

## Computations in large $N$ matrix mechanics

Charles Schwartz\*

*Department of Physics, University of California, Berkeley, California 94720*

(Received 21 July 1999; published 18 October 1999)

The algebraic formulation of large  $N$  matrix mechanics recently developed by Halpern and Schwartz leads to a practical method of numerical computation for both action and Hamiltonian problems. The new technique posits a boundary condition on the planar connected parts  $X_w$ , namely, that they should decrease rapidly with increasing order. This leads to algebraic and/or variational schemes of computation which show remarkably rapid convergence in numerical tests on some many-matrix models. The method allows the calculation of all moments of the ground state, in a sequence of approximations, and excited states can be determined as well. There are two unexpected findings: a large  $d$  expansion and a new selection rule for certain types of interactions. [S0556-2821(99)02422-4]

PACS number(s): 11.15.Pg, 11.15.Tk

### I. INTRODUCTION

Large  $N$  matrix mechanics [1] differs from ordinary quantum mechanics (QM) in that the canonical commutator

$$i[p, q] = I, \quad (1.1)$$

in the one-matrix case, is replaced by the relation

$$i[\pi, \phi] = |0\rangle\langle 0| \quad (1.2)$$

where  $|0\rangle$  is the ground state in the reduced Hilbert space. The original matrix-valued coordinates  $\phi_{rs}$ ,  $r, s = 1, \dots, N$ , are represented by the single operator  $\phi$  in this reduced Hilbert space [2].

The solution of the one-matrix large  $N$  Hamiltonian problem with an arbitrary potential  $V(\phi)$  was given some years ago [3], and only a couple of two-matrix problems in the action formalism have previously been solved [4,5].

The many-matrix problem involves several *noncommuting* operators  $\phi_m$  and their conjugate momenta. Following Halpern and Schwartz [6], this system is described at equal times by a symmetric free algebra which involves a pair (tilde and no tilde) for each Hermitian operator

$$[\tilde{\phi}_m, \phi_n] = [\tilde{\pi}_m, \pi_n] = 0, \quad m, n = 1, \dots, d \quad (1.3a)$$

$$i[\tilde{\pi}_m, \phi_n] = i[\pi_m, \tilde{\phi}_n] = \delta_{mn}|0\rangle\langle 0| \quad (1.3b)$$

$$\tilde{\phi}_m|0\rangle = \phi_m|0\rangle, \quad \tilde{\pi}_m|0\rangle = \pi_m|0\rangle \quad (1.3c)$$

and the ground state energy is given by

$$E_0 = N^2 \langle 0 | \frac{1}{2} \sum_{m=1}^d \pi_m \pi_m + V(\phi) | 0 \rangle \quad (1.4)$$

where  $(\phi)$  refers to the set of operators  $\{\phi_m\}$ . We shall use the summation convention in what follows.

In ordinary quantum mechanics systems of several interacting bodies are most commonly attacked from the Schrödinger equation in coordinate space, using the direct product basis  $|q_1, q_2, \dots, q_d\rangle$ . That approach is not available in the large  $N$  reduced Hilbert space because of the noncommutativity of the operators  $\phi_m$ . A basis of states in this reduced space may be written as

$$|w\rangle \equiv \phi^w |0\rangle \quad (1.5)$$

where we use the “word” notation for ordered products of operators

$$\phi^w = \phi_{m_1} \phi_{m_2} \cdots \phi_{m_n}, \quad w = m_1 m_2 \cdots m_n, \quad m_i = 1, \dots, d \quad (1.6)$$

and we write  $[w] = n$  for the length of the word  $w$ . See Appendix A for a collection of relevant definitions and formulas.

The new approximation technique presented in this paper lies close to the Heisenberg (matrix) formulation rather than the Schrödinger (wave function) formulation and makes use of the set of polynomials  $T_w(\phi)$  introduced in Ref. [7]:

$$[1 - \beta_m \phi_m + X(\beta)]^{-1} = \sum_w \beta^w T_w(\phi) \quad (1.7a)$$

$$X(\beta) = \sum_w \beta^w X_w, \quad X_0 = 0, \quad \langle 0 | T_w(\phi) | 0 \rangle = \delta_{w,0} \quad (1.7b)$$

where the  $\beta_m$  are a dummy set of (noncommuting) parameters and the numbers  $X_w$  were identified as the planar connected parts defined in earlier diagrammatic studies [8]. Various properties of these  $X_w$  are given in Appendix A, including their relation to the ordinary moments  $Z_w \equiv \langle 0 | \phi^w | 0 \rangle$  of the ground state.

The core idea of the present work is to truncate the set of these  $X$ 's,

$$\text{set } X_w = 0 \text{ for all } [w] > n, \quad (1.8)$$

\*Email address: schwartz@physics.berkeley.edu

TABLE I.  $X_n$  and  $Z_n$  for the  $F = \phi^3$  action problem.

$n$	2	4	6	10	20
$X_n$	.544331	-.0925926	.0403208	.0143736	-.00311591
$Z_n$	.544331	.500000	.544331	.816497	3.95996
$X_n/Z_n$	1.00000	-.185185	.074074	.017604	.000787

and solve the (now finite) set of algebraic equations, calling this the “ $n$ th order approximation.” Then increase  $n$ , step by step, and see whether the numerical results appear to converge. This is an intuitive-experimental approach for now, since we have no mathematical proof that this method should work.

With even a small number of the  $X$ ’s determined, one can approximate all the moments of the ground state and the accuracy of these results increases systematically as one proceeds to higher orders of approximation. The excited states of a Hamiltonian system are also amenable to this method.

The recent algebraic developments by Halpern and Schwartz [6,7] provide a wealth of formal definitions and relations for many-matrix problems, unifying the study of both action and Hamiltonian systems. These start with the definitions of generalized creation and annihilation operators in the reduced Hilbert space,

$$\pi_m|0\rangle = iF_m(\phi)|0\rangle, \quad \langle 0|\pi_m = -i\langle 0|F_m(\phi) \quad (1.9a)$$

$$B_m = F_m(\phi) + i\pi_m, \quad B_m|0\rangle = \langle 0|B_m^\dagger = 0 \quad (1.9b)$$

$$B_m B_n^\dagger = E_{mn}(\phi) \quad (1.9c)$$

$$E_{mn}(\phi)|0\rangle = 2i[\tilde{\pi}_n, F_m(\phi)]|0\rangle \quad (1.9d)$$

which is the interacting Cuntz algebra. [In the case of non-interacting harmonic oscillators, we have  $E_{mn} \propto \delta_{mn}$  and Eqs. (1.9b), (1.9c) reduce to the original Cuntz algebra.]

In the practical work of this paper there is a basic distinction between the two types of problems. For action problems we start out knowing the functions  $F_m(\phi)$  explicitly and this lets us work directly with the algebraic equations for the connected parts  $X_w$  derived in Ref. [7] (see Sec. V). For Hamiltonian problems we do not know  $F_m(\phi)$  beforehand and so part of the method presented here involves a constructive representation of these operators, for which task we use the polynomials  $T_w(\phi)$  (see Sec. VI).

