitution $\psi_1 \to \beta \psi_1$, is

$$\{\alpha, \psi_1^* \psi_2 - \psi_2^* \psi_1\} = 2(\psi_2 \alpha \psi_1^* - \psi_1 \alpha \psi_2^*). \tag{3.4}$$

Using the above formulas, we can rewrite the momentum-space Lagrangian of Sec. II as follows:

$$L_1 = \left\{ \frac{-i}{2}, L \right\} = -\phi^* i \dot\phi + \frac{1}{2} \{\Omega, \phi^* i \phi\}. \tag{3.5}$$

IV. ANOTHER LAGRANGIAN

We have been able to find a Lagrangian for the extended wave equation, but it is nonlocal in time.

Define

$$\psi = \psi(\mathbf{x}, t), \quad \bar\psi = \psi^*(\mathbf{x}, -t), \tag{4.1}$$

$$\epsilon = \epsilon(t) = +1 (t > 0) \text{ or } -1 (t < 0) \tag{4.2}$$

and assume that the potential φ is even under $t \to -t$ while $\mathbf{A}$ and W are odd.

Then we construct the following (real) action:

$$\mathcal{A} = \int d^4 x \left[\bar\psi \frac{\partial \psi}{\partial t} + \left\{ \frac{i}{2}, \epsilon \bar\psi (\mathbf{u} \cdot \nabla + e\varphi) \psi \right\} + e \bar\psi \mathbf{u} \cdot \mathbf{A} \psi - \frac{m}{2} \{e^{i\epsilon e W} k, \epsilon \bar\psi \psi\} \right], \tag{4.3}$$

which, upon variation of the four real components of ψ, gives the extended wave equation as follows:

$$\epsilon \frac{\partial \psi}{\partial t} i = \mathbf{u} \cdot \nabla \psi + e\varphi \psi - \epsilon e \mathbf{u} \cdot \mathbf{A} \psi i + m \psi e^{i\epsilon e W} j. \tag{4.4}$$

For $t > 0$, this is exactly the original equation [Eq. (1.2)], and if we now change $t \to -t$, we find the very same equation for $\psi(\mathbf{x}, -t)$.

One may well wonder what is the correct choice of time $t = 0$ and how to interpret the discontinuity which this Lagrangian posits at that point.

V. MULTIPARTICLE STATES

In Ref. 1, Sec. 12, we saw a particular way to construct two-or-more-particle wave functions for a generalization of our quaternionic wave equation, and that will be explored further below.

In response to some comments received, I did make an attempt to expand the established mathematics of "tensor products," as applied to Hilbert spaces over an Abelian field (like ordinary complex numbers), so that it might accommodate quaternionic quantum theory. This effort is summarized in the Appendix, and it may be called a limited success.

Let me start by repeating the previous approach used, with a slight difference of notation, which I will explain a bit later on. Here, we will make frequent use of the notation for two-sided multiplication by quaternionic numbers and functions: $(a \| b)\psi = a\psi b$.

Write the free wave equation, Eq. (1.1), as

$$\mathcal{D}\psi = \left[\left(\frac{\partial}{\partial t} \| k \right) - (\mathbf{u} \cdot \nabla \| j) + m \right]\psi = 0, \tag{5.1}$$

with our regular plane wave solution written as

$$\psi(\mathbf{x},t) = \psi^{op}_{\mathbf{p},\eta}(\mathbf{x},t)\phi = (\exp(\eta\mathbf{u} \cdot \hat{p}\mathbf{p} \cdot \mathbf{x}) \| \exp((i\eta p + km)t))\phi. \tag{5.2}$$

