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INFINITE-DIMENSIONAL FREE ALGEBRA AND THE FORMS OF THE MASTER FIELD

M. B. HALPERN*

Department of Physics, University of California, Berkeley, California 94720, USA

and

Theoretical Physics Group, Ernest Orlando Lawrence Berkeley National Laboratory, University of California, Berkeley, California 94720, USA

C. SCHWARTZ

Department of Physics, University of California, Berkeley, California 94720, USA

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We find an infinite-dimensional free algebra which lives at large N in any SU(N)-invariant action or Hamiltonian theory of bosonic matrices. The natural basis of this algebra is a free-algebraic generalization of Chebyshev polynomials and the dual basis is closely related to the planar connected parts. This leads to a number of free-algebraic forms of the master field including an algebraic derivation of the Gopakumar–Gross form. For action theories, these forms of the master field immediately give a number of new free-algebraic packagings of the planar Schwinger–Dyson equations.

1. Introduction

Recently,[1] we have studied the algebras of phase-space master fields in general matrix models, obtaining in particular a number of new free algebras which generalize the Cuntz algebra. Among these generalizations, our starting point in this paper is the set of *interacting Cuntz algebras*[a]

$$\begin{aligned} B_m = \sqrt{2}A_m &= F_m(\phi) + i\pi_m \,, \\ B_m^\dagger &= F_m(\phi) - i\pi_m \,, \\ E_{mn}(\phi) &= 2C_{mn}(\phi) \,, \end{aligned} \tag{1.1a}$$

*E-mail: halpern@physics.berkeley.edu
[a]Certain powers of $\sqrt{2}$ are scaled out here relative to the operators A and C of Ref. 1.

$$B_m B_n^\dagger = E_{mn}\,, \tag{1.1b}$$

$$B_m^\dagger (E^{-1})_{mn} B_n = 1 - |0\rangle\langle 0|\,, \tag{1.1c}$$

$$B_m|0\rangle = \langle 0|B_m^\dagger = 0\,, \quad m,\ n = 1\cdots d \tag{1.1d}$$

which occur at large N in general bosonic matrix models, and may also occur in matrix models with fermions. The fields ϕ_m and π_m are the master field and the reduced momenta respectively, and the operators F_m and E_{mn} are determined by the potential. The Cuntz algebra is the special case of (1.1) obtained in the case of matrix oscillators.

In the present paper, we will generalize these algebras in two directions. First, we recall that the "fifth-time" formulation (see for example Ref. 2) maps any Euclidean action theory into a higher-dimensional theory[b] with a Hamiltonian formulation. This allows us to read Ref. 1 as a unified free-algebraic treatment of action and phase-space master fields (see Sec. 2). The unified formulation includes and extends Haan's[6] early free-algebraic formulation of action master fields, and one sees in particular that the interacting Cuntz algebra (1.1) occurs in the same way for action and phase-space master fields. The operators F_m and E_{mn} of the algebra (1.1) are straightforward to compute explicitly for the action case.

The second direction is the main subject of this paper. For action and/or phase-space master fields, the interacting Cuntz algebra can be extended to an infinite-dimensional free algebra (see Secs. 3–6), whose structure, especially in the action case, controls the large N theory. The annihilation operators of this algebra are defined as composites of the interacting Cuntz operators

$$B_w = B^w = B_{m_1}\cdots B_{m_n}\,, \quad w = m_1\cdots m_n\,, \quad [w] = n\,, \tag{1.2a}$$

$$B_{\bar w} = B^{\bar w} = B_{m_n}\cdots B_{m_1}\,, \quad \bar w = m_n\cdots m_1\,, \quad [\bar w] = n\,, \tag{1.2b}$$

where w is any word, composed of letters m_i, and $[w]$ is the length of w. When a word w is written as a subscript it is a label but when written as a superscript it is to be read as an exponent, producing an ordered product, as in (1.2a). The null word is 0, with $[0] = 0$ and $B^0 = 1$. This word notation, and the rule

$$ww' = m_1\cdots m_n m_1'\cdots m_{n'}'\,, \qquad [ww'] = [w] + [w'] \tag{1.3}$$

will be followed uniformly below.

Surprisingly, the composite annihilation operators (1.2) and the corresponding creation operators turn out to be linear in the reduced momenta π_m,

$$B_{mw} = F_{mw}(\phi) + i\pi_m G_w(\phi)\,, \qquad {B_{mw}}^\dagger = F_{mw}(\phi)^\dagger - iG_{\bar w}(\phi)\pi_m \tag{1.4}$$

and this fact underlies the simple form of the infinite-dimensional free algebra below. The operators G_w and F_w can be obtained in terms of F_m and E_{mn} (see Sec. 3),

[b] In an evident parallel with the AdS/CFT correspondence,[3–5] the fifth-time formulation also gives the large N action theory as a classical solution of the higher-dimensional theory (see Subsec. 2.1 and App. A).

and G_w, F_w turn out to be free-algebraic generalizations of Chebyshev polynomials (see Subsecs. 3.2 and 3.3).

The infinite-dimensional free algebra is then

$$B_w B_{w'} = B_{ww'}\,, \qquad {B_w}^\dagger {B_{w'}}^\dagger = {B_{w'w}}^\dagger\,, \tag{1.5a}$$

$${B_{mw}}^\dagger B_{nw'} = G_{\bar{w}m} B_{nw'} - G_{\bar{w}} B_{mnw'}\,, \tag{1.5b}$$

$${B_{mw}}^\dagger (E^{-1})_{mn} B_{nw'} = {B_w}^\dagger G_{w'} + G_{\bar{w}} B_{w'} - G_{\bar{w}w'} - G_{\bar{w}}|0\rangle\langle 0|G_{w'}\,, \tag{1.5c}$$

$$\begin{aligned} B_{mw}{B_{nw'}}^\dagger &= \sum_{w''} \big(B_{mw''} f_{w'',w,n,w'} + f^*_{w'',w',m,w} {B_{nw''}}^\dagger\big) \\ &\quad + E_{mw;nw'}(\phi)\,, \end{aligned} \tag{1.5d}$$

$$B_w|0\rangle = \delta_{w,0}|0\rangle\,, \qquad \langle 0|{B_w}^\dagger = \langle 0|\delta_{w,0}\,, \tag{1.5e}$$

where $E_{mw,nw'}(\phi)$ and the structure constants f will be given in Sec. 6. The interacting Cuntz algebra (1.1) is a subalgebra of (1.5), and (1.5b) includes a new relation for $B_m^\dagger B_n$. In the case of oscillators and/or free action theories, the Cuntz algebra itself is a subalgebra of the infinite-dimensional algebra (see App. B).

The creation operators of this algebra provide us with a natural basis

$${B_w}^\dagger|0\rangle = G_{\bar{w}}(\phi)|0\rangle \tag{1.6}$$

comprised of the G_w's themselves, and the dual basis, orthonormal to (1.6), turns out to involve the planar connected parts X_w in a very simple way.

This leads us to a number of forms of the master field (see Sec. 7), including the basic form

$$\phi_m = \sum_w X_{mw} G_{\bar{w}}(\phi) \tag{1.7}$$

and the dual basis form

$$\phi_m = \phi_m^\dagger = \bar{B}_m\big(1 + \bar{X}(B^\dagger)\big)\,, \qquad \bar{B}_m = (E^{-1})_{mn} B_n\,, \tag{1.8}$$

where $\bar{B}$ and $B^\dagger$ satisfy a Cuntz algebra and $\bar{X}(B^\dagger)$ is a generating function of planar connected parts. The dual basis form (1.8) is the Hermitian counterpart of the non-Hermitian form obtained diagrammatically by Gopakumar and Gross.[7] We also give the forms of the master field in terms of the planar correlators and the planar 1PI parts.

For action theories, these forms of the master field immediately give a number of new free-algebraic forms (see Sec. 8) of the planar Schwinger–Dyson equations, including, surprisingly, the basic form (1.7) itself and the dual basis system

$$B_m^\dagger + E_{mn}\big(\bar{B}(1+\bar{X})\big)\bar{B}_n = G_m\big(\bar{B}(1+\bar{X})\big) \tag{1.9}$$

both of which can be used for computation of the planar connected parts. Systems similar to (1.9) follow for the planar correlators and the planar effective action, and, although they are packaged differently, these systems (including (1.9)) are closely related to the free-algebraic equations derived diagrammatically in Ref. 8.

We conclude that the interacting Cuntz algebra (1.1) and the infinite-dimensional free algebra (1.5) provide an algebraic framework which underlies and extends much of what is known about large N, and we are optimistic that these algebras will provide a foundation for the future study of the master field.

2. Unification of Action and Phase-Space Master Fields

2.1. *Fifth-time formulation and Euclidean quantum field theory*

We consider a general SU(N)-invariant matrix model with Euclidean action S

$$\langle \mathrm{Tr}\, \phi^w \rangle = \eta^{-1} \int (d\phi) e^{-S}\, \mathrm{Tr}[\phi^w]\,, \quad \eta = \int (d\phi) e^{-S}\,, \tag{2.1a}$$

$$S = N\, \mathrm{Tr}\left[\mathcal{S}\left(\frac{\phi}{\sqrt{N}}\right)\right], \quad \phi^w = \phi^{m_1} \cdots \phi^{m_n}\,, \quad m = 1 \cdots d \tag{2.1b}$$

and follow the fifth-time formulation[2] to interpret the model as a quantum system, with a (fifth time) Hamiltonian formulation, in one higher dimension. The resulting picture is a pedestrian version of operator Euclidean quantum field theory.