In Sec. II we test the idea on a simple example: a one-matrix action problem. In Sec. III we try to give some understanding of why this method apparently works well. Counting of the variables in many-matrix problems and making use of symmetry to keep things manageable is discussed in Sec. IV, followed in Sec. V by some algebraic results for a model action problem with  $d$  interacting matrices. The plan of attack for many-matrix Hamiltonian problems is set out in Sec. VI and numerical results for a set of model potentials are presented in Sec. VII. We note not only the extremely rapid convergence found in these examples but also an unexpected selection rule. Section VIII presents more details of

this computational program, and a related method for calculating excited states is given in Sec. IX. Several appendixes discuss further details and possible extensions of this work.

## II. FIRST TEST: ONE-MATRIX ACTION PROBLEM

We start with a simple problem: a one-matrix action at large  $N$ . As given in Ref. [7] for the quartic action ( $F = \phi^3$ ), we have the following equation for the connected parts  $X_n$ :

$$X(X+1)^2 - \beta^2 X_2 - \frac{1}{2}\beta^4 = 0, \quad X = \sum_{n>0} \beta^n X_n \quad (2.1)$$

which leads to the recursion formula

$$X_n = \frac{1}{2} \delta_{n,4} - \sum_{p=2}^{n-2} X_p \left( \sum_{q=2}^{n-p-2} X_q X_{n-p-q} + 2X_{n-p} \right), \quad n = 4, 6, \dots \quad (2.2)$$

For one-matrix problems we replace the word label  $w$  by  $n = [w]$ . We can compare this with the Schwinger-Dyson equations for the ordinary moments  $Z_n = \langle 0|\phi^n|0\rangle$ , which may be written as

$$2Z_{n+4} = \sum_{m=0}^n Z_m Z_{n-m}, \quad Z_0 = 1 \quad (2.3)$$

and only even  $n$  enter because of the parity symmetry in this problem. If we have the value of  $X_2 = Z_2$  [which we know from other analysis to be  $(2/3)^{3/2}$ ], then we can compute all the higher ones from these recursion formulas. Table I shows some numerical results and we see that the ratio  $X_n/Z_n$  decreases fairly rapidly as  $n$  increases.

Now we want to turn this process around and calculate the value of  $X_2$  from the recursion formula (2.2) using the idea that  $X_n$  should decrease rapidly at large  $n$ —a sort of boundary condition. That is, we consider  $X_2$  as an unknown parameter and then search for that value that will allow us to truncate Eqs. (2.2) with  $X_{n+2} = 0$ , and then we step up the value of  $n$  and repeat the process. Table II contains the results of this computation and we see that the residual error at each level of approximation decreases quite rapidly as we increase  $n$ .

We view this as a sort of eigenvalue problem for the connected parts  $X_n$  and recognize a certain similarity here with the familiar procedure for numerical integration of the one-dimensional Schrödinger equation in some given potential. While that other problem involves a continuous variable

TABLE II. Compute  $X_2$  by truncation:  $X_{n+2}=0$ .

$n+2$	4	6	8	10	20
Approx. $X_2$	.500000	.534522	.541429	.543344	.544321
Error	-.044331	-.009809	-.002902	-.000987	-.000010

$\psi(x)$  obeying a linear (differential) equation our current problem involves a discrete set  $X_n$  obeying a nonlinear (algebraic) equation.

**III. WHY SHOULD THIS METHOD WORK?**

To understand what is going on here it may help to consider the ordinary moments

$$Z_n = \langle 0 | \phi^n | 0 \rangle = \int dq \rho(q) q^n \tag{3.1}$$

for a one-matrix problem. These  $Z_n$ , for a typical ground state density  $\rho(q)$ , are a rather monotonous sequence of numbers. The infinite set of coupled equations for these moments (Schwinger-Dyson equations in one language) contains all the information about the ground state; but one would not try to truncate this infinite system of equations by setting the  $Z_n$  equal to zero after some cutoff  $n = n^*$ .

(In earlier work [9] on moment equations for the one- and two-body Schrödinger equation, the asymptotic behavior of these moments as  $n \rightarrow \infty$  was inferred from the differential equation for the wave function and this allowed a backward iteration procedure.)

Now, by contrast, observe the definition of the planar connected parts, again for the one-matrix problem:

$$X_{n+1} = \langle 0 | \phi T_n(\phi) | 0 \rangle = \int dq \rho(q) q T_n(q) \tag{3.2}$$

where the polynomials  $T_n$  have the property

$$\langle 0 | T_n(\phi) | 0 \rangle = 0, \quad n > 0. \tag{3.3}$$

Clearly, the  $X_n$  are just an algebraic combination of the  $Z_n$ . But Eq. (3.3) tells us that the polynomials  $T_n$  are oscillatory within the domain of integration, and this suggests that the  $X_n$ , given by (3.2), can be thought of as something like the Fourier coefficients of the density  $\rho(q)$ . Therefore, if the ground state is reasonably smooth and the polynomials  $T_n$  are reasonably ‘‘appropriate,’’ then we would expect that the higher Fourier coefficients (the  $X_n$ ) could decrease rapidly. This is the motivation to try a truncation scheme on the  $X$ ’s.

A further advantage of the  $X$ ’s is that they are directly sensitive to the interactions in many-matrix problems. In Ref. [7] it was shown that in many-matrix problems without interactions, the  $X_w$  vanish if there is any mixing of letters in the word  $w$ .

Once one has determined, approximately, even a small number of the  $X$ ’s, this allows one to give approximate values for *all* of the  $Z$ ’s in any one- or many-matrix problem by use of the general algebraic relation (A2) between the generating functions for these two sets of parameters.

With these encouraging results, we go on to study the problems of many matrices in large  $N$  action and Hamiltonian systems.

**IV. MANY MATRICES: COUNTING THE VARIABLES**

With  $d$  matrices, the number of words of length  $n$  is  $d^n$  and this number grows very rapidly. If we have some symmetries in the action or the Hamiltonian, then we can reduce the number of independent variables  $X_w$  that we have to handle at each level of approximation. In this paper we consider model problems with the following invariance properties of the ground state  $|0\rangle$ .

*Parity symmetry.* Change the sign of  $\phi_m$  (and  $\pi_m$ ) for any  $m$ .

*Permutation symmetry.* Make any permutation among the  $d$  labels  $m, n, \dots$

In addition, there is the general invariance of the  $X_w$  (as of the trace operation in the unreduced space) under a cyclic permutation of the letters in the word  $w$ .

With these conditions, the number of independent  $X_w$ ’s is greatly reduced—to what we shall call a set of ‘‘basic words’’ at each level  $n$ —as shown in Table III.

At each level of approximation (signified by the maximum word length  $n$ ) we shall deal with a number of basic words (the dimension  $D$  of our parameter space). From Table III we read off these dimensions: for  $d=2$ ,  $D=1,4,8,20,48, \dots$ ; for  $d=3$ ,  $D=1,4,13,54, \dots$ ; for  $d=5$ ,  $D=1,4,13,72, \dots$ ; for  $d=9$ ,  $D=1,4,13, \dots$ . The first task of the computer program is to make a table of all  $d^n$  words at each  $n$ , identify each word with an equivalence class according to the symmetries described above and assign one member of each class as a basic word  $w_i$ ,  $i=1, \dots, D$ .

**V. MANY-MATRIX ACTION PROBLEMS**

**A. General algebraic machinery**

For action problems, we have the dual basis system of equations derived by Halpern and Schwartz [7]:

TABLE III. Count of  $d^n \rightarrow$  basic words.