With x standing for the four coordinates $\mathbf{x}$, t, the propagator is written as

$$G_+(x,x') = - (\theta(t-t') \| k) \sum_{\mathbf{p},\eta} \frac{1}{(2\pi)^3} \psi^{op}_{\mathbf{p},\eta}(\mathbf{x}-\mathbf{x}',t-t'), \tag{5.3}$$

so that we have

$$\mathcal{D}G_+ = \delta^4(x-x'). \tag{5.4}$$

Two of these definitions, Eqs. (5.1) and (5.3), differ from what was given before by the extra quaternion k seen multiplying from the right. The reason for this becomes apparent when we note how the Lorentz transformation is applied to the wave equation,

$$\mathcal{D} \to \mathcal{L}^{-1}\mathcal{D}\mathcal{L}, \quad \mathcal{L} = \exp\left(\left(\frac{1}{2}\mathbf{u} \cdot \mathbf{v} \| i \right) \right). \tag{5.5}$$

This is important for maintaining Lorentz covariance in some other steps, which involved products of these $\mathcal{D}$ operators.

If we can read Eq. (5.4) as saying $\mathcal{D}G_+ = 1$, can we also have $G_+\mathcal{D} = 1$? We investigate

$$G_+\mathcal{D}\psi(\mathbf{x},t) = - \int d^4x' (\theta(t-t') \| k) \sum_{\mathbf{p},\eta} \frac{1}{(2\pi)^3} (\exp(\eta\mathbf{u} \cdot \hat{p}\mathbf{p} \cdot (\mathbf{x}-\mathbf{x}')) \| \exp(\Omega_{p,\eta}(t-t'))) \left[\left(\frac{\partial}{\partial t'} \| k \right) \right.$$
$$\left. - (\mathbf{u} \cdot \nabla' \| j) + m \right] \psi(\mathbf{x}',t'). \tag{5.6}$$

If we execute partial integration on the (primed) space and time derivatives, we do get exactly the answer $\psi(\mathbf{x},t)$, provided that we understand one detail of the mathematical notation here. The symbol $\|$ that stands to the left of the $(t-t')$ argument in the Green function does *not* mean that these coordinates are to be placed to the right of the time derivative operator which follows. That "right-multiplication" instruction applies only to numbers (quaternions) and not to coordinates. Our mathematical notation may need to be improved somewhat to make this rule transparent, but we shall deal with what we now have.

Let us now see what sense we can make of the most general two-particle wave function, for the free particles, written in our "nested" way,

$$\Psi(\mathbf{x}_1,t_1;\mathbf{x}_2,t_2) = \sum_{\mathbf{p},\eta} \exp(\eta\mathbf{u} \cdot \hat{p}\mathbf{p} \cdot \mathbf{x}_1)\psi_{\mathbf{p},\eta}(\mathbf{x}_2,t_2)\exp((i\eta p + km)t_1), \tag{5.7}$$

$$\psi_{\mathbf{p},\eta}(\mathbf{x}_2,t_2) = \sum_{\mathbf{p}',\eta'} \exp(\eta'\mathbf{u}\cdot\hat{p}'\mathbf{p}'\cdot\mathbf{x}_2)\phi_{\mathbf{p},\eta,\mathbf{p}',\eta'} \exp((i\eta'p'+km)t_2). \tag{5.8}$$

Let us now study $\rho = \Psi^*\Psi$, which we understand to be a function of two sets of space-time coordinates. A detailed calculation, which I will not write out here, leads to the nice result

$$\frac{\partial\rho}{\partial t_1} = \nabla_{x_1}\cdot\mathbf{j}, \quad \mathbf{j} = \left\{\frac{-i}{2},\Psi^*\mathbf{u}\Psi\right\}. \tag{5.9}$$

This looks exactly like the one-particle conservation law [Eq. (1.3)], but it has a broader interpretation now. If we identify ρ as a probability density, then this is a joint probability density that is conserved—looking at coordinates of particle number 1—for any given wave function of particle number 2. Such a result in ordinary (complex) quantum theory would be rather obvious and would be interpreted in terms of the factorizeability of the theory for two noninteracting particles.

For the quaternionic theory, this result is not trivial, especially in view of the previous difficulties encountered by other authors: the so-called failure of clustering (see Ref. 2).