In the Hamiltonian formulation, the matrix fields ϕ^m are operators and the action averages are reinterpreted as ground state averages:

$$\langle \mathrm{Tr}\, \phi^w \rangle = \langle .0 |\, \mathrm{Tr}\, \phi^w | 0. \rangle\,, \qquad |0.\rangle = \psi_0(\phi) = \eta^{-\frac{1}{2}} e^{-\frac{S}{2}}\,, \tag{2.2}$$

where the dot in the (unreduced) ground state follows the notation of Ref. 1. We may also introduce momentum operators and equal fifth-time commutators as

$$\pi^m_{rs} = \frac{1}{i} \frac{\partial}{\partial \phi^m_{sr}}\,, \quad \left[\phi^m_{rs}, \pi^n_{tu}\right] = i\delta^{mn}\, \delta_{st}\, \delta_{ru}\,, \tag{2.3a}$$

$$(\phi^m_{rs})^\dagger = \phi^m_{sr}\,, \quad (\pi^m_{rs})^\dagger = \pi^m_{sr}\,, \quad r,\ s = 1 \cdots N \tag{2.3b}$$

and, following Ref. 1, we use the momenta to construct matrix creation and annihilation operators

$$B^m_{rs} = \sqrt{2} A^m_{rs} = F^m_{rs} + i\pi^m_{rs}\,, \qquad B^m_{rs}|0.\rangle = 0\,, \tag{2.4a}$$

$$(B^{\dagger m})_{rs} = \sqrt{2} (A^{\dagger m})_{rs} = F^m_{rs} - i\pi^m_{rs}\,, \quad \langle .0 | (B^{\dagger m})_{rs} = 0\,, \tag{2.4b}$$

$$F^m_{rs} = \frac{1}{2} \frac{\partial S}{\partial \phi^m_{sr}}\,. \tag{2.4c}$$

Reference 1 also tells us that the quantities

$$E^{mn}_{rs} = 2C^{mn}_{rs} = [B^m_{rt}, (B^{\dagger n})_{ts}] = \frac{\partial^2 S}{\partial \phi^m_{tr} \partial \phi^n_{st}} \tag{2.5}$$

will be useful at large N.

As for the fifth-time Hamiltonian itself, we may choose any of a very large number of operators, for example

$$H_5 = \frac{1}{2}\, \mathrm{Tr}(B^{\dagger m} B^m)\,, \qquad H_5 |0.\rangle = 0 \tag{2.6}$$

so long as the choice provides us with a healthy Hilbert space and its ground state is $|0.\rangle$ in (2.2). The equal fifth-time averages of any such higher-dimensional system will be the original Euclidean action averages, and, moreover, the large N action averages are controlled by the phase-space master fields,[1] which are classical solutions of the higher-dimensional theory. The parallel with the AdS/CFT correspondence[3–5] is clear, if only we are clever enough to choose both an interesting action theory and an interesting higher-dimensional extension. Except for a simple example based on (2.6) in App. A, further consideration of this issue is beyond the scope of the present paper, and we will not choose any specific form for H_5 here.

2.2. *Reduced formulation*

We may now go over to reduced states and operators for the large N action theory, drawing heavily on the results of Ref. 1. Important relations given there include

$$\langle .0|\,\mathrm{Tr}\left[\left(\frac{\phi}{\sqrt{N}}\right)^{w}\right]|0.\rangle = N\langle 0|\phi^{w}|0\rangle \equiv N\langle\phi^{w}\rangle\,, \tag{2.7}$$

where ϕ_m is the master field, ϕ^w are products of the master field in the word notation (1.2a), and the undotted vacuum is the reduced ground state. The reduced equal (fifth) time algebra involves the tilde operators introduced in Ref. 1

$$[\phi_m, \tilde{\pi}_n] = [\tilde{\phi}_m, \pi_n] = i\delta_{m,n}|0\rangle\langle 0|\,, \tag{2.8a}$$

$$[\phi_m, \tilde{\phi}_n] = [\pi_m, \tilde{\pi}_n] = 0\,, \tag{2.8b}$$

$$[\phi_m, \pi_m] = i(d-1+|0\rangle\langle 0|)\,, \tag{2.8c}$$

$$\phi_m^{\dagger} = \phi_m\,, \qquad \pi_m^{\dagger} = \pi_m\,, \tag{2.8d}$$

$$\tilde{\phi}_m|0\rangle = \phi_m|0\rangle\,, \quad \tilde{\pi}_m|0\rangle = \pi_m|0\rangle\,, \tag{2.8e}$$

where the operators π_m are the reduced momenta.

The reduced creation and annihilation operators corresponding to (2.4) are

$$B_m = \sqrt{2}A_m = F_m(\phi) + i\pi_m\,, \quad B_m^{\dagger} = \sqrt{2}A_m^{\dagger} = F_m(\phi) - i\pi_m\,. \tag{2.9}$$

These operators satisfy the interacting Cuntz algebra[1]

$$B_m B_n^{\dagger} = E_{mn}(\phi) = 2C_{mn}(\phi)\,, \tag{2.10a}$$

$$B_m^{\dagger}(E^{-1})_{mn}B_n = 1 - |0\rangle\langle 0|\,, \tag{2.10b}$$

$$B_m|0\rangle = \langle 0|B_m^{\dagger} = 0 \tag{2.10c}$$

at equal (fifth) time, as well as the relations

$$[\tilde{B}_m, B_n] = [\tilde{B}_m^{\dagger}, B_n^{\dagger}] = 0\,, \tag{2.11a}$$

$$[\tilde{\phi}_p, B_m B_n^{\dagger}] = 0\,, \tag{2.11b}$$

$$B_m B_n^{\dagger}|0\rangle = 2i[\tilde{\pi}_n, F_m]|0\rangle = 2C_{mn}(\phi)|0\rangle = E_{mn}(\phi)|0\rangle\,, \tag{2.11c}$$

which will be useful below.

It should be noted that Haan's[6] Euclidean master field relation appears in our notation as

$$(F_m + i\tilde{\pi}_m)|0\rangle = 0\,. \tag{2.12}$$

Although this relation follows from (2.8e), (2.9) and (2.10c), the operators $F_m + i\tilde{\pi}_m$ do not satisfy any simple algebra.

2.3. *Sharpening a tool*

In Ref. 1, the $BB^\dagger$ relation (2.10a) was proven by analysis of the ground state wave function (and follows from (2.2) in the action case), but a conjecture was offered which would give this result directly in the reduced operator formulation. Here we prove this conjecture, assuming only the completeness of the basis $\phi^w|0\rangle$.

Theorem.

$$\textit{If } [X, \tilde{\phi}_m] = [Y, \tilde{\phi}_m] = 0,\ \forall m \textit{ and } X|0\rangle = Y|0\rangle,\ \textit{then } X = Y\,. \tag{2.13}$$

Proof. Introduce the complete set of states

$$|w\rangle \equiv \phi^w|0\rangle = \tilde{\phi}^{\bar{w}}|0\rangle \tag{2.14}$$

and follow the steps

$$X|w\rangle = X\tilde{\phi}^{\bar{w}}|0\rangle = \tilde{\phi}^{\bar{w}}X|0\rangle = \tilde{\phi}^{\bar{w}}Y|0\rangle = Y\tilde{\phi}^{\bar{w}}|0\rangle = Y|w\rangle\,. \tag{2.15}$$

In practice, this theorem can be read as

$$[\tilde{\phi}_m, O_1(\phi,\pi)] = 0\,, \quad \forall m \quad \to \quad O_1(\phi,\pi) = O_2(\phi)\,, \tag{2.16a}$$

$$O_1(\phi,\pi)|0\rangle = O_2(\phi)|0\rangle\,, \tag{2.16b}$$

where $O_2(\phi)$ is determined by the ground state condition (2.16b). This is the form conjectured in Ref. 1. As a first application of this theorem, the relation (2.10a) of the interacting Cuntz algebra follows immediately from (2.11).

2.4. *Action examples*

Using Apps. C and E of Ref. 1, and in particular the results,

$$(E^{mn})_{rs} = \frac{\partial^2 S}{\partial\phi^m_{tr}\partial\phi^n_{st}} \underset{N}{=} B^m_{rt}(B^{\dagger n})_{ts}\,, \tag{2.17a}$$

$$\frac{1}{N}\,\mathrm{Tr}\left[h\left(\frac{\phi}{\sqrt{N}}\right)\right] \underset{N}{=} \langle h(\phi)\rangle \tag{2.17b}$$

we may compute the operators F_m and E_{mn} of the interacting Cuntz algebra (2.9) and (2.10) for any action:

(1) Standard one-matrix model

$$S = \mathrm{Tr}\left(\frac{m^2}{2}\phi^2 + \frac{\lambda}{4N}\phi^4\right), \tag{2.18a}$$

$$F = \frac{1}{2}(m^2\phi + \lambda\phi^3), \quad E = m^2 + \lambda(\langle\phi^2\rangle + \langle\phi\rangle\phi + \phi^2). \tag{2.18b}$$

(2) General one-matrix model

$$S = N\sum_{n=1}^{\infty}\frac{S_n}{n}\,\mathrm{Tr}\left[\left(\frac{\phi}{\sqrt{N}}\right)^n\right], \tag{2.19a}$$

$$F = \frac{1}{2}\sum_{n=1}^{\infty} S_n\phi^{n-1}, \quad E = \sum_{n=2}^{\infty} S_n \sum_{m=0}^{n-2}\langle\phi^m\rangle\phi^{n-m-2}. \tag{2.19b}$$

(3) Two-matrix model

$$S = \mathrm{Tr}\left[\frac{m_1^2}{2}(\phi^1)^2 + \frac{m_2^2}{2}(\phi^2)^2 + \frac{\lambda_1}{4N}(\phi^1)^4 + \frac{\lambda_2}{4N}(\phi^2)^4 + g\phi^1\phi^2\right], \tag{2.20a}$$

$$F_1 = \frac{1}{2}(m_1^2\phi_1 + \lambda_1\phi_1^3 + g\phi_2), \quad F_2 = \frac{1}{2}(m_2^2\phi_2 + \lambda_2\phi_2^3 + g\phi_1), \tag{2.20b}$$

$$E_{11} = m_1^2 + \lambda_1(\phi_1^2 + \langle\phi_1\rangle\phi_1 + \langle\phi_1^2\rangle), \quad E_{12} = g, \tag{2.20c}$$

$$E_{22} = m_2^2 + \lambda_2(\phi_2^2 + \langle\phi_2\rangle\phi_2 + \langle\phi_2^2\rangle), \quad E_{21} = g. \tag{2.20d}$$

(4) General action

$$S = N\sum_{w} S_w\,\mathrm{Tr}\left[\left(\frac{\phi}{\sqrt{N}}\right)^w\right], \tag{2.21a}$$

$$F_m = \frac{1}{2}\sum_{w} S_w \sum_{w=umv}\phi^{vu}, \quad E_{mn} = \sum_{w} S_w \sum_{w\sim numv}\langle\phi^u\rangle\phi^v, \tag{2.21b}$$

where the notation $w \sim w'$ means that the two words are equivalent under a cyclic permutation of their letters.

For actions with even powers of ϕ only, we may set the odd vev's to zero. We also find that the simple forms

$$F_m = F_m(\phi_m), \qquad E_{mn} = E_m(\phi_m)\delta_{m,n} \tag{2.22}$$

follow for matrix models of independent matrices (free random variables[9]). The special case of free actions and/or oscillators (which give the Cuntz algebra) is discussed in App. B.