$n$	$d=2$	$d=3$	$d=5$	$d=9$
2	4 $\rightarrow$ 1	9 $\rightarrow$ 1	25 $\rightarrow$ 1	81 $\rightarrow$ 1
4	16 $\rightarrow$ 3	81 $\rightarrow$ 3	625 $\rightarrow$ 3	6561 $\rightarrow$ 3
6	64 $\rightarrow$ 4	729 $\rightarrow$ 9	15625 $\rightarrow$ 9	531441 $\rightarrow$ 9
8	256 $\rightarrow$ 12	6561 $\rightarrow$ 41	390625 $\rightarrow$ 59	
10	1024 $\rightarrow$ 28	59049 $\rightarrow$ 257		
12	4096 $\rightarrow$ 94			

$$B_m^\dagger = G_m(\phi) - E_{mn}(\phi)\bar{B}_n, \quad \phi_m = \bar{B}_m[1 + \bar{X}(B^\dagger)] \quad (5.1a)$$

$$\bar{X}(B^\dagger) = \sum_w X_w B^{\dagger w} = \sum_w X_w B^{\dagger \bar{w}}. \quad (5.1b)$$

Here, the operators  $\bar{B}_m, B_m^\dagger$  obey the simple Cuntz algebra

$$\bar{B}_m B_n^\dagger = \delta_{mn} \quad (5.2)$$

and the role of these operators is to generate an infinite set of coupled algebraic equations for the connected parts  $X_w$ , as will be shown by example below. The functions  $G_m = 2F_m$  and  $E_{mn}$ , defined earlier in Eqs. (1.9a), (1.9d), are immediately known once we specify the action  $S$ . Then we shall proceed with the sequence of truncation approximations, generalizing the one-matrix example of Sec. II.

### B. Model problem

We take for our model problem here the  $d$ -matrix action

$$S = -\frac{1}{4N} \sum_{m,n=1}^d \text{Tr}([\phi_m, \phi_n])^2 \quad (5.3)$$

in the unreduced Hilbert space. This gives us the reduced operators

$$G_m(\phi) = \sum_{n \neq m} (\phi_m \phi_n \phi_n + \phi_n \phi_n \phi_m - 2\phi_n \phi_m \phi_n) \quad (5.4a)$$

$$E_{mm}(\phi) = \sum_{n \neq m} (\phi_n \phi_n + X_{nn}) \quad (5.4b)$$

$$E_{m \neq n}(\phi) = \phi_m \phi_n - 2\phi_n \phi_m \quad (5.4c)$$

where we note that this  $S$  has the symmetries mentioned in the previous section and this leads to the simplifications  $X_m = 0$ ,  $X_{mn} = \delta_{mn} X_{11}$ .

Equations (5.1a) now look like

$$\begin{aligned} B_m^\dagger = & \sum_{n \neq m=1}^d \{(\bar{B}_n \bar{B}_n \bar{B}_m \bar{X} + \bar{B}_m (\bar{B}_n \bar{B}_n \bar{X} - X_{11}) - 2\bar{B}_n \bar{B}_m \bar{B}_n \bar{X}) \\ & + (\bar{B}_n \bar{X} \bar{B}_n \bar{B}_m \bar{X} + \bar{B}_m \bar{X} \bar{B}_n \bar{B}_n \bar{X} - 2\bar{B}_n \bar{X} \bar{B}_m \bar{B}_n \bar{X}) \\ & + (\bar{B}_n \bar{B}_n \bar{X} \bar{B}_m \bar{X} + \bar{B}_m \bar{B}_n \bar{X} \bar{B}_n \bar{X} - 2\bar{B}_n \bar{B}_m \bar{X} \bar{B}_n \bar{X}) \\ & + (\bar{B}_n \bar{X} \bar{B}_n \bar{X} \bar{B}_m \bar{X} + \bar{B}_m \bar{X} \bar{B}_n \bar{X} \bar{B}_n \bar{X} - 2\bar{B}_n \bar{X} \bar{B}_m \bar{X} \bar{B}_n \bar{X})\}. \end{aligned} \quad (5.5)$$

This system of equations is equivalent to the Schwinger-Dyson set of equations but it is packaged to emphasize the role of the  $X$ 's and it leads directly to our sequence of approximations. The first line of terms in Eq. (5.5) has only one  $\bar{X}$  and its first few terms are

$$\begin{aligned} & (X_{mnnp} + X_{nnmp} - 2X_{nmnp})B_p^\dagger \\ & + (X_{mnnpqr} + X_{nnmpqr} - 2X_{nmnpqr})B_r^\dagger B_q^\dagger B_p^\dagger \end{aligned} \quad (5.6)$$

where the usual constraint on the sum ( $n \neq m$ ) is understood. The second and third lines have two  $\bar{X}$ 's and their first few terms are

$$\begin{aligned} & 2X_{11}X_{mp}B_p^\dagger + (2X_{11}X_{mpqr} + X_{np}X_{mnqr} + X_{mp}X_{nnqr} \\ & - 2X_{np}X_{nmqr} + X_{nnpq}X_{mr} + X_{nmpq}X_{nr} \\ & - 2X_{mnpq}X_{nr})B_r^\dagger B_q^\dagger B_p^\dagger \end{aligned} \quad (5.7)$$

and the fourth line, with three  $\bar{X}$ 's, starts off as

$$(X_{np}X_{nq}X_{mr} + X_{mp}X_{nq}X_{nr} - 2X_{np}X_{mq}X_{nr})B_r^\dagger B_q^\dagger B_p^\dagger. \quad (5.8)$$

Collecting the linear terms in  $B^\dagger$  gives us the equation

$$1 = 2(d-1)(X_{1122} - X_{1212} + X_{11}^2) \quad (5.9)$$

where we have used the symmetry properties to list the basic words: (11) at  $n=2$  and (1111), (1122), (1212) at  $n=4$ . This equation is exact and leads to our lowest (second) order approximation: we set all  $X$ 's with word length greater than 2 equal to zero and we get

$$X_{11} \approx 1/\sqrt{2(d-1)}. \quad (5.10)$$

Next, we collect the cubic terms in  $B^\dagger$ . For our fourth order approximation we drop all  $X_w$ 's with  $[w] > 4$ :

$$\begin{aligned} 0 = & X_{11}\{(2d-2 + \epsilon_{mp} + \epsilon_{mr})X_{mpqr} - 2\epsilon_{mp}X_{mqrp} \\ & - 2\epsilon_{mr}X_{mrpq} + (\delta_{mp}\delta_{qr} + \delta_{mr}\delta_{pq})[(d-1)X_{1122} + X_{1111} \\ & - X_{mmqq}]\} + X_{11}^3[\delta_{pq}\delta_{mr}\epsilon_{mp} + \delta_{qr}\delta_{mp}\epsilon_{mq} - 2\delta_{pr}\delta_{mq}\epsilon_{mp}] \end{aligned} \quad (5.11)$$

where  $\epsilon_{pq} = 1 - \delta_{pq}$ . These equations are now evaluated for varying choices of the labels  $m, p, q, r$ , which must be paired. We find

$$X_{1111} + X_{1122} = 0 \quad \text{for } m=p=q=r \quad (5.12a)$$

$$X_{1111} + 3(d-1)X_{1122} - 2X_{1212} + X_{11}^2 = 0 \quad \text{for } m=p \neq q=r \quad (5.12b)$$

$$2X_{1122} - dX_{1212} + X_{11}^2 = 0 \quad \text{for } m=q \neq p=r. \quad (5.12c)$$

The solution of this set of equations (for  $d \neq 2$ ) is

$$X_{1111} = -X_{1122} = \frac{1}{3}X_{1212} = \frac{1}{3d+2}X_{11}^2 \quad (5.13)$$

and, putting these results back into Eq. (5.9), we find the fourth order approximation for  $X_{11}$ :

$$X_{11} \simeq \left[ 2(d-1) \left( 1 - \frac{4}{3d+2} \right) \right]^{-1/2}. \quad (5.14)$$

For  $d=2$ , Eqs. (5.12) are indeterminate, but for this case a scaling argument leads to the conclusion that the system is not bounded.