This does not solve all such problems, however, since our starting wave function [Eq. (5.7)] has coordinates 1 and 2 ordered in a particular way. The result [Eq. (5.9)] does not hold true for coordinate 2 in the derivative operations. However, if we first integrate over coordinate $\mathbf{x}_1$, then we do recover a one-particle density which is conserved.

VI. MORE ON QUATERNIONIC AMPLITUDE AND LINEARITY

In Ref. 1, Sec. 7, we noted that the amplitude constant ϕ of a plane wave solution of the free equation stood in a particular place in the middle of the complete wave function,

$$\psi = \exp(\eta\mathbf{u}\cdot\hat{p}\mathbf{p}\cdot\mathbf{x})\phi\exp((i\eta p+km)t), \tag{6.1}$$

and not casually at the right or the left of the wave function as one might place it for convenience in ordinary (complex) quantum theory. One may ask whether this feature is generally true, even for the full interacting wave equation.

Let us write the most general quaternionic wave equation as $\partial\psi/\partial t = H\psi$. Then the general initial value problem can be resolved, formally at least, as

$$\psi(t) = \exp(Ht)\psi(0), \tag{6.2}$$

where $\psi(0)$ may be specified arbitrarily and we still have a proper solution of the wave equation. Note that we have suppressed the space coordinates, so these initial data $\psi(0)$ can really be any function of space. Also, the operator H is likely to involve right-multiplying quaternions, so this $\psi(0)$ really stands in the midst of a complicated set of other things, just like the constant ϕ in Eq. (6.1). If the operator H should contain time dependent terms, we also know how to rewrite the propagator as a time-ordered exponential of the integral $\int_0^t H(t')dt'$.

These initial quaternionic data $\psi(0)$ generalizes the simple constant ϕ which we called the amplitude of the plane wave, and from this identification we see the generalized meaning of linearity and superposition for quaternionic quantum theory.

Let us further scrutinize the fundamental meaning of the principle of linearity (or superposition), which is so central to quantum theory. In the usual (complex) theory, we say that if ψ_1 and ψ_2 are solutions of the wave equation (or state vectors in the Hilbert space), then

$$\psi = c_1\psi_1 + c_2\psi_2, \tag{6.3}$$

for arbitrary complex numbers c_1, c_2, shall also be a solution of the wave equation (or a state vector). When this definition of linearity is carried over into quaternionic theory, it causes much trouble (see the Appendix, for example), even aside from the question of whether to write those constants on the left or on the right.

Let us ask what is really required in physics by examining the familiar example of the two-slit interference experiment, which is discussed in any textbook. Let $\psi_1(x_1)$ be the amplitude in the open slit number 1, and let $\psi_2(x_2)$ be the amplitude in the open slit number 2. Then we want to see how each of those wave functions propagates to some point x on the observation screen,

$$\psi_1(x) = G(x;x_1)\psi_1(x_1), \quad \psi_2(x) = G(x;x_2)\psi_2(x_2), \tag{6.4}$$

where the propagators $G(x,x')$ are derived from the pertinent wave equation. Finally, the principle of superposition tells us merely that we should add these two amplitudes at the observation screen,

$$\psi(x) = \psi_1(x) + \psi_2(x). \tag{6.5}$$

Whatever phase difference arises between these two waves (thus producing the interference pattern) comes from the propagators and it is thus already built into the correct wave functions at point x.

The mathematical statement of superposition (6.5) is much simpler than statement (6.3), and, I claim, this is all we need for an acceptable quantum theory. This makes a big difference in any investigation of the mathematics of quaternionic quantum theory.

ACKNOWLEDGMENTS

APPENDIX: TENSOR PRODUCTS

A long-standing problem in earlier studies of quaternionic quantum theory has been the description of multiparticle states. First I will review how we do this in the usual (complex) quantum theory and then explore how the usual mathematics of tensor products may be extended to the realm of quaternions. We start with two particles, given coordinates x_1 and x_2.