3. Annihilation Operators

In Secs. 3–7 below, action and phase-space master fields are discussed on an equal footing.

3.1. *Linear in* π

We turn now to the construction of the infinite-dimensional free algebra, beginning with the composite annihilation operators B_w:

$$B_w \equiv B^w = B_{m_1} B_{m_2} \cdots B_{m_n}\,, \quad B_w|0\rangle = \delta_{w,0}|0\rangle\,. \tag{3.1}$$

These operators automatically satisfy the product rule

$$B_w B_{w'} = B_{ww'} \tag{3.2}$$

and moreover we find with (2.8) and (2.10c) that

$$[\tilde{\phi}_p, B_m] = -\delta_{p,m}|0\rangle\langle 0|\,, \tag{3.3a}$$

$$[\tilde{\phi}_p, B_{mn}] = -\delta_{p,m}|0\rangle\langle 0|B_n = -\delta_{p,m}|0\rangle\langle 0|2F_n(\phi)\,, \tag{3.3b}$$

$$\langle 0|\xi(\phi)\pi_m = \langle 0|\{[\tilde{\xi}(\phi), \pi_m] - iF_m(\phi)\tilde{\xi}(\phi)\} = \langle 0|\xi_m(\phi)\,, \tag{3.3c}$$

where the operators $\xi_m(\phi)$ are determined in principle as in Ref. 1. It follows that

$$[\tilde{\phi}_p, B_{mw}] = -\delta_{p,m}|0\rangle\langle 0|B_w = -\delta_{p,m}|0\rangle\langle 0|G_w(\phi)\,, \tag{3.4}$$

where the operators G_w are to be determined. The theorem in (2.13) then tells us that the annihilation operators are linear in the reduced momenta π_m

$$B_{mw} = F_{mw}(\phi) + i\pi_m G_w(\phi)\,, \tag{3.5a}$$

$$G_0 = 1\,, \qquad G_m = 2F_m\,, \tag{3.5b}$$

where the operators F_w are also to be determined. In what follows, we will discuss this surprising result from a number of viewpoints.

3.2. *Determination of* F_w *and* G_w

In this subsection, we give an independent inductive proof of the formula (3.5a) which also determines the coefficients F_w and G_w recursively in terms of the known operators F_m and E_{mn}.

To begin, we rewrite the interacting Cuntz relation (2.10a) in terms of reduced momenta, using (2.9):

$$B_m B_n^\dagger = E_{mn}(\phi) \leftrightarrow \pi_m\pi_n + i\pi_m F_n - iF_m\pi_n + F_m F_n - E_{mn} = 0\,. \tag{3.6}$$

The π form of this relation will be called the *first master constraint* below. It allows us to eliminate $\pi_m\pi_n$ in favor of terms linear in π, and hence to verify for example that $B_{mn} = B_m B_n$ is indeed linear in π. A proof by induction then starts with

$$B_m B_{nw} = B_{mnw} \leftrightarrow (F_m + i\pi_m)(F_{nw} + i\pi_n G_w) = F_{mnw} + i\pi_m G_{nw}\,, \tag{3.7}$$

where we have assumed the form (3.5a) and the left side of (3.7) is a special case of (3.2).

Using (3.6) in (3.7), one then obtains the recursion relations

$$G_{mw} = F_{mw} + F_m G_w \,, \tag{3.8a}$$

$$F_{mnw} = F_m(F_{nw} + F_n G_w) - E_{mn} G_w \tag{3.8b}$$

which can be rearranged into the more useful forms

$$G_{mnw} = G_m G_{nw} - E_{mn} G_w \,, \tag{3.9a}$$

$$F_{mw} = G_{mw} - F_m G_w \,. \tag{3.9b}$$

These relations are easily iterated to any desired order, and we list here the results

$$G_0 = 1\,, \quad G_m = 2F_m\,, \quad G_{mn} = G_m G_n - E_{mn}\,, \tag{3.10a}$$

$$G_{mnp} = G_m G_n G_p - G_m E_{np} - E_{mn} G_p\,, \tag{3.10b}$$

$$G_{mnpq} = G_m G_n G_p G_q - G_m G_n E_{pq} - G_m E_{np} G_q - E_{mn} G_p G_q + E_{mn} E_{pq}\,, \tag{3.10c}$$

$$F_{mn} = 2F_m F_n - E_{mn}\,, \quad F_{mnp} = 4F_m F_n F_p - F_m E_{np} - 2E_{mn} F_p \tag{3.10d}$$

for the first few words of F and G.

More generally, the recursion relations can be used to prove the following properties:

$$G_w{}^\dagger = G_{\bar{w}}\,, \qquad F_{mw}{}^\dagger = G_{\bar{w}m} - G_{\bar{w}} F_m\,, \tag{3.11}$$

$$\mathcal{G} = \frac{1}{1 - \alpha_m G_m(\phi) + \alpha_m \alpha_n E_{mn}(\phi)} = \sum_w \alpha^w G_w(\phi)\,, \tag{3.12a}$$

$$\big(1 - \alpha_m F_m(\phi)\big)\mathcal{G} = \sum_w \alpha^w F_w(\phi)\,, \quad F_0 = 1\,, \tag{3.12b}$$

$$G_{wm} G_{nw'} = G_{wmnw'} + G_w E_{mn} G_{w'}\,. \tag{3.13}$$

Here α_m (with products α^w) is a free-algebraic source or "place marker" whose only property is that it commutes with ϕ_m and π_m.

The generating functions (3.12a) and (3.12b) show that G_w and F_w are free-algebraic generalizations of Chebyshev polynomials (see also Subsec. 3.3 and App. B).

We also mention the relations

$$B_{mw} = -B_m^\dagger G_w + G_{mw}\,, \qquad B_{mw}{}^\dagger = -G_{\bar{w}} B_m + G_{\bar{w}m} \tag{3.14}$$

which are a useful alternative to the basic equation (3.5a), and the relations

$$G_{wmn} = G_{wm} G_n - G_w E_{mn}\,, \quad G_0 = 1\,, \quad G_m = 2F_m\,, \tag{3.15a}$$

$$F_{wmn} = F_{wm} G_n - F_w E_{mn}\,, \quad F_0 = 1\,, \quad F_m = 1F_m \tag{3.15b}$$

which show a complete symmetry of the recursion relations for G_w and F_w, except for their initial conditions. The relations

$$\begin{aligned} \pi_m G_w|0\rangle &= iF_{mw}|0\rangle \,, \\ B_m^\dagger G_w|0\rangle &= G_{mw}|0\rangle \,, \end{aligned} \tag{3.16a}$$

$$[i\pi_m, \tilde{G}_{nw}]|0\rangle = E_{mn}G_w|0\rangle \,, \tag{3.16b}$$

$$[i\pi_m, \tilde{F}_n]|0\rangle = C_{mn}(\phi)|0\rangle \tag{3.16c}$$

also follow from the discussion above. The relation (3.16c), which is a special case of (3.16b), was given in Ref. 1.

3.3. *One-matrix models*

In the case of general one-matrix (action or Hamiltonian) models the operators F and E commute, and $w \to [w]$, giving the simpler forms

$$G_{n+2} = G_1 G_{n+1} - EG_n \,, \quad G_0 = 1 \,, \quad G_1 = 2F \,, \tag{3.17a}$$

$$F_{n+1} = G_{n+1} - FG_n \,, \quad F_0 = 1 \,, \quad F_1 = F \,, \tag{3.17b}$$

$$G_n = E^{\frac{n}{2}} \frac{\sin((n+1)\theta)}{\sin\theta} \,, \quad F_n = E^{\frac{n}{2}} \cos(n\theta) \,, \tag{3.17c}$$

$$\rho = \frac{\sqrt{E}}{\pi} \sin\theta \,, \quad \cos\theta = \frac{F}{\sqrt{E}} \,, \quad E = 2C = F^2 + \pi^2\rho^2 \,, \tag{3.17d}$$

$$G_m G_n = \sum_{k=0}^{\min_{m,n}} E^k G_{m+n-2k} \tag{3.17e}$$

which include the Chebyshev polynomials themselves in (3.17c). The finite operator product expansion in (3.17e) follows immediately from this form. According to Ref. 1, the quantity ρ in (3.17d) is the ground state density of the action or Hamiltonian system.

Another special case with simplifications is that of many oscillators and/or free actions (see App. B).

3.4. *Master constraints*

Using (3.5a), the composition law

$$B_{mw}B_n = B_{mwn} \tag{3.18}$$

can be written out in two equivalent forms, called the *master constraints*,

$$\pi_m G_w \pi_n + i\pi_m {F_{n\bar{w}}}^\dagger + F_{mw}(-i\pi_n) + F_{mwn} - F_{mw}F_n = 0 \,, \tag{3.19a}$$

$$B_m^\dagger G_w B_n = B_m^\dagger G_{wn} + G_{mw}B_n - G_{mwn} \tag{3.19b}$$

and (3.19a) contains the first master constraint (3.6) as the special case when $w = 0$.

More generally, the form (3.19a) of the master constraints allow us to eliminate quadratic forms $\pi_m G_w \pi_n$ in favor of forms linear in the reduced momenta, and similarly for $B_m^\dagger G_w B_n$ in (3.19b).

In Hamiltonian theories, constraints are constants of the motion and the first master constraint, which is equivalent to $B_m B_n^\dagger - E_{mn} = 0$, was noted as a set of d^2 constants of the motion in Ref. 1. It is shown in App. C that all the higher master constraints are in fact composites of the first master constraint, so there are no new independent constants of the motion in this list.

4. Creation Operators

4.1. *Creation operators and the natural basis*

The creation operators of the infinite-dimensional free algebra are defined as the Hermitian conjugates of the annihilation operators

$$B_w{}^\dagger = B_{m_n}^\dagger \cdots B_{m_1}^\dagger = B^\dagger{}_{\bar{w}}\,, \tag{4.1a}$$

$$B_{mw}{}^\dagger = F_{mw}(\phi)^\dagger - iG_{\bar{w}}(\phi)\pi_m\,, \tag{4.1b}$$

$$\langle 0|B_w{}^\dagger = \langle 0|\delta_{w,0} \tag{4.1c}$$

and therefore satisfy the product rule

$$B_w{}^\dagger B_{w'}{}^\dagger = B_{w'w}{}^\dagger\,. \tag{4.2}$$

The set of all these creation operators on the ground state is a natural complete[1] basis, and we see from (3.9b), (3.16a) and (4.1b) that this basis can be expressed in terms of the polynomial G_w's as

$$(B_{\bar{w}})^\dagger|0\rangle = B^{\dagger\, w}|0\rangle = \tilde{B}^{\dagger\, \bar{w}}|0\rangle = G_w(\phi)|0\rangle\,, \tag{4.3a}$$

$$\langle G_w(\phi)\rangle = \delta_{w,0}\,. \tag{4.3b}$$

In what follows, the states on the right and left of (4.3a) will be called the *natural basis* and its operator form respectively. Further discussion of completeness is given in Subsec. 5.4.