It was very pleasing to find, in the fourth order calculation above, that the number of independent equations was just equal to the number of unknowns and we found a unique solution. Will this circumstance continue at higher orders of approximation? I have no general answer.

One should program a computer to carry the above sequence of approximations to higher order; only algebraic work is required at each step. I have not done this yet, giving priority to the more difficult Hamiltonian problems, reported in Sec. VI.

### C. Large $d$ expansion

From the result above one is led to speculate that this truncation sequence of approximations may be related to a ‘‘large  $d$ ’’ expansion. The algebraic calculations described above have been carried out to the sixth order, with 9 equations in 9 unknowns, and solved in the approximation that  $d \gg 1$ . This leads to the following result:

$$(X_{11})^{-2} = 2(d-1) \left[ 1 - \frac{4}{3d+2} - \frac{185}{81d^2} + O(d^{-3}) \right]. \quad (5.15)$$

We do not have a systematic theory of such a large  $d$  approximation but the following crude attempt may be instructive. Look back at the formula for  $G_m$ , Eq. (5.4a), and replace the operator pair  $\phi_n \phi_n$  by its ground state average, which is  $X_{11}$ . This butchered  $G_m$  is then

$$G_m \sim 2\omega \phi_m, \quad \omega = (d-1)X_{11} \quad (5.16)$$

which is the formula for a system of noninteracting harmonic oscillators. The oscillator result  $X_{mn} = \delta_{mn}/(2\omega)$  then gives immediately the leading term in Eq. (5.15). The higher order terms in  $1/d$  are then expected to come from a perturbation theory expansion about this oscillator approximation. Also, if one looks at the computer results for the Hamiltonian problems (Sec. VII), one may discern a suggestion of more rapid convergence for larger values of  $d$ .

## VI. MANY-MATRIX HAMILTONIAN PROBLEMS

### A. Choosing the model problems

We shall study the Hamiltonians for  $d$  bosonic matrices, given in the unreduced Hilbert space as

$$H = \frac{1}{2} \sum_{m=1}^d \text{Tr}(\pi_m \pi_m) + N \text{Tr} \left[ V \left( \frac{\phi}{\sqrt{N}} \right) \right] \quad (6.1)$$

with the following choices of the potential:

$$V_1(\phi) = \frac{1}{4} \sum_{m=1}^d \phi_m^4 \quad (6.2a)$$

$$V_2(\phi) = \frac{1}{4} \left( \sum_{m=1}^d \phi_m^2 \right)^2 \quad (6.2b)$$

$$V_3(\phi) = \frac{1}{4} \sum_{m < n=1}^d \phi_m^2 \phi_n^2 \quad (6.2c)$$

$$V_4(\phi) = -\frac{1}{8} \sum_{m < n=1}^d [\phi_m, \phi_n]^2 \quad (6.2d)$$

or, if desired, any linear combination of them. The first potential, which is just the non-interacting case, is used for verification of the computational procedure. The third and fourth potentials have ‘‘flat directions,’’ which make them particularly interesting. (Will the calculations converge nicely, indicating a bound state, or will they not?) All four potentials have the symmetries (parity and permutation) described in Sec. IV. The additional  $\text{SO}(d)$  symmetry of  $V_2$  and  $V_4$  is not used at the outset but will be noted in the results.

The following subsections outline the method and further details are given in Sec. VIII and in Appendixes A and B.

### B. Construction of $F_m(\phi)$

A central construct of our previous work [6,7] is the reduced operator  $F_m(\phi)$ , defined in Eq. (1.9a). We will represent this quantity by a finite linear expansion in the polynomials  $T_w(\phi)$ ,

$$F_m(\phi) = \sum_w R_w^{(m)} T_w(\phi), \quad (6.3)$$

at each level of approximation and then see how to determine the coefficients  $R$ . (See Sec. VIII A for more details.)

For any reduced operator  $A$  which depends on the  $\phi$ 's one has the identity

$$2\langle 0|A(\phi)F_m(\phi)|0\rangle = \langle 0|i[\tilde{\pi}_m, A(\phi)]|0\rangle \quad (6.4)$$

which is proved using the definitions (1.9a) and (1.3c). Choosing  $A = T_{w'}$ , and using the formulas (A7) and (1.7b) this gives

$$\langle 0|T_{w'}(\phi)F_m(\phi)|0\rangle = \frac{1}{2} \delta_{w',m} \quad (6.5)$$

for any word  $w'$ . We impose these relations on the approximate expansion (6.3) and obtain

$$\sum_w K_{w',w} R_w^{(m)} = \frac{1}{2} \delta_{w',m} \quad (6.6)$$

where



TABLE IV. Calculated values of  $E/d$  for potential  $V_2$ .

$n$	$D$	$d=2$	$d=3$	$d=5$	$d=9$
2	1	.429	.472	.5408	.6412
4	4	.42672	.47035	.53921	.64007
6	8,13	.426672	.4703152	.539189	.640058
8	20,54	.42667093	.47031461		
10	48	.426670885			

$$K_{w',w} \equiv \langle 0 | T_{w'}(\phi) T_w(\phi) | 0 \rangle. \quad (6.7)$$

This matrix  $K$  is numerically evaluated in terms of the  $X$ 's, as detailed in Appendix B, and then we determine the expansion coefficients  $R$  from a straightforward matrix inversion calculation. Of course, we make this a square (and positive) matrix, as detailed in Eqs. (8.5), (8.6). This completes the first part of the fitting problem, which we would term the kinematic part since it assures that we are doing our best, at any given level of approximation, to represent the basic commutator algebra (1.3b).

Now we turn to the second part, which involves the dynamics of any particular Hamiltonian.

### C. Minimizing the energy

The kinetic energy of the ground state can be expressed as

$$\begin{aligned} K.E./N^2 &= \frac{1}{2} \langle 0 | \pi_m \pi_m | 0 \rangle = \frac{1}{2} \langle 0 | F_m F_m | 0 \rangle \\ &= \frac{1}{4} \langle 0 | i[\tilde{\pi}_m, F_m] | 0 \rangle = \frac{1}{4} R_m^{(m)} = \frac{d}{4} R_1^{(1)} \end{aligned} \quad (6.8)$$

using the methods and results of the previous subsection.

The potential energy of the ground state is expressed directly in terms of the  $X$ 's using Eq. (A3b):

$$\langle 0 | \phi_m^4 | 0 \rangle = X_{1111} + 2X_{11}^2 \quad (6.9a)$$

$$\langle 0 | \phi_m^2 \phi_n^2 | 0 \rangle = X_{1122} + X_{11}^2, \quad m \neq n \quad (6.9b)$$

$$\langle 0 | \phi_m \phi_n \phi_m \phi_n | 0 \rangle = X_{1212}, \quad m \neq n \quad (6.9c)$$

where we have used the specified symmetries to write these formulas in terms of the four basic words at the second and fourth orders.