The usual tensor product formalism has us write a composite vector as

$$\psi(x_1,x_2) = f(x_1)g(x_2) \rightarrow |\psi\rangle = f \otimes g. \tag{A1}$$

This composite is supposed to reside in a new Hilbert space with the inner product rule

$$\langle\psi'|\psi\rangle = \langle f'|f\rangle\langle g'|g\rangle, \tag{A2}$$

which we say is *factorizeable*. We also note the complete *linearity* of this inner product under $f \rightarrow c_1 f_1 + c_2 f_2$ for arbitrary complex numbers c_1, c_2, and similarly for g, f', g'. Furthermore, we have composite operators that act on such states as

$$(A \otimes B)(f \otimes g) = (Af) \otimes (Bg), \tag{A3}$$

which property we may call *separability* of operators.

In the complex case, any numbers that occur as multipliers in the operators or in the state vectors can be factored out and written anyplace we wish, since they commute with all other operators and with one another. In the quaternionic case that simplicity no longer holds. In earlier study of this tensor product mathematics, Horwitz and Biedenharn[3] concluded that it was impossible to extend this to quaternionic quantum mechanics. Their primary criterion was the linearity condition, which we see obviously fails when we have quaternionic functions f and g. This same conclusion is given in Adler's book.[2]

However, as discussed above, we now have a more enlightened view of linearity and superposition in dealing with quaternionic wave functions. So we shall require only the simple sort of linearlty, $f \rightarrow f_1 + f_2$, which also allows for real multipliers.

This still leaves us with plenty of other criteria to be checked out, and this is what we shall do here.

The first place we see trouble in trying to extend the above formalism to quaternions is Eq. (A2), since the two single-particle inner products do not commute with one another and that violates the general rule in any Hilbert space

$$\langle \psi' | \psi \rangle^* = \langle \psi | \psi' \rangle. \tag{A4}$$

1. First attempt

Let us now try an alternative rule for inner products,

$$\langle \psi' | \psi \rangle = \frac{1}{2}[\langle f' | f \rangle \langle g' | g \rangle + \langle g' | g \rangle \langle f' | f \rangle], \tag{A5}$$

which at least satisfies Eq. (A4). But there is more to check. Consider the particular composite vector

$$|\Psi_o\rangle = f \otimes g + (fi) \otimes (gi), \tag{A6}$$

which, using the rule [Eq. (A5)], turns out to have norm $\langle \Psi_o | \Psi_o \rangle = 0$. Yet, when we take the inner product $\langle f' \otimes g' | \Psi_o \rangle$, we find that it is nonzero for general quaternionic functions f and g. This is an intolerable situation; and so we reject this rule [Eq. (A5)].

2. Second attempt

Alternatively, we consider the rule

$$\langle \psi' | \psi \rangle = \langle g' | \langle f' | f \rangle | g \rangle, \tag{A7}$$

which involves one number (the inner product $\langle f' | f \rangle$) sitting inside an inner product of two other vectors, $\langle g' | \cdots | g \rangle$. For $f' = f$, $g' = g$ this is simply $\langle \psi | \psi \rangle = \langle f | f \rangle \langle g | g \rangle$.

Searching for a zero norm composite vector we now come up with

$$|\Psi_o\rangle = f \otimes g + (fi) \otimes (ig). \tag{A8}$$

Of course, for simply complex functions it is perfectly obvious that this is just $[1-1]f \otimes g$. Is this a problem, however, for our quaternionic situation? We do not see the same problem as found above, in the first attempt, since, according to the rule [Eq. (A7)], the general inner product $\langle f' \otimes g' | \Psi_o \rangle = 0$.

This suggests a mathematical scheme for our tensor products where it is stated that $fa \otimes g = f \otimes ag$ for any scalar a. This looks like the ordinary rule for associativity in multiplication; however, this does cause us a problem with the idea of *separability*.