4.2. $B^\dagger B$ *relations*

Using (3.5a), (4.1b) and the first master constraint (3.6), we find the $B^\dagger B$ algebra

$$B_{mw}{}^\dagger B_{nw'} = G_{\bar{w}m} B_{nw'} - G_{\bar{w}} B_{mnw'} \tag{4.4}$$

and the relations

$$B_m^\dagger (E^{-1})_{mn} B_n = 1 - |0\rangle\langle 0|\,, \tag{4.5a}$$

$$B_{mpw}{}^\dagger (E^{-1})_{mn} B_{nqw'} = G_{\bar{w}p} B_{qw'} - G_{\bar{w}} B_{pqw'} - G_{\bar{w}p}|0\rangle\langle 0|G_{qw'} \tag{4.5b}$$

also follow immediately from the interacting Cuntz algebra and the composition laws (3.18) and (4.2).

A more symmetric version of (4.4) and (4.5) is

$$B_w{}^\dagger B_{w'} = B_w{}^\dagger G_{w'} + G_{\bar{w}} B_{w'} - G_{\bar{w}w'}\,, \tag{4.6a}$$

$$B_{mw}{}^\dagger (E^{-1})_{mn} B_{nw'} = B_w{}^\dagger B_{w'} - G_{\bar{w}}|0\rangle\langle 0|G_{w'}\,, \tag{4.6b}$$

where (4.6a) can be used to "linearize" (4.6b). These forms follow directly from (3.14) and the interacting Cuntz algebra.

4.3. *Local and nonlocal*

In Ref. 1, many reduced operators were called nonlocal because they involved arbitrarily-high powers of the reduced momenta π_m, and others were called local because they involved no more than two powers of the reduced momenta. The results above blur this distinction.

As an example,[1] consider the (Hermitian) isotropic oscillator Hamiltonian H, which may now be re-expressed in terms of the generators of the infinite-dimensional free algebra:

$$H \equiv \sum_{w\neq 0} A_w{}^\dagger A_w = \sum_{w\neq 0} \frac{1}{2^{[w]}} B_w{}^\dagger B_w \tag{4.7a}$$

$$= \sum_{m,w} \frac{1}{2^{[w]+1}} (G_{\bar{w}m} B_{mw} - G_{\bar{w}} B_{mmw}) \tag{4.7b}$$

$$= \sum_{m,w} \frac{1}{2^{[w]+1}} (B_{mw}{}^\dagger G_{mw} - B_{mmw}{}^\dagger G_w)\,. \tag{4.7c}$$

The starting point is "nonlocal" because each of the Cuntz operators in the products $A_w = A^w = A_{m_1} \cdots A_{m_n}$ is linear in the reduced momentum, while (4.7b) and its Hermitian conjugate (4.7c) are "local but nonpolynomial" because they are linear in the reduced momenta.

Although we will not discuss it explicitly here, the phenomenon of this section also generates new large N field identifications (see Ref. 1) in the unreduced theory.

5. Dual Basis

5.1. *Definition*

We wish to find new polynomials $\{T_w(\phi)\}$ which are vev-orthogonal to the set $\{G_w(\phi)\}$

$$\langle T_w(\phi) G_{\bar{w}'}(\phi)\rangle = \delta_{w,w'}\,, \qquad T_0(\phi) = 1 \tag{5.1}$$

and we will refer to the set of states $\{\langle 0|T_w(\phi)\}$ as the *dual basis.*

Towards the construction of these polynomials, we first postulate a generating function for the T's

$$Y = \frac{1}{1 - \beta_m \phi_m + X(\beta)} = \sum_w \beta^w T_w(\phi)\,, \tag{5.2a}$$

$$X(\beta) = \sum_w \beta^w X_w\,, \qquad X_0 = 0\,, \tag{5.2b}$$

where β_m is another free-algebraic source (like α_m above) and the quantity $X(\beta)$ is to be determined. Note that the relations

$$\langle T_w \rangle = \delta_{w,0}\,, \qquad \langle Y \rangle = 1 \tag{5.3}$$

follow from (5.1) and (5.2) respectively.

Next, follow the steps

$$\langle 0|Y\tilde{B}_m^\dagger = \langle 0|[Y, -i\tilde{\pi}_m] = \langle 0|Y[1 - \beta_n\phi_n + X, i\tilde{\pi}_m]Y \tag{5.4a}$$

$$= \langle 0|Y\beta_m|0\rangle\langle 0|Y = \beta_m\langle 0|Y\,, \tag{5.4b}$$

where we have used (2.8) and (5.2). Repeating this, we obtain

$$\langle 0|Y(\tilde{B}^\dagger)^w|0\rangle = \beta^w\langle Y\rangle = \beta^w \tag{5.5}$$

which, with (4.3a), gives us the desired result (5.1).

To compute T_w and X_w explicitly, multiply (5.2a) on the left by the inverse of Y to obtain

$$1 = \sum_w \beta^w T_w - \sum_{m,w} \beta^{mw}\phi_m T_w + \sum_{w,w'} \beta^{ww'} X_w T_{w'}\,. \tag{5.6}$$

Then, equating coefficients of each β word, we find the recursion relation for T_w

$$T_{mw} = \phi_m T_w - \sum_{w=w_1w_2} X_{mw_1} T_{w_2}\,, \qquad T_0 = 1\,. \tag{5.7}$$

Multiplying in the other order leads to

$$T_{wm} = T_w\phi_m - \sum_{w=w_1w_2} T_{w_1} X_{w_2m} \tag{5.8}$$

and the vev's of these equations

$$X_{mw} = \langle \phi_m T_w \rangle = \langle T_w \phi_m \rangle = X_{wm} \tag{5.9}$$

determine the X_w's and show that they have cyclic symmetry in the letters of their words.

5.2. *Examples*

Because the T's and X's are unfamiliar, we list the first few words of each:

$$\begin{aligned} T_0 &= 1\,, \quad T_m = \phi_m - X_m\,, \\ T_{mn} &= (\phi_m - X_m)(\phi_n - X_n) - X_{mn}\,, \end{aligned} \tag{5.10a}$$

$$\begin{aligned} T_{mnp} &= (\phi_m - X_m)(\phi_n - X_n)(\phi_p - X_p) \\ &\quad - (\phi_m - X_m)X_{np} - X_{mn}(\phi_p - X_p) - X_{mnp}\,, \end{aligned} \tag{5.10b}$$

$$X_0 = 0\,, \quad X_m = \langle \phi_m \rangle\,, \quad X_{mn} = \langle \phi_m \phi_n \rangle - X_m X_n\,, \tag{5.11a}$$

$$X_{mnp} = \langle \phi_m \phi_n \phi_p \rangle - X_m X_{np} - X_n X_{mp} - X_p X_{mn} - X_m X_n X_p\,, \tag{5.11b}$$

$$\begin{aligned} X_{mnpq} &= \langle \phi_m \phi_n \phi_p \phi_q \rangle - X_m X_{npq} - X_n X_{mpq} - X_p X_{mnq} \\ &\quad - X_q X_{mnp} - X_{mn} X_{pq} - X_{mq} X_{np} - X_n X_m X_{pq} \\ &\quad - X_n X_p X_{mq} - X_n X_q X_{mp} - X_p X_m X_{qn} \\ &\quad - X_q X_p X_{mn} - X_q X_m X_{np} - X_m X_n X_p X_q\,. \end{aligned} \tag{5.11c}$$

One sees that the X_w's so far match the planar connected parts discussed in Refs. 10 and 8, and one also sees that $T_w(\phi)$, with $\langle T_w \rangle = \delta_{w,0}$, may be considered as a type of normal ordered product $:\phi^w:$ of the reduced fields.

5.3. *More general results*

From the recursive definitions (5.7)–(5.9) we find

$$T_w{}^\dagger = T_{\bar{w}}\,, \quad X_w{}^* = X_{\bar{w}}\,, \tag{5.12a}$$

$$\langle G_{\bar{w}} T_{w'} \rangle = \delta_{w,w'} \tag{5.12b}$$

as well as the following relations

$$T_w = \phi^w - \sum_{w = w_1 w_2 w_3} T_{w_1} X_{w_2} \phi^{w_3}\,, \tag{5.13}$$

$$T_w T_{w'} = \sum_{w''} C_{w,w',w''} T_{w''}\,, \quad [w''] \le [w] + [w']\,, \tag{5.14}$$

$$\langle T_m T_w \rangle = X_{mw}(1 - \delta_{w,0})\,, \tag{5.15a}$$

$$\langle T_{mn} T_w \rangle = X_{mnw}(1 - \delta_{w,0}) + \sum_{\substack{w = w_1 w_2 \\ w_1, w_2 \neq 0}} X_{nw_1} X_{mw_2}\,, \tag{5.15b}$$

$$\vdots$$

$$Z(j) \equiv \sum_w \langle \phi^w \rangle j^w = 1 + X\big(jZ(j)\big)\,. \tag{5.16}$$

In particular, (5.13) can also be iterated to obtain the T's. The relation in (5.14) is an operator product expansion, whose sum obeys the selection rule shown because the T's are finite polynomials in ϕ. The list of relations begun in (5.15) shows correspondingly higher powers of X_w when extended to more general words.

The final relation (5.16), with j another free-algebraic source, is proven in App. D. This is the standard relation[10,8] between the generating functions Z and X of planar and connected planar correlators respectively, and completes the identification of X_w as the planar connected part with $[w]$ legs.