TABLE V. Calculated values of  $X_{11}$  for potential  $V_2$ .

$n$	$D$	$d=2$	$d=3$	$d=5$	$d=9$
2	1	.437	.397	.347	.292
4	4	.4428	.4010	.34912	.29365
6	8,13	.443007	.401106	.349171	.293667
8	20,54	.4430170	.4011103		
10	48	.44301744			

TABLE VI. Computed results for the one-matrix problem:  $V_1$ .

$n$	$D$	$E$	$X_{11}$
2	1	.375	.50
4	2	.3717	.5100
6	3	.371638	.51057
8	4	.3716339	.510611
10	5	.37163373	.5106136

We program the computer to evaluate the ground state energy  $E = E_0/N^2$  at the  $n$ th order approximation with any assigned numerical values for the quantities  $X_w$  for  $[w] \leq n$ . The final step of this scheme is to vary this set of  $X$ 's so as to minimize  $E$ . This procedure is without mathematical justification; it just seems like the natural thing to do.

What is more, this part of the method is far from straightforward as a computational task because the energy  $E$  is a very nonlinear function of the many variables  $X$ . In Sec. VIII B we describe the techniques used to search for this minimum. The numerical results are presented next.

## VII. NUMERICAL RESULTS

The tables that follow give the outputs of the computations and are designed to show at a glance the convergence of the approximation scheme described above.

Table IV shows the energy ( $E/d$ ) calculated for the potential  $V_2$ , for several values of  $d$  and at several levels of approximation, and Table V gives the corresponding values of  $X_{11} = \langle 0 | \phi_1^2 | 0 \rangle$ .

We note how rapidly these numbers converge as one goes down each column in the tables. For each step increasing the order of approximation, we see a *one or two orders of magnitude* increase in accuracy, somewhat better for  $E$  than for  $X$ . Also, one sees in these tables that the first approximation (a ‘‘back of the envelope’’ computation) is accurate to about 1%. Such is the power of the  $X$ . For comparison, Table VI presents results for the one-matrix problem,  $d=1$  and  $V_1$ , computed by the same program. We see that the results of the many-matrix computations (above) converge about as rapidly as the one-matrix results, although the amount of work required to obtain the former is much greater.

Table VII gives the  $E/d$  results computed for the potential  $V_3$  and one sees rapid convergence here as well.

In Table VIII we see the results for the potential  $V_4$ , which has the greatest amount of ‘‘flat directions’’ among our models. Here the rate of convergence is noticeably

TABLE VII. Calculated values of  $E/d$  for potential  $V_3$ .

$n$	$D$	$d=2$	$d=3$	$d=5$	$d=9$
2	1	.236	.298	.375	.4725
4	4	.2312	.29470	.373207	.471358
6	8,13	.231036	.294625	.3731823	.47134965
8	20,54	.2310258	.29462242		
10	48	.23102504			

TABLE VIII. Calculated values of  $E/d$  for potential  $V_4$ .

$n$	$D$	$d=2$	$d=3$	$d=5$	$d=9$
2	1	.24	.30	.38	.47
4	4	.224	.289	.370	.4690
6	8,13	.2232	.2890	.36944	.468940
8	20,54,72	.22299	.28895	.369431	
10	48	.222964			

slower than in the previous models, but still looks convincingly good.

Also, in the several tables above, one sees a suggestion of more rapid convergence for larger values of  $d$ ; see the discussion of the large  $d$  expansion in Sec. V C.

In another experiment, we studied the one-matrix problem with potential

$$V(\phi) = \frac{1}{2}\phi^2 - \frac{g}{4}\phi^4 \quad (7.1)$$

as the parameter  $g$  approached the value  $\sqrt{8}/3\pi$  where the bound state disappears. The numerical procedure searching to minimize the energy worked well until one approached very close to this critical value; then it failed dramatically.

Other  $X_w$  values are also produced in these computations, albeit with a somewhat lesser accuracy. Table IX has some of these for the potential  $V_2$ .

If there is rotational symmetry in the ground state, one can derive the following relation among the fourth order  $X$ 's,

$$X_{1111} = 2X_{1122} + X_{1212} \quad (7.2)$$

and the data in Table IX satisfy this relation, as does the corresponding data for the potential  $V_4$ , which is also rotationally invariant.

There is another, unexpected, phenomenon seen in the data of Table IX: namely, that  $X_{1212} = 0$ . An increasing number of other  $X_w$ 's also vanish when one looks at higher orders. This result also appears for the potential  $V_3$ , but not for  $V_4$ . When a particular  $X_w$  goes to zero, so does the corresponding coefficient  $R_w$ . The empirical rule is this: Write out the word  $w$  and remove any pair of matching adjacent letters; repeat this process; the  $X_w$  will vanish unless this process can reduce the original word to null. I do not have a full explanation for this newly discovered selection rule but it appears to be related to the fact that these potentials [see Eqs. (6.2b) and (6.2c)] involve only pairs  $(\phi_m \phi_m)$  of each operator. This new symmetry is particular to large  $N$  matrix mechanics with its noncommuting coordinate operators; it would not arise in ordinary quantum mechanics.

 TABLE IX. Computed values of some other  $X_w$  for  $V_2$ .

$X_w$	$d=2$	$d=3$	$d=5$	$d=9$
$X_{1111}$	-.0132659	-.0082358	-.004201	-.001798
$X_{1122}$	-.0066329	-.0041179	-.002101	-.000899
$X_{1212}$	0.0	0.0	0.0	0.0

From an experimental (numerical) perspective, but lacking any formal proof, it appears that these types of large  $N$  problems are now solvable. It will be important for others to repeat this work independently in order to verify these results.

## VIII. DETAILS OF THE COMPUTATIONAL PROGRAM

### A. Full $F_m$

The expression (6.3) for  $F_m(\phi)$  needs to be refined. The motivation for what follows comes from Appendix E in Ref. [6] where the ground state wave function (the action) is modeled and one sees the consequent structure of  $F_m(\phi)$ .

Corresponding to each basic word  $w_i$  we want to have a group of terms [in the  $T_w(\phi)$ ] with a common coefficient  $R_i^{(m)}$ :

$$F_m(\phi) = \sum_{i=1}^D R_i^{(m)} F_{m,i}(\phi). \quad (8.1)$$

For the first stage in this construction we define

$$\partial_m T_w(\phi) \equiv \sum_{w \sim mw'} T_{w'}(\phi) \quad (8.2)$$

which, one can show, will guarantee that the flatness condition [6]

$$[\tilde{\pi}_m, F_n(\phi)] - [\pi_n, \tilde{F}_m(\phi)] = 0 \quad (8.3)$$

is satisfied.

For the second stage we take all permutations among the  $m = 1, \dots, d$  letters that occur in the basic words  $w_i$ .

$$F_{m,i}(\phi) = \frac{1}{c(w_i)(d-1)!} \partial_m \sum_{perm's} \text{permute } T_{w_i}(\phi) \quad (8.4)$$

where the constant  $c(w)$ , the number of subcycles in the word  $w$ , is defined in Appendix A. The normalization constants used above are convenient but not essential.