We have said that we want to have $(A \otimes B)(f \otimes g) = (Af) \otimes (Bg)$, which is to say that we can assign a quaternionic operator to a specific subspace. But we have now acknowledged that $(fa) \otimes g = f \otimes (ag)$. These two properties conflict, as may be seen in

$$(I \otimes B)((fa) \otimes g) = (fa) \otimes (Bg) = f \otimes (aBg), \tag{A9}$$

$$(I \otimes B)((fa) \otimes g) = (I \otimes B)(f \otimes (ag)) = f \otimes (Bag). \tag{A10}$$

This is consistent only if a always commutes with B, which our quaternions do not admit. So this second attempt is also a failure.

3. Third attempt

As a variant of the second attempt, we consider the inner product rule

$$\langle\psi'|\psi\rangle = \frac{1}{2}[\langle g'|\langle f'|f\rangle|g\rangle + \langle f'|\langle g'|g\rangle|f\rangle], \tag{A11}$$

which asserts that the order of the components f, g should not matter. Again, take a general superposition $|\Psi\rangle = f\otimes g + f'\otimes g'$ and calculate the norm, which can be rearranged as follows:

$$N^2 \equiv \langle f|f\rangle\langle g|g\rangle, \quad N'^2 \equiv \langle f'|f'\rangle\langle g'|g'\rangle, \quad M \equiv NN',$$

$$X \equiv \frac{1}{M}\langle g'|\langle f'|f\rangle|g\rangle, \quad Y \equiv \frac{1}{M}\langle f'|\langle g'|g\rangle|f\rangle,$$

$$\langle\Psi|\Psi\rangle = (N-N')^2 + \frac{M}{2}[|1+X|^2 + |1+Y|^2 + (1-|X|^2) + (1-|Y|^2)]. \tag{A12}$$

Each of the five terms in Eq. (A12) is non-negative, so for this norm to vanish, each of those five terms must vanish. The X terms in Eq. (A12) will vanish if we choose $f' = f\mu$ and $g' = \nu g$ with $\mu\nu = -1$. This leads to the troublesome situation we found above in the second attempt. Now, however, we also need the Y terms in Eq. (A12) to vanish, and these two requirements can be met, for general quaternionic functions f, g, only with the choices $f' = f$, $g' = -g$ or $f' = -f$, $g' = g$, which gives us trivially, $|\Psi\rangle = 0$.

This inner product rule [Eq. (A11)] is unfamiliar; it says, for example, that the vectors $1\otimes 1$ and $i\otimes j$ are orthogonal to each other. Nevertheless, the mathematics appears to be consistent.

So, it appears that we have now overcome some of the previous difficulties, and with this last choice for an inner product rule [Eq. (A11)], we can indeed achieve both the enlightened version of linearity as well as the attractive idea of separability given by Eq. (A3), both within the confines of a proper Hilbert space. I do not yet have a complete proof that this is correct, but I am encouraged by the idea that what we required is merely a decent method of bookkeeping. We can keep the individual Hilbert spaces effectively separated but still have a proper composite Hilbert space with this tensor product.

Thus, if we have two operators, A acting on the vector f and B acting on the vector g, then we can write them, in the tensor product notation, as $A\otimes I$ and $I\otimes B$, respectively; these two operators now commute with each other regardless of any quaternions embedded in them.

How to extend this rule from 2 to n components in the tensor product is an open question. It might stay at two terms in the inner product: a given order and its reverse. Or it might become 2^{n-1} terms by a process of induction, or it might become $n!$, with all permutations of ordering.

What about the criterion of factorizeability? We have clearly had to give up on that. I think this is more a matter of physics than of mathematics. If this quaternionic theory has any relation to physical reality, then it appears that we shall have to acknowledge some sort of persistent entanglement of many-particle states. That is not nice, according to our conventional ideas and experience, but it is not unimaginable. The example of two-particle density studied in Sec. V, above, gives us a mixed picture of such entanglement.

[1] C. Schwartz, J. Math. Phys. **47**, 122301 (2006), e-print hep-th/0607121.
[2] S. L. Adler, *Quaternionic Quantum Mechanics and Quantum Fields* (Oxford University Press, New York, 1995), Chap. 9.
[3] L. P. Horwitz and L. C. Biedenharn, Ann. Phys. (N.Y.) **157**, 432 (1984).