5.4. *Completeness*

The dual basis $\{\langle 0|T_w(\phi)\}$ is complete because the $\{\phi^w|0\rangle\}$ basis is complete, and therefore the basis $\{B^{\dagger w}|0\rangle = G_w(\phi)|0\rangle\}$ is also complete.[c] This gives the completeness statements

$$\mathbf{1} = \sum_w G_w(\phi)|0\rangle\langle 0|T_{\bar{w}}(\phi) = \sum_w T_w(\phi)|0\rangle\langle 0|G_{\bar{w}}(\phi) \tag{5.17}$$

and various consequences such as

$$\delta_{w,w'} = \sum_{w''}\langle T_{\bar{w}}T_{w''}\rangle\langle G_{\bar{w}''}G_{w'}\rangle\,. \tag{5.18}$$

Moreover, either set of polynomials $\{G_w(\phi)\}$ or $\{T_w(\phi)\}$ give a complete basis[d] for expansion of any polynomial in ϕ

$$\mathcal{F}(\phi) = \sum_w G_w(\phi)\langle T_{\bar{w}}(\phi)\mathcal{F}(\phi)\rangle = \sum_w T_w(\phi)\langle G_{\bar{w}}(\phi)\mathcal{F}(\phi)\rangle\,. \tag{5.19}$$

We have already encountered such an expansion in (5.14).

Another operator product expansion which will be useful below is

$$G_w G_{w'} = \sum_{w''} G_{w''}\langle T_{\bar{w}''}G_w G_{w'}\rangle\,. \tag{5.20}$$

The sum on the right of (5.20) is generally an infinite number of terms, but a finite number in the case of oscillators/free actions (see App. B). It will also be useful to consider expansions of products of the master field:

$$\phi_m = \sum_w X_{mw}G_{\bar{w}}(\phi) = \sum_w G_w(\phi)X_{\bar{w}m}\,, \tag{5.21a}$$

$$\phi_m\phi_n = \sum_w \Big(X_{mnw} + \sum_{w=w_1w_2} X_{nw_1}X_{mw_2}\Big)G_{\bar{w}}(\phi)\,. \tag{5.21b}$$

$$\vdots$$

The proof of these follow readily from (5.19) and (5.15).

[c] A different argument for the completeness of $B^{\dagger w}|0\rangle$ was given in Ref. 1.

[d] There are questions which need further study concerning the domain of convergence of expansions in the G_w's when infinite sums are involved. For example, in the case of one matrix with a pure ϕ^4 action the functions $G_w(\phi)$ have no linear term in ϕ and yet Eq. (5.21a) says that an infinite sum of such functions is equal to ϕ.

5.5. *Operator form of the dual basis*

Recall that ${B_{\bar{w}}}^{\dagger}|0\rangle$ is the operator form of the basis $G_w(\phi)|0\rangle$. To obtain the operator form of the dual basis $\langle 0|T_w(\phi)$, we first define a new set of operators $\bar{B}_m$

$$\bar{B}_m \equiv (E^{-1})_{mn} B_n \,. \tag{5.22}$$

The interacting Cuntz algebra (2.10) implies that these operators satisfy a (dual basis) Cuntz algebra

$$\bar{B}_m B_n^{\dagger} = \delta_{m,n} \,, \qquad B_m^{\dagger} \bar{B}_m = 1 - |0\rangle\langle 0| \,, \tag{5.23a}$$

$$\bar{B}_m |0\rangle = \langle 0| B_m^{\dagger} = 0 \tag{5.23b}$$

although $\bar{B}_m$ and $B_m^{\dagger}$ are not Hermitian conjugates. This curious fact will play a central role in the discussion of Sec. 7.

Next, we consider the product of any number of $\bar{B}$'s

$$\bar{B}_w = \bar{B}^w = \bar{B}_{m_1} \cdots \bar{B}_{m_n} \tag{5.24}$$

and note that

$$\langle 0|\bar{B}^{\bar{w}'} G_w |0\rangle = \langle 0|\bar{B}^{\bar{w}'} B^{\dagger w}|0\rangle = \delta_{w,w'} \,. \tag{5.25}$$

It follows that

$$\langle 0|(T_{\bar{w}'} - \bar{B}_{\bar{w}'}) G_w |0\rangle = 0 \,, \quad \forall\ w \tag{5.26}$$

and this gives the operator form of the dual basis

$$\langle 0|\bar{B}_w = \langle 0|T_w(\phi) \,, \tag{5.27a}$$

$$\mathbf{1} = \sum_w {B_w}^{\dagger}|0\rangle\langle 0|\bar{B}_w \tag{5.27b}$$

because the basis $G_w|0\rangle$ is complete.

The operator form of the dual basis gives us a number of new forms for the planar connected parts

$$X_{\bar{w}mn} = \langle T_{\bar{w}m}\phi_n\rangle = \langle \bar{B}_{\bar{w}m}\phi_n\rangle = \langle \bar{B}_{\bar{w}}(E^{-1})_{mn}\rangle = \langle T_{\bar{w}}(E^{-1})_{mn}\rangle \,, \tag{5.28}$$

where we have used (2.8e) and (3.3a). Then the useful relation

$$\left(E^{-1}(\phi)\right)_{mn} = \sum_w X_{mn\bar{w}} G_w(\phi) \tag{5.29}$$

follows immediately from (5.19).

For free random variables, we can say more. Taken together, the form of E in (2.22) and the final form of X in (5.28) show that $X_{\bar{w}mn} \propto \delta_{m,n}$ in this case. Then, the cyclic symmetry of X_w tells us that the only nonzero planar connected parts are the "single letter" X's

$$X_{w(m)} \equiv X_{m\cdots m} \neq 0 \,, \quad m = 1 \cdots d \,. \tag{5.30}$$

This simple fact means that the computation of the planar connected parts (see Sec. 8) is one-dimensional and, via Eq. (5.16), the relation (5.30) explains many intricate identities among the planar parts.

6. $BB^\dagger$

We have so far verified all the relations of the infinite-dimensional free algebra (1.5) except for the $BB^\dagger$ relation (1.5d). This relation requires a combination of several of the principles we have discussed above, and will be developed in stages.

Note first the relations

$$B_{wm}B_n^\dagger = B_w E_{mn}\,, \qquad B_m {B_{wn}}^\dagger = E_{mn}{B_w}^\dagger\,, \tag{6.1a}$$

$$B_{wm}{B_{w'n}}^\dagger = B_w E_{mn}{B_{w'}}^\dagger \tag{6.1b}$$

which follow from (2.10a) alone. In (6.1b), we see that this direction soon fails to produce relations linear in B_w and ${B_w}^\dagger$.

To obtain relations linear in B and $B^\dagger$, we consider the product $B_{mw}{B_{nw'}}^\dagger$ using the forms (3.14) of these operators in terms of the interacting Cuntz operators. Among the four resulting terms, the only term quadratic in B, $B^\dagger$ is $B_m^\dagger G_w G_{w'} B_n$. This term may be "linearized" by first using the completeness relation (5.20) and then using the master constraints in the form (3.19b). We find two alternative forms of the result:

$$\begin{aligned} B_{mw}{B_{nw'}}^\dagger &= -\left[B_m^\dagger G_w G_{\bar{w}'n} + G_{mw}G_{\bar{w}'}B_n - G_{mw}G_{\bar{w}'n}\right] \\ &\quad + \sum_{w''}\langle T_{\bar{w}''}G_w G_{\bar{w}'}\rangle\left[B_m^\dagger G_{w''n} + G_{mw''}B_n - G_{mw''n}\right], \end{aligned} \tag{6.2}$$

$$\begin{aligned} B_{mw}{B_{nw'}}^\dagger &= \left[B_{mw}G_{\bar{w}'n} + G_{mw}{B_{nw'}}^\dagger - G_{mw}G_{\bar{w}'n}\right] \\ &\quad - \sum_{w''}\langle T_{\bar{w}''}G_w G_{\bar{w}'}\rangle\left[B_{mw''n} + {B_{n\bar{w}''m}}^\dagger - G_{mw''n}\right]. \end{aligned} \tag{6.3}$$

These forms are linear in the operators B, $B^\dagger$ but the coefficients are functions of ϕ.

A form which is strictly linear in the generators B_w, ${B_w}^\dagger$ can be derived from (6.2) by again using the expansion (5.20) for the products of two G's and then using the formulas (3.14) in reverse. The result is

$$\begin{aligned} B_{mw}{B_{nw'}}^\dagger &= \sum_{w''}(B_{mw''}f_{w'',w,n,w'} + f^*_{w'',w',m,w}{B_{nw''}}^\dagger) \\ &\quad + E_{mw;nw'}(\phi)\,, \end{aligned} \tag{6.4a}$$

$$f_{w'',w,n,w'} = \langle T_{\bar{w}''}G_w G_{\bar{w}'n}\rangle - \sum_u \delta_{w'',un}\langle T_{\bar{u}}G_w G_{\bar{w}'}\rangle\,, \tag{6.4b}$$

$$\begin{aligned} E_{mw;nw'} &= G_{mw}G_{\bar{w}'n} + \sum_{w''}\left[G_{mw''n}\langle T_{\bar{w}''}G_w G_{w'}\rangle\right. \\ &\quad \left. - G_{mw''}\langle T_{\bar{w}''}G_w G_{\bar{w}'n}\rangle - G_{w''n}\langle T_{\bar{w}''}G_{mw}G_{\bar{w}'}\rangle\right]. \end{aligned} \tag{6.4c}$$

One may compare this general structure with the simple oscillator results in App. B.

7. Forms of the Master Field

7.1. *Basic form*

The form (5.21a) of the master field in terms of the basis G_w

$$\phi_m = \sum_w X_{mw} G_{\bar{w}}(\phi) \tag{7.1}$$

will be called the *basic form* of the master field. All the other forms of the master field below follow from the basic form.