Now we construct the matrix elements

$$t_{i,j} = \langle 0 | F_{m,i}^\dagger(\phi) F_{m,j}(\phi) | 0 \rangle \quad (\text{no sum}) \quad (8.5)$$

where these are linear combinations of the  $K_{w,w'}$  defined in Eq. (6.7) and Eq. (6.6) is replaced by

$$\sum_{j=1}^D t_{i,j} R_j^{(m)} = \frac{1}{2} \delta_{i,1}, \quad i = 1, \dots, D. \quad (8.6)$$

In order to save computing time in evaluating each  $t_{i,j}$  it is important to find and to count repeated evaluations of the same  $K$  elements. I am not sure that I have done this job completely in my program.

### B. Searching

The hardest part of this program is searching for the minimum energy in the parameter space of the basic word con-

nected parts:  $X_{w_i} \equiv x_i$ ,  $i=1, \dots, D$ . The first method used fits a quadratic function to  $E(x)$  evaluated at  $D(D+1)/2$  nearby points and then finds the extremum:

$$b_i = E(x_i + \delta) - E(x),$$

$$a_{i,j} = E(x_i + \delta, x_j + \delta) - E(x) - b_i - b_j \quad (8.7a)$$

$$\sum_{j=1}^D a_{i,j} v_j = b_i - \frac{1}{2} a_{i,i}, \quad x'_i = x_i - \delta v_i. \quad (8.7b)$$

If one is close enough to the minimum, iterating this procedure should converge rapidly. For most of the data presented in Sec. VII this method worked, although I am sure that more sophisticated techniques could have been more efficient. For the largest size computation carried out ( $d=5$ ,  $n=8$ ,  $D=72$ ) the time for each evaluation of the energy was about 1 min and each iteration of this search procedure took about 44 h on a common desktop microcomputer.

Sometimes, however, this approach failed. For the potential  $V_4$ , beyond the sixth order calculation (for  $d=2$  and  $d=3$ ) this method diverged or led to impossible output (see the next subsection). What succeeded in those cases was a second method: start by solving the numerical problem for some other potential (like  $V_3$  where the first search method worked well) and then gradually change a coupling constant  $g$  inserted into the potential and solve again, repeating in small steps until one arrives at the desired result. At each new step one can start efficiently with a sort of perturbation theory

$$\sum_{j=1}^D a_{i,j} \Delta_j = \delta^2 \frac{\partial^2 E(x)}{\partial g \partial x_i}, \quad x'_i = x_i - (\delta g) \Delta_i \quad (8.8)$$

which involves the matrix  $a_{i,j}$ , Eq. (8.7a), which one has already calculated at the previous step.

Just because the numerical search appears to converge is no proof that we have found the correct solution. In work on the potential  $V_4$  for  $d=2$  we had some results at the sixth order ( $D=8$ ) which first appeared well converged by the first searching method, but a later check on the rotational symmetry (7.2) showed that this was a false solution. Repeating this calculation using the second search method described above led to satisfactory results. The fact that the false energy value was off only in the fifth decimal place stands as a cautionary note on this new numerical technique.

Another numerical searching procedure is suggested by the algebraic work in Sec. V. One could vary only the subset of  $X_w$ 's with  $[w]=n^*$ , keeping all others fixed, then cycle through the choices of  $n^*$ .

It should be repeated that this is all experimental work that is in need of sound mathematical justification and guidance. The multidimensional energy surface  $E(x)$  is a very complicated nonlinear function of the parameters  $x$ . In fact, there are singularities which may lie not far away from the desired minimum. One can see the simplest example of this situation in the  $2 \times 2$  matrix equation (8.6) for the  $d=1$  case.

### C. Constraints

The quantities  $X_w$  cannot be regarded as completely independent variables. For example, in the one-matrix case one has

$$\langle 0 | (\phi^2 - \langle \phi^2 \rangle)^2 | 0 \rangle \geq 0 \quad (8.9)$$

which leads to the inequality  $X_4 \geq -(X_2)^2$ .

Using the general Schwarz inequality, we can write

$$|\langle 0 | T_w T_{w'} | 0 \rangle|^2 \leq \langle 0 | T_w^- T_w | 0 \rangle \langle 0 | T_{w'}^- T_{w'} | 0 \rangle \quad (8.10)$$

for all words  $w$  and  $w'$ . This implies many constraints upon the allowed values of the  $X$  parameters as we search to minimize the energy. It is unclear how best to implement these constraints; in the computations reported here I only checked that the matrix (8.5) satisfied

$$|t_{i,j}|^2 \leq t_{i,i} t_{j,j}, \quad t_{i,i} > 0 \quad \forall i, j \quad (8.11)$$

at each evaluation. A failure of this test signals that the search has strayed into forbidden territory.

An entirely different sort of constraint comes from the use of a purely real (rather than complex) representation for the  $\phi$  operators. This implies that we should have  $X_w = X_w^* = X_w^-$ . With the extensive symmetry of the problems studied here many of these constraints are automatic, but at the 10th order for  $d=2$  and at the 8th order for  $d>2$ , one finds some basic words that do not satisfy  $\bar{w} \approx w$ . Rather than imposing this constraint, we are satisfied to find that this equality comes out in the numerical results.

### IX. EXCITED STATES

After the ground state problem is solved, we consider excited (adjoint) states in the reduced Hilbert space:

$$H|0\rangle = E_0|0\rangle, \quad H|E\rangle = E|E\rangle \quad (9.1)$$

where it should be remembered that we do not know the form of the reduced Hamiltonian  $H$  [6] but only that it generates time translations. With the postulate

$$|E\rangle = U|0\rangle \quad (9.2)$$

for some operator  $U$  we find the identity

$$(E - E_0) \langle 0 | U^\dagger U | 0 \rangle = -i \langle 0 | U^\dagger \dot{U} | 0 \rangle. \quad (9.3)$$

Now we make the construction, as with  $F$  before,

$$U = \sum_w r_w T_w(\phi), \quad U^\dagger = \sum_w r_w^* T_w^-(\phi) \quad (9.4)$$

and we have, using Eq. (A9),

$$\dot{U} = \sum_w r_w \sum_{w=w_1 m w_2} T_{w_1} \pi_m T_{w_2} \quad (9.5)$$

where the  $r_w$  are as yet undetermined constants.

We can now write Eq. (9.3) as



$$E - E_0 = \left( \sum_{w,w'} r_w^* L_{w,w'} r_{w'} \right) / \left( \sum_{w,w'} r_w^* K_{w,w'} r_{w'} \right) \quad (9.6)$$

where the matrix  $K_{w,w'}$  was defined earlier and from Eq. (A10) we have

$$L_{w,w'} \equiv -i \langle 0 | T_w \dot{T}_{w'} | 0 \rangle = \frac{1}{2} \sum_{w=umv} \sum_{w'=u'm'v'} K_{u,v'} K_{u',v} \quad (9.7)$$

Finally, vary the coefficients  $r$  to find stationary values of Eq. (9.6) and we get a traditional linear matrix problem, where  $E - E_0$  is an eigenvalue of the matrix  $L$  with respect to the metric matrix  $K$ .