7.2. *In terms of interacting Cuntz operators*

The basis G_w is a set of polynomials (see Sec. 3) in G_m and E_{mn}, which may in turn be written as

$$G_m = B_m + B_m^\dagger \,, \qquad E_{mn} = B_m B_n^\dagger \,. \tag{7.2}$$

These relations allow us to express the G_w's and hence the master field (7.1) in terms of interacting Cuntz operators:

$$\phi_m = X_m + X_{mn}(B_n + B_n^\dagger) + X_{mpn}\big(B_n B_p + B_n^\dagger(B_p + B_p^\dagger)\big) + \cdots \,. \tag{7.3}$$

7.3. *In terms of ordinary Cuntz operators*

Recall the construction[1] of ordinary Cuntz operators from the interacting Cuntz operators

$$a_m = \big(E^{-\frac{1}{2}}\big)_{mn} B_n \,, \quad a_m^\dagger = B_n^\dagger \big(E^{-\frac{1}{2}}\big)_{nm} \,, \tag{7.4a}$$

$$a_m a_n^\dagger = \delta_{m,n} \,, \quad a_m^\dagger a_m = 1 - |0\rangle\langle 0| \,, \quad a_m|0\rangle = \langle 0|a_m^\dagger = 0 \,, \tag{7.4b}$$

where $a^\dagger$ is the Hermitian conjugate of a. This allows us to express the master field (7.3) in terms of ordinary Cuntz operators:

$$\phi_m = X_m + X_{mn}\Big(\big(E^{\frac{1}{2}}\big)_{nq} a_q + a_q^\dagger \big(E^{\frac{1}{2}}\big)_{qn}\Big) + \cdots \,. \tag{7.5}$$

7.4. *Dual basis form*

To express the master field in this form, follow the steps

$$\bar{B}_m = (E^{-1})_{mn} B_n = \sum_w X_{mn\bar{w}} G_w(F_n + i\pi_n) = \sum_w X_{mn\bar{w}}(G_{wn} - B^{\dagger wn}) \,, \tag{7.6}$$

where we have used the form (4.1b) for ${B_w}^\dagger$ and the identities (3.9b) and (5.29). Adding X_m, we obtain the dual basis form of the master field

$$\phi_m = \sum_w X_{mw} G_{\bar{w}} = \bar{B}_m + \sum_w X_{m\bar{w}} B^{\dagger w} \,. \tag{7.7}$$

Recall from (5.23) that the operators $\bar{B}_m$, $B_m^\dagger$ (with $\bar{B}_m^\dagger \neq B_m^\dagger$) also satisfy an ordinary Cuntz algebra. Other ways of writing the dual basis form include

$$\phi_m = \bar{B}_m + \sum_w X_{mw} {B_w}^\dagger = \bar{B}_m \big(1 + \bar{X}(B^\dagger)\big) , \tag{7.8a}$$

$$\bar{X}(\beta) \equiv \sum_w \beta^w X_{\bar{w}} , \quad \langle 0|\bar{X}(B^\dagger) = 0 . \tag{7.8b}$$

Here, the first form in (7.8a) emphasizes that the master field is linear in the generators of the infinite-dimensional free algebra, and $\bar{X}(B^\dagger)$ in the second form is an alternate generating function of the planar connected parts (see App. D).

Note that the forms (7.7) and (7.8a) of the master field (and other forms throughout this section which are equal to ϕ_m in (7.1)) appear to involve the reduced momenta π_m in the creation and annihilation operators. However, as the reader is encouraged to verify, all such π terms cancel.

7.5. *Second dual basis form*

In spite of appearances, the dual basis form (7.8a) of the master field is Hermitian (as are all the previous forms), which tells us that

$$\phi_m = \phi_m^\dagger = \bar{B}_m^\dagger + \sum_w X_{m\bar{w}} B^w , \qquad \bar{B}_m^\dagger = B_n^\dagger (E^{-1})_{nm} . \tag{7.9}$$

The operators B, $\bar{B}^\dagger$, with $B^\dagger \neq \bar{B}^\dagger$, form another (second dual basis) Cuntz algebra

$$\begin{aligned} B_m \bar{B}_n^\dagger &= \delta_{m,n} , \\ \bar{B}_m^\dagger B_m &= 1 - |0\rangle\langle 0| , \\ B_m|0\rangle &= \langle 0|\bar{B}_m^\dagger = 0 , \end{aligned} \tag{7.10a}$$

$${\bar{B}_w}^\dagger |0\rangle = T_{\bar{w}}(\phi)|0\rangle , \tag{7.10b}$$

$$\mathbf{1} = \sum_w {\bar{B}_w}^\dagger |0\rangle\langle 0| B_w \tag{7.10c}$$

and we see that ${\bar{B}_w}^\dagger$ creates the ket form of the dual basis.

7.6. *Non-Hermitian forms*

Because the two sets of operators $(a_m, a_n^\dagger)$ and $(\bar{B}_m, B_n^\dagger)$ both satisfy the Cuntz algebra, the two sets are related by a similarity transformation S

$$S a_m S^{-1} = \bar{B}_m = (E^{-1}(\phi))_{mn} B_n = \big(E^{-\frac{1}{2}}(\phi)\big)_{mn} a_n , \tag{7.11a}$$

$$S a_m^\dagger S^{-1} = B_m^\dagger = a_n^\dagger \big(E^{\frac{1}{2}}(\phi)\big)_{nm} \tag{7.11b}$$

and S cannot be unitary because $B^\dagger$ is not the Hermitian conjugate of $\bar{B}$.

Then we see that the dual form of the master field in (7.8a) is the Hermitian counterpart of the non-Hermitian Gopakumar–Gross form M_m of the master field:

$$\phi_m = S M_m S^{-1}\,, \tag{7.12a}$$

$$M_m = a_m + \sum_w X_{m\bar{w}} a^{\dagger w} = a_m\big(1 + \bar{X}(a^\dagger)\big) \neq M_m^\dagger\,. \tag{7.12b}$$

Our algebraic derivation of (7.12b) complements the diagrammatic derivation[e] in Ref. 7. The one-matrix form $M = a + \Sigma_m c_{m+1} a^{\dagger m}$ of the non-Hermitian master field was determined earlier in Ref. 9.

The Hermitian conjugate of the Gopakumar–Gross form, which also serves as a master field, is related to the second dual form (7.9) of the Hermitian master field as follows:

$$S^{-1\,\dagger} a_m S^\dagger = B_m = \big(E^{\frac{1}{2}}\big)_{mn} a_n\,, \tag{7.13a}$$

$$S^{-1\,\dagger} a_m^\dagger S^\dagger = \bar{B}_m^\dagger = B_n^\dagger \big(E^{-1}\big)_{nm} = a_n^\dagger \big(E^{-\frac{1}{2}}\big)_{nm}\,, \tag{7.13b}$$

$$\phi_m = \phi_m^\dagger = S^{-1\,\dagger} M_m^\dagger S^\dagger\,, \tag{7.13c}$$

$$M_m^\dagger = a_m^\dagger + \sum_w X_{m\bar{w}} a^w = \big(1 + \bar{X}(a)\big) a_m^\dagger\,. \tag{7.13d}$$

These relations are nothing but the Hermitian conjugate of (7.11) and (7.12).

7.7. *In terms of planar correlators*

The relation (D.10) can be read as

$$\bar{Z}(j) = 1 + \bar{X}(B^\dagger)\,, \quad \bar{Z}(j) = \sum_w j^{\bar{w}} \langle \phi^w \rangle\,, \tag{7.14a}$$

$$B_m^\dagger = j_m \bar{Z}(j)\,, \quad j_m = B_m^\dagger \bar{Z}^{-1}(j)\,, \tag{7.14b}$$

$$\bar{B}_m j_n = \delta_{m,n} \bar{Z}^{-1}(j)\,, \quad \langle 0|\bar{Z}(j) = \langle 0|\,, \tag{7.14c}$$

where $\bar{Z}(j)$ is an alternate generating function for planar correlators. The "quantum source" j_m lives in a fourth Cuntz algebra

$$\frac{\partial}{\partial j_m} \equiv \bar{Z}(j)\bar{B}_m\,, \quad \bar{B}_m = \bar{Z}^{-1}(j)\frac{\partial}{\partial j_m}\,, \tag{7.15a}$$

$$\frac{\partial}{\partial j_m} j_n = \delta_{m,n}\,, \quad j_m \frac{\partial}{\partial j_m} = 1 - |0\rangle\langle 0|\,, \tag{7.15b}$$

$$\frac{\partial}{\partial j_m}|0\rangle = \langle 0| j_m = 0 \tag{7.15c}$$

which follows from (7.14) and the Cuntz algebra (5.23) of $\bar{B}$ and $B^\dagger$.

[e]Unfortunately, Gopakumar and Gross give the similar but incorrect result $M_m = a_m + \sum_w X_{mw} a^{\dagger w}$, as we ourselves did in an earlier version. To check that (7.12b) is in fact the correct form, evaluate $\langle M_m M_n M_p \rangle$ using the Cuntz algebra for a, $a^\dagger$.

This gives the forms of the master field

$$\phi_m = \bar{B}_m \bar{Z}(j) = \bar{Z}^{-1}(j) \frac{\partial}{\partial j_m} \bar{Z}(j) , \tag{7.16a}$$

$$\phi^w = \bar{Z}^{-1}(j) \left(\frac{\partial}{\partial j} \right)^w \bar{Z}(j) \tag{7.16b}$$

in terms of the planar correlators.

7.8. *In terms of planar 1PI parts*

The master field can also be written as a function of the planar connected one particle irreducible (1PI) parts. To see this, we first decompose the dual basis form of the master field (7.8a) into its classical part Φ_m and its quantum part $\bar{B}_m$

$$\phi_m = \Phi_m + \bar{B}_m , \tag{7.17a}$$

$$\Phi_m(B^\dagger) \equiv \bar{B}_m \bar{X}(B^\dagger) = \sum_w X_{m\bar{w}} B^{\dagger w} , \tag{7.17b}$$

$$\Phi_m B_m^\dagger = B_m^\dagger \Phi_m = \bar{X}(B^\dagger) . \tag{7.17c}$$

Our definition of the classical part Φ_m agrees with the field called Φ in Ref. 8, but the identities in (7.17c) are new.

The planar effective action $\Gamma(\Phi)$ is defined as

$$\Gamma(\Phi) \equiv \Phi_m B_m^\dagger = B_m^\dagger \Phi_m = \bar{X}(B^\dagger) = \sum_w \Gamma_w \Phi^w , \tag{7.18a}$$

$$\bar{B}_m \Gamma(\Phi) = \Phi_m , \qquad \langle 0 | \Gamma(\Phi) = 0 , \tag{7.18b}$$

where Γ_w is the cyclically symmetric planar 1PI part with $[w]$ legs. This definition of $\Gamma(\Phi)$ follows Ref. 10 but differs by a minus sign from the definition of Ref. 8, and we note in particular that the Legendre transform defined in Ref. 8

$$\bar{X}(B^\dagger) = -\Gamma(\Phi) + B_m^\dagger \Phi_m + \Phi_m B_m^\dagger \tag{7.19}$$

is satisfied trivially by (7.18a).

Then the master field can be written as

$$\phi_m = \Phi_m + \bar{B}_m = \bar{B}_m \big(1 + \Gamma(\Phi)\big) \tag{7.20}$$

by changing variables from $B^\dagger$ to Φ. But this is only half the job because we also want to find the Cuntz algebra in which Φ_m resides.