The evaluation of the matrix  $K$  and thus also of  $L$  is done entirely in terms of the  $X_w$ 's, which were already solved with the ground state problem. Thus (although I have not done any explicit numerical calculations for excited states) the complete spectrum of  $H$  can be calculated. The lowest order approximation,  $U = T_m(\phi)$ , gives  $E_m - E_0 = 1/(2X_{mm})$ .

#### ACKNOWLEDGMENTS

I am grateful to M. B. Halpern for his advice on numerous occasions and I also thank K. Bardakci and M. Rieffel for helpful conversations.

#### APPENDIX A: USEFUL FORMULAS OLD AND NEW

Further conventions on the word notation are the following:

$w = 0$  is the null word.

$w = m$  means that the word  $w$  consists of a single letter  $m$ .

$w \sim w'$  means that the two words differ by at most a cyclic permutation of their letters.

$w \approx w'$  means that the two words are equivalent under some larger symmetry.

$w_1 w_2 = w_3$  means that the second word is appended to the first word and the result is the third word.

$w = umv$  means that the word  $w$  is decomposed as indicated.

$\bar{w}$  is the word formed by reversing the sequence of letters in the word  $w$ .

$c(w)$ , the number of subcycles in the word  $w$ , is defined as the largest integer  $k$  such that  $w = u^k$  for any word  $u$  with  $[u] > 0$ .

Basic relations among  $T(\phi)$  and  $X$  are the following [7]:

$$T_{mw} = \phi_m T_w - \sum_{w=w_1 w_2} X_{mw_1} T_{w_2} \quad (A1a)$$

$$T_{wm} = T_w \phi_m - \sum_{w=w_1 w_2} T_{w_1} X_{w_2 m} \quad (A1b)$$

$$X_{mw} = \langle 0 | \phi_m T_w | 0 \rangle = \langle 0 | T_w \phi_m | 0 \rangle = X_{wm} \quad (A1c)$$

$$T_w^\dagger = T_{\bar{w}}, \quad X_w^* = X_{\bar{w}} \quad (A1d)$$

The relation between  $X$  and  $Z$  is the following:

$$Z(j) = 1 + X(jZ(j)). \quad (A2)$$

Examples (for the case of parity symmetry, which means that each letter must appear an even number of times or else the  $Z$  and  $X$  vanish) are the following:

$$Z_{mm} = \langle 0 | \phi_m \phi_n | 0 \rangle = \delta_{mn} X_{mm} \quad (A3a)$$

$$Z_{mnpq} = Z_{npqm} = \begin{cases} X_{mmmm} + 2X_{mm}^2 & \text{if } m = n = p = q, \\ X_{mmpq} + X_{mm} X_{pp} & \text{if } m = n \neq p = q, \\ X_{mnmn} & \text{if } p = m \neq n = q. \end{cases} \quad (A3b)$$

For one-matrix problems the label  $w$  is replaced by  $n = [w]$ . For systems with the parity selection rule,

$$T_0 = 1, \quad T_1 = \phi, \quad T_2 = \phi^2 - X_2, \quad X_2 = \langle \phi^2 \rangle \quad (A4a)$$

$$T_3 = \phi^3 - 2\phi X_2, \quad T_4 = \phi^4 - 3\phi^2 X_2 - X_4 + X_2^2 \quad (A4b)$$

$$X_4 = \langle \phi^4 \rangle - 2X_2^2, \quad X_6 = \langle \phi^6 \rangle - 6X_4 X_2 - 5X_2^3. \quad (A4c)$$

Below are some new relations involving  $T(\phi)$  that are used in the present work. Start with the generating function

$$Y = 1/(1 - \beta_m \phi_m + X(\beta)) = \sum_w \beta^w T_w(\phi) \quad (A5)$$

and calculate the commutator

$$i[\tilde{\pi}_m, Y] = Y \beta_m |0\rangle \langle 0| Y. \quad (A6)$$

Now expand in powers of  $\beta$  and match terms to find

$$i[\tilde{\pi}_m, T_w(\phi)] = \sum_{w=w_1 m w_2} T_{w_1}(\phi) |0\rangle \langle 0| T_{w_2}(\phi). \quad (A7)$$

The other version of this relation,

$$i[\pi_m, \tilde{T}_w] = \sum_{w=w_1 m w_2} \tilde{T}_{w_2} |0\rangle \langle 0| \tilde{T}_{w_1}, \quad (A8)$$

comes from Eq. (D.11) in Ref. [7]. In a very similar way one gets the time derivative equation

$$\frac{d}{dt} T_w(\phi) = \sum_{w=w_1 m w_2} T_{w_1} \pi_m T_{w_2} \quad (A9)$$

where we have used  $(d/dt)\phi_m = \pi_m$ . Combining the last two equations leads to

$$i \langle 0 | T_{w'} \frac{d}{dt} T_w | 0 \rangle = -\frac{1}{2} \sum_{w=umv} \sum_{w'=u'm'v'} \langle 0 | T_u T_{v'} | 0 \rangle \times \langle 0 | T_{u'} T_v | 0 \rangle \quad (A10)$$

which is surprisingly simple.

**APPENDIX B: EVALUATING  $\langle T_w T_{w'} \rangle$** 

We seek some recursive procedure for evaluation of the matrix elements

$$K_{w,w'} = \langle 0 | T_w(\phi) T_{w'}(\phi) | 0 \rangle = K_{w',w} \quad (\text{B1})$$

in terms of the connected parts  $X_w$ . Using Eqs. (A1a) and (A1b) it is relatively easy to find the following relations:

$$K_{wm,w'} = K_{w,mw'} + \sum_{w''=uv} X_{mu} K_{w,v} - \sum_{w''=uv} X_{vm} K_{u,w'} \quad (\text{B2})$$

with the boundary conditions  $K_{w,0} = K_{0,w} = \delta_{w,0}$ . This looks very nice as a recursive computer program but it turns out to be expensive: the time required grows exponentially as one increases the size of the words. One could save time by building a table of all the  $K$  matrix elements one might need, but that requires enormous amounts of space.

An alternative method is given by the following formula:

$$K_{w,w'} = \sum_{w''=uv} \sum_{w'''=u'v'} K_{u,v'} X_{vu'} \quad (\text{B3})$$

$[v] > 0, \quad [u'] > 0, \quad K_{0,0} = 1$

which may be derived by combining Eq. (A1a) with the expansion

$$\phi_m = \sum_w X_{mw} G_w^-(\phi) \quad (\text{B4})$$

from Ref. [7] and also using the identity

$$\langle 0 | T_w T_{w'} G_{w''}^- | 0 \rangle = \sum_{w'''=uv} \langle 0 | T_{w_1} T_{w_2} | 0 \rangle \delta_{w,uw_1} \delta_{w',w_2v} \quad (\text{B5})$$

which is similar to the Ward identities derived in Appendix E of Ref. [7].

The program uses Eq. (B3) to build a small table of  $K$ 's each time one of them is called for and the time for this grows as  $n^4$  rather than exponentially. Still, this is the main time consuming part of the computations.