This is most easily done in the case $X_m = 0$ (no tadpoles), which we assume below. In this case, one has the additional relations

$$\Phi_m = B_n^\dagger \gamma_{nm} = \gamma_{mn} B_n^\dagger , \tag{7.21a}$$

$$\gamma_{mn}(B^\dagger) = \sum_w X_{mn\bar{w}} B^{\dagger w} , \quad \gamma_{mn} | 0 \rangle = (E^{-1})_{mn} | 0 \rangle , \tag{7.21b}$$

$$\bar{B}_m \Phi_n = \gamma_{mn} \tag{7.21c}$$

and γ_{mn} is invertible because it begins with X_{mn}. This gives us the Cuntz algebra of Φ_m:

$$\frac{\partial}{\partial \Phi_m} \equiv (\gamma^{-1})_{mn} \bar{B}_n \,, \tag{7.22a}$$

$$\frac{\partial}{\partial \Phi_m} \Phi_n = \delta_{m,n} \,, \quad \Phi_m \frac{\partial}{\partial \Phi_m} = 1 - |0\rangle\langle 0| \,, \tag{7.22b}$$

$$\frac{\partial}{\partial \Phi_m} |0\rangle = \langle 0| \Phi_m = 0 \tag{7.22c}$$

and we may now express the dual basis Cuntz operators as

$$B_m^\dagger = \Phi_n (\gamma^{-1})_{nm} = (\gamma^{-1})_{mn} \Phi_n \,, \tag{7.23a}$$

$$\bar{B}_m = \gamma_{mn} \frac{\partial}{\partial \Phi_n} \,. \tag{7.23b}$$

Moreover, the relation

$$B_m^\dagger = \frac{\partial}{\partial \Phi_m} \Gamma(\Phi) \tag{7.24}$$

now follows from (7.22a), (7.15b) and (7.23a).

Our next task is to find the Φ dependence of $\gamma_{mn}(B^\dagger)$. Note first that

$$\Gamma(\Phi) = \Phi_m \Phi_n (\gamma^{-1})_{nm} \tag{7.25}$$

follows from (7.18a) and (7.23a), and this gives us the desired result

$$(\gamma^{-1}(\Phi))_{mn} = \frac{\partial}{\partial \Phi_m} \frac{\partial}{\partial \Phi_n} \Gamma(\Phi) \,. \tag{7.26}$$

Using (7.23b) and (7.24) in (7.20), we have found the forms of the master field

$$\phi_m = \gamma_{mn}(\Phi) \frac{\partial}{\partial \Phi_n} \big(1 + \Gamma(\Phi)\big) = \Phi_m + \gamma_{mn}(\Phi) \frac{\partial}{\partial \Phi_n} \tag{7.27}$$

in the Φ, $\frac{\partial}{\partial \Phi}$ basis.

Comparing these two forms of the master field (or the two forms of $B^\dagger$ in (7.23a)), we also find the consistency relation

$$\frac{\partial}{\partial \Phi_m} \Gamma(\Phi) = \left(\frac{\partial}{\partial \Phi_m} \frac{\partial}{\partial \Phi_n} \Gamma(\Phi) \right) \Phi_n \tag{7.28}$$

but this is only the statement that Γ_w is cyclically symmetric.

8. Forms of the Schwinger–Dyson Equations

In this section, we use the forms of the master field to quickly derive a number of new free-algebraic forms of the large N Schwinger–Dyson equations for action theories.[f] The first form in Subsec. 8.1 is novel, and the rest, although packaged

[f] Another form of the Schwinger–Dyson equations follows as null state Ward identities of the infinite-dimensional free algebra (see App. E).

differently, are closely related to known free-algebraic formulations.[10,8,11,12] In all our formulations, the dynamical input is stored in the operators $G_m(\phi)$, $E_{mn}(\phi)$ of the interacting Cuntz algebra (2.9) and (2.10).

8.1. *The basic form as a computational system*

We consider first the basic form of the master field

$$\phi_m = \sum_w X_{mw} G_{\bar{w}}(\phi) \tag{8.1}$$

which, by matching ϕ dependence on left and right, is itself a computational system for the planar connected parts.

We illustrate this by studying the classical limit of the system. Reinstating $\hbar$ temporarily, we find that

$$G_m = O(\hbar^0)\,, \qquad E_{mn} = O(\hbar) \tag{8.2}$$

because E^{mn} in (2.5) is a commutator. The classical limit of (8.1)

$$\phi_m \simeq \sum_w X_{mw} G^{\bar{w}}\,, \quad G_w \simeq G^w = G_{m_1} \cdots G_{m_n} \tag{8.3}$$

is then obtained by neglecting all E terms in the G_w's (see Eq. (3.9a)).

For definiteness, we consider the solution of this equation for the general quartic interaction

$$G_m = 2\omega_m \phi_m + \lambda_{mnpq} \phi_n \phi_p \phi_q\,, \tag{8.4}$$

where λ_{mnpq} is cyclically symmetric in its indices. In this case, (8.3) contains only odd powers of ϕ and we may set the coefficients of each odd power to zero, obtaining the list of equations

$$\phi: \; \phi_m = X_{mn} 2\omega_n \phi_n\,, \tag{8.5a}$$

$$\phi^3: \quad 0 = X_{mn} \lambda_{npqr} \phi_p \phi_q \phi_r + X_{mnpq} 2\omega_q \phi_q 2\omega_p \phi_p 2\omega_n \phi_n\,. \tag{8.5b}$$

$$\vdots$$

The master field ϕ_m is a free variable (with no relations), so the unique solution of this list is easily obtained:

$$X_{mn} = \frac{1}{2\omega_m} \delta_{m,n}\,, \quad X_{mnpq} = -\frac{\lambda_{mqpn}}{2\omega_m 2\omega_n 2\omega_p 2\omega_q}\,, \tag{8.6a}$$

$$X_{mnpqrs} = \frac{1}{\prod 2\omega} \sum_t \frac{1}{2\omega_t} (\lambda_{msrt}\lambda_{tqpn} + \lambda_{nmst}\lambda_{trqp} + \lambda_{pnmt}\lambda_{tsrq})\,. \tag{8.6b}$$

$$\vdots$$

These results are recognized as the tree-graph contributions to the planar connected parts.

For the special case of free random variables, the basic form (8.1) decouples into d one-matrix problems with $\bar{X} = X$

$$\phi_m = \sum_{w(m)} X_{w(m)} G_{w(m)}(\phi_m) \tag{8.7}$$

according to Eqs. (2.22), (5.30) and (3.9a). The one-matrix bases $G_{w(m)}(\phi_m)$ have the decoupled form discussed in Subsec. 3.3.

Other relations of this type, e.g. Eq. (5.29), may also be considered as computational systems.

8.2. *The dual basis system*

The planar connected parts $\bar{X}(B^\dagger)$ satisfy

$$B^\dagger_m + E_{mn}(\phi)\bar{B}_n = G_m(\phi)\,, \tag{8.8a}$$

$$\phi_p = \bar{B}_p\big(1 + \bar{X}(B^\dagger)\big) \tag{8.8b}$$

which we record together as the *dual basis system*

$$B^\dagger_m + E_{mn}\big(\bar{B}(1+\bar{X})\big)\bar{B}_n = G_m\big(\bar{B}(1+\bar{X})\big)\,. \tag{8.9}$$

To derive this system, start with $G_m = B_m + B^\dagger_m$, go to the dual basis with (5.22) and use the dual basis form (8.8b) of the master field.

We have checked that the system (8.9), although packaged differently, is equivalent to the Schwinger–Dyson equations derived diagrammatically for the planar connected parts in Ref. 8. In particular, our Cuntz operators $\bar{B}_m$ act on $\bar{X}(B^\dagger)$ as the operator $\frac{\delta}{\delta J_m}$ of Ref. 8 acts on their $W(J)$, but the two operators are not the same because

$$[\bar{B}_m, c] = 0\,, \quad \frac{\delta}{\delta J_m} c = 0 \tag{8.10}$$

for any c-number c. The E term in (8.9) collects the results of this difference. In what follows, we make some additional remarks on the structure of the dual basis system.

We begin by discussing this system in the case of one matrix, where right multiplication by powers of $B^\dagger$ gives the simple equation[g]

$$E\left(\frac{\psi}{\beta}\right) - G\left(\frac{\psi}{\beta}\right)\beta + \beta^2 = 0\,, \quad \psi(\beta) = 1 + X(\beta)\,, \quad \psi(0) = 1 \tag{8.11}$$

for any interaction. (We have replaced $B^\dagger$ by a commuting source β.) In the special case of the quartic interaction (see (2.18)), this reads

$$\lambda\psi^2(\psi - 1) + \beta^2\big(m^2(\psi-1) - \lambda X_2 - \beta^2\big) = 0 \tag{8.12}$$

[g]This equation gives the large β form $X(\beta) \sim c^{-\frac{1}{p}}\beta^{1+\frac{2}{p}}$ when $G(\phi) \sim c\phi^p$ at large ϕ.

and, except that X_2 appears as an unknown, this is the cubic equation found in Ref. 10 for this interaction. In fact, the equation determines X_2 along with the rest of $X(\beta)$ in a perturbative or semiclassical expansion. To begin the perturbation theory, set $\lambda = 0$ to find $\psi(\beta) = 1 + \frac{\beta^2}{m^2}$. More general perturbation theory is discussed in App. F.

For the special case of free random variables, the dual basis system (8.9) decouples into d one-matrix systems

$$B_m^\dagger + E_m(\phi_m)\bar{B}_m = G_m(\phi_m)\,, \tag{8.13a}$$

$$\phi_m = \bar{B}_m + \sum_{w(m)} X_{mw(m)} B^{\dagger w(m)} \tag{8.13b}$$

which comprise d decoupled systems of the form (8.11).

The classical limit of the full system (8.9) is

$$B_m^\dagger \simeq G_m(\phi)\,, \quad \phi_p \simeq \bar{B}_p \bar{X}(B^\dagger) = \sum_w X_{p\bar{w}} B^{\dagger w} \tag{8.14}$$

because $(1 + \bar{X})$ in (8.8b) should be replaced by the dimensionless combination $(1 + \bar{X}/\hbar)$. As an example, the classical limit (8.14) reads

$$\begin{aligned} 0 &= (B_m^\dagger - 2\omega_m X_{mn} B_n^\dagger) \\ &\quad - (2\omega_m X_{mnpq} B_q^\dagger B_p^\dagger B_n^\dagger + \lambda_{mnpq} X_{nr} B_r^\dagger X_{ps} B_s^\dagger X_{qt} B_t^\dagger) + \cdots \end{aligned} \tag{8.15}$$

for the general quartic interaction (8.4). Setting each power of $B^\dagger$ to zero separately, we find the same tree graphs (8.6) for the planar connected parts.