**APPENDIX C: SOME ALTERNATIVE COMPUTATIONAL SCHEMES**

One alternative scheme is to start out by fitting the quantity  $E_{mn}(\phi)$  instead of  $F_m(\phi)$ :

$$E_{mn}(\phi) = \sum_w R_w^{(mn)} T_w(\phi). \quad (\text{C1})$$

The definition (1.9d) is

$$E_{mn}(\phi) | 0 \rangle = 2i [\tilde{\pi}_n, F_m(\phi)] | 0 \rangle \quad (\text{C2})$$

and using Eq. (A7), we find

$$R_w^{(mn)} = 2R_{wn}^{(m)} \quad (\text{C3})$$

upon comparison with Eq. (6.3). Next, we use the formal expansion from Ref. [7],

$$(E^{-1})_{mn} = \sum_w X_{wmn} G_w^-(\phi), \quad (\text{C4})$$

to write the system of conditions

$$\langle 0 | T_{w'} (E_{mn} (E^{-1})_{np} - \delta_{m,p}) | 0 \rangle = 0 \quad (\text{C5a})$$

$$\sum_{w,w''} X_{npw''} \langle 0 | T_{w'} T_w G_{w''}^- | 0 \rangle R_{wn}^{(m)} = \frac{1}{2} \delta_{m,p} \delta_{w',0} \quad (\text{C5b})$$

and one can show, using Eq. (B3), that this reduces to equations identical to Eq. (6.6). So this method is not an alternative at all.

A second alternative scheme does away with minimizing the energy and works instead from the equations of motion

$$\dot{\pi}_m | 0 \rangle = i \dot{F}_m(\phi) | 0 \rangle = -V'_m(\phi) | 0 \rangle. \quad (\text{C6})$$

Using the representation (6.3) for  $F_m$ , this leads us to a new system of equations

$$-i \sum_w \langle 0 | T_{w'} \dot{T}_w | 0 \rangle R_w^{(m)} = \langle 0 | T_{w'} V'_m | 0 \rangle \quad (\text{C7})$$

where the matrix elements on the left side are the quantities  $L_{w',w}$  defined in Sec. IX. One now has *two* sets of matrix equations—Eqs. (6.6) and (C7)—determining the same set of expansion coefficients: call the solutions  $R$  and  $R'$ . One would now seek a set of values for the parameters  $X_{w_i}$  that would make these two sets of solutions the same. Computationally, the way to do this would presumably be to minimize the error,

$$\text{error} = \sum_i |R_i - R'_i|^2, \quad (\text{C8})$$

and this defines another nonlinear search procedure. But what weight function ought optimally to be put into this error calculation?

A third alternative is to use the monomials  $\phi^w$  instead of the polynomials  $T_w(\phi)$  as a basis for the fitting of the operators  $F_m$  or  $U$ . This leads to much simpler formulas for the matrix elements of  $K$  and  $L$ , expressed in terms of the moments  $Z_w = \langle 0 | \phi^w | 0 \rangle$ . Then one would use the relation (A2) to evaluate each  $Z_w$  in terms of the chosen set of parameters  $X_w$ . I believe that this approach has drawbacks in both speed and numerical accuracy, but it should be explored.

**APPENDIX D: IS THIS METHOD USEFUL IN ORDINARY QM?**

With the apparent success of this approximation method in large  $N$  matrix mechanics, one goes back to ordinary quantum mechanics to see if we have a new useful calculational technique. The formalism developed in Ref. [7] is eas-

ily modified to fit the standard commutation relation

$$i[p_i, q_j] = \delta_{ij} I \quad (\text{D1})$$

with the following construction:

$$Y = e^{\beta_i q_i - X(\beta)} = \sum_{\mu} T_{\mu}(q) \quad (\text{D2a})$$

$$X(\beta) = \sum_{\mu} C_{\mu} \beta^{\mu} X_{\mu} \quad (\text{D2b})$$

$$Z(\beta) = \langle 0 | e^{\beta_i q_i} | 0 \rangle = \sum_{\mu} C_{\mu} \beta^{\mu} \langle 0 | q^{\mu} | 0 \rangle. \quad (\text{D2c})$$

Here  $\mu$  represents the unordered set of occupation numbers  $\{\mu_i\}$  (remember that the  $q_i$ 's commute with one another now) and

$$C_{\mu} = 1 / \prod_i (\mu_i)!. \quad (\text{D3})$$

In the simple one-matrix case we have

$$Z(\beta) = \sum_{n=0}^{\infty} \beta^n Z_n / n!, \quad Z_n = \langle 0 | q^n | 0 \rangle = \int dq q^n \rho(q) \quad (\text{D4a})$$

$$X(\beta) = \sum_{n=1}^{\infty} \beta^n X_n / n! = \ln[Z(\beta)] \quad (\text{D4b})$$

and we want to test whether the ratio  $X_n/Z_n$  decreases rapidly with  $n$ , as we saw for the large  $N$  situation in Sec. II. For the case of a harmonic oscillator, we have the same result in both theories; namely,  $X_n$  vanishes for  $n > 2$ .

One simple (non-oscillator) model that allows analytic calculations is a constant density  $\rho(q)$  over some finite range

TABLE X. Results for the Schrödinger equation (D5).

$n$	$D$	$E$	$X_{11}$
2	1	.429	.437
4	2	.4217	.4525
6	3	.4210	.45512
8	4	.42086	.45571

of  $q$ . Here we find that the ratio  $X_n/Z_n$  decays rapidly with  $n$  for the large  $N$  situation but this ratio grows very rapidly for the ordinary quantum mechanics situation.

We have also applied the method of this paper to the quantum mechanical nonlinear oscillator,

$$H = \frac{1}{2} p^2 + \frac{1}{4} q^4. \quad (\text{D5})$$

Numerical results for the ground state are shown in Table X. The convergence seen here is fairly good, although not as good as for the similar large  $N$  problem, shown in Table VI. (The accuracy shown here is comparable to that obtained with conventional variational calculations of this Schrödinger equation, at the same levels of approximation.)

It must be reported, however, that the results shown in Table X were not obtained easily. The problem of nearby singularities in the energy surface, mentioned in Sec. VIII B, was more severe in this ordinary quantum mechanics problem than in the large  $N$  problems. For the calculations through  $D=3$ , I used the second searching method, starting from the harmonic oscillator and then moving gradually to the quartic potential in steps of size  $1/8$ . For  $D=4$ , I had to decrease the step size to  $1/16$ , and for  $D=5$ , I gave up after failing in the search procedure with step size  $1/32$ .

In conclusion, I am still in doubt about the answer to the question posed in the heading of this appendix.

- 
- [1] M. B. Halpern, Nucl. Phys. **B188**, 61 (1981).  
[2] K. Bardakci, Nucl. Phys. **B178**, 263 (1981).  
[3] M. B. Halpern and C. Schwartz, Phys. Rev. D **24**, 2146 (1981).  
[4] M. Staudacher, Phys. Lett. B **305**, 332 (1993).  
[5] V. A. Kazakov and P. Zinn-Justin, Nucl. Phys. **B546**, 647 (1999).  
[6] M. B. Halpern and C. Schwartz, Int. J. Mod. Phys. A **14**, 3059 (1999). This paper provides a brief overview of related work in

- the field.  
[7] M. B. Halpern and C. Schwartz, "Infinite Dimensional Free Algebra and the Forms of the Master Field," Int. J. Mod. Phys. A (to be published), hep-th/9903131.  
[8] P. Cvitanovic, P. G. Lauwers, and P. N. Scharbach, Nucl. Phys. **B203**, 385 (1982).  
[9] R. Blankenbecler, T. DeGrand, and R. I. Sugar, Phys. Rev. D **21**, 1055 (1980).