An equivalent form of the dual basis system (8.9) is

$$a_m^\dagger + E_{mn}(M)a_n = G_m(M)\,, \quad M_p = a_p\big(1 + \bar{X}(a^\dagger)\big) \tag{8.16}$$

in terms of ordinary Cuntz operators and the Gopakumar–Gross form of the master field.

The other forms of the planar Schwinger–Dyson equations below are the forms taken by Eq. (8.9) in different bases.

8.3. *Equation for the planar correlators*

The generating function $\bar{Z}(j)$ of planar correlators satisfies

$$j_m \bar{Z}(j) - G_m\left(\bar{Z}^{-1}(j)\frac{\partial}{\partial j}\bar{Z}(j)\right) + E_{mn}\left(\bar{Z}^{-1}(j)\frac{\partial}{\partial j}\bar{Z}(j)\right)\bar{Z}^{-1}(j)\frac{\partial}{\partial j_n} = 0\,. \tag{8.17}$$

To derive this, use (8.8a), (7.14), (7.15a) and the form (7.16a) of the master field. This can be simplified to

$$\left(\bar{Z}(j)j_m - G_m\left(\frac{\partial}{\partial j}\right)\right)\bar{Z}(j) + E_{mn}\left(\frac{\partial}{\partial j}\right)\frac{\partial}{\partial j_n} = 0 \tag{8.18}$$

for any polynomial interaction.

For the one-matrix case ($\bar{Z} = Z$) the system (8.18) reduces to the quadratic equation

$$\big(jZ(j)\big)^2 - G\left(\frac{1}{j}\right) jZ(j) + E\left(\frac{1}{j}\right) = 0\,, \quad Z(0) = 1 \tag{8.19}$$

for any interaction. This equation may also be obtained from Eq. (8.11) and

$$\psi(B^\dagger) = Z(j)\,, \quad B^\dagger = jZ(j)\,, \quad \frac{\psi(B^\dagger)}{B^\dagger} = \frac{1}{j} \tag{8.20}$$

which is the one-dimensional form of $\psi = 1 + X$ and (7.14).

Again, the relations (8.18) or (8.19) are equivalent to those given in Ref. 8, although ours are packaged differently. In particular, our "derivative" with respect to j is a Cuntz operator satisfying

$$\frac{\partial}{\partial j_m} c = c \frac{\partial}{\partial j_m} \tag{8.21}$$

when c is a c-number, and *not* the rule $\frac{\delta c}{\delta j_m} = 0$ satisfied by the derivative in Ref. 8. The difference between these two operators is again collected in the E term of (8.18).

8.4. *Equation for the planar effective action*

The planar effective action $\Gamma(\Phi)$ satisfies

$$\frac{\partial}{\partial \Phi_m}\Gamma(\Phi) + E_{mn}\left(\Phi + \gamma\frac{\partial}{\partial\Phi}\right)\gamma_{np}(\Phi)\frac{\partial}{\partial\Phi_p} = G_m\left(\Phi + \gamma\frac{\partial}{\partial\Phi}\right). \tag{8.22}$$

To derive this system, use (7.23b), (7.24) and (7.27) in (8.8a). Although packaged differently, this system is equivalent to the equation for Γ given in Ref. 8. (Again, our Cuntz operator $\frac{\partial}{\partial\Phi}$ commutes with c-numbers and so is not equal to the operator $\frac{\delta}{\delta\Phi}$ of Ref. 8.)

For the classical limit of (8.22), we know to neglect E and the quantum part $\bar{B} = \gamma\frac{\partial}{\partial\Phi}$ of the master field. This gives immediately the classical limit of the planar effective action

$$\Gamma(\Phi) \simeq \Phi_m G_m(\Phi) \tag{8.23}$$

for any theory.

For the general one-matrix model, the system (8.22) simplifies to

$$\{\Gamma(\Phi) - \Phi G(\Phi[1 + \Gamma^{-1}(\Phi)])\}\Gamma(\Phi) + \Phi^2 E(\Phi[1 + \Gamma^{-1}(\Phi)]) = 0\,. \tag{8.24}$$

This equation can also be obtained directly from (8.11) by the transformation

$$\psi = 1 + X(B^\dagger) = 1 + \Gamma(\Phi)\,, \tag{8.25a}$$

$$B^\dagger = \beta = \frac{\Gamma(\Phi)}{\Phi} \tag{8.25b}$$

which is the one-dimensional form of (7.18). The relations (8.25) were pointed out in Ref. 10, and we have checked for the quartic case (2.18) that the resulting cubic equation is in agreement with that given there.

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Appendix A. Large N as Higher-Dimensional Classical Solution

The fifth-time formulation[2] of any Euclidean action theory allows us to compute the large N limit of the action theory as a classical solution of a higher-dimensional theory, in parallel with the AdS/CFT correspondence.[3–5] There is great latitude in the choice of the fifth-time theory, but any choice will give the same large N averages for the original theory. Moreover, other methods of higher-dimensional extension are known (see e.g. Ref. 13) and others still can be invented.

As an illustration, we consider the action theory

$$S = \mathrm{Tr}\left(\frac{1}{2}m^2\phi^2 + \frac{\lambda}{4N}\phi^4\right) \tag{A.1}$$

and we will choose the higher-dimensional extension (overdot is fifth-time derivative)

$$H_5 = \frac{1}{2}\,\mathrm{Tr}(B^\dagger B) = \frac{1}{2}\,\mathrm{Tr}(\pi^2) + V_5\,, \tag{A.2a}$$

$$\begin{aligned} V_5 = {} & \frac{1}{8}\,\mathrm{Tr}\left[\left(m^2\phi + \frac{\lambda}{N}\phi^3\right)^2\right] \\ & - \frac{1}{4}\left[m^2N^2 + 2\lambda\,\mathrm{Tr}(\phi^2) + \frac{\lambda}{N}(\mathrm{Tr}\,\phi)^2\right], \end{aligned} \tag{A.2b}$$

$$S_5 = \int dt\left(\mathrm{Tr}\left(\frac{1}{2}\dot\phi^2\right) - V_5\right) \tag{A.2c}$$

which is a special case of the simple H_5 in Eq. (2.6).

Now we may follow Ref. 14 to consider the phase-space master field, which solves the higher-dimensional classical equations of motion. Using Apps. C and E of Ref. 1 and in particular Eq. (2.17b) of the present paper, we find the reduced classical equations of motion

$$\dot\phi = \pi\,, \qquad \dot\pi = -V'\,, \tag{A.3a}$$

$$V = \frac{1}{8}(s')^2 - \frac{\lambda}{2}\phi(\phi + \langle\phi\rangle)\,, \quad s \equiv \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 \tag{A.3b}$$

and the ground state density

$$\rho(\phi) = \frac{1}{\pi}\sqrt{2\big(\epsilon - V(\phi)\big)}\,, \quad \int d\phi\,\rho(\phi) = 1 \tag{A.4}$$

from which the original action averages can be computed. (One may set $\langle\phi\rangle = 0$ by symmetry.)

We note that, relative to the discussion of Ref. 10, the higher-dimensional extension has done the relevant Hilbert inversion for us

$$\frac{1}{2}s'(\phi) = F(\phi) = \int dq \, \frac{\mathcal{P}}{\phi - q}\rho(q) \tag{A.5}$$

(F is given in (2.18b)) and moreover the extension has given us the ground state density ρ in the higher-dimensional form (A.4). Using (2.6), these features persist for the higher-dimensional solution of any one-matrix action theory.

Finally, Eqs. (3.17d), (A.3b) and (A.4) tell us that

$$E = F^2 + \pi^2\rho^2 = 2\epsilon + \lambda\phi^2 \tag{A.6}$$

and we obtain

$$\langle\phi^2\rangle = \frac{2\epsilon - m^2}{\lambda} \tag{A.7}$$

on comparison with the form of E in (2.18b).

Appendix B. Oscillators/Free Actions

A number of simplifications occur for oscillator Hamiltonians and/or free action theories, which we treat together here in the oscillator notation (for free action theories, $S = \frac{1}{2}\Sigma_n m_n^2 \operatorname{Tr}(\phi^n\phi^n)$, replace $2\omega_n$ by m_n^2)

$$G_m = 2\omega_m\phi_m\,, \quad E_{mn} = 2\omega_m\delta_{m,n}\,, \quad X_{mn} = \frac{1}{2\omega_m}\delta_{m,n}\,. \tag{B.1}$$

All other planar connected parts are zero.

Comparing the generating functions (3.12a) and (5.2a), we find that the basis polynomials G_w and the dual basis polynomials T_w are proportional

$$G_w(\phi) = (2\omega)^w T_w(\phi)\,. \tag{B.2}$$

It follows that

$$\langle G_{\bar{w}}G_{w'}\rangle = (2\omega)^w\delta_{w,w'}\,, \quad \langle T_{\bar{w}}T_{w'}\rangle = \left((2\omega)^{-1}\right)^w\delta_{w,w'}\,, \tag{B.3a}$$

$$G_{wm}G_{nw'} = G_{wmnw'} + 2\omega_m\delta_{m,n}G_wG_{w'}\,, \tag{B.3b}$$

$$T_{wm}T_{nw'} = T_{wmnw'} + X_{mn}T_wT_{w'}\,, \tag{B.3c}$$

where (B.3a) and (B.3b) follow from (5.1) and (3.13) respectively, while (B.3c) follows from (B.3b). The solution of the recursion relation (B.3b) is the finite operator product expansion

$$G_wG_{w'} = \sum_u \delta_{w,w_1u}\,\delta_{w',\bar{u}w_2}(2\omega)^u G_{w_1w_2} \tag{B.4}$$

which is a free-algebraic generalization of a familiar decomposition rule for the product of two Chebyshev polynomials (see also the general one-dimensional operator product expansion in Eq. (3.17e)). Using (B.2) in (B.4), one also obtains the explicit form (in this case) of the $T_w T_{w'}$ operator product expansion in (5.14).

In this case, the interacting Cuntz algebra becomes the Cuntz algebra

$$\{a_m, a_m^\dagger\} \equiv \frac{\{B_m, B_m^\dagger\}}{\sqrt{2\omega_m}} \tag{B.5}$$

and the infinite-dimensional free algebra (1.5) has corresponding simplifications due to the simple forms of G and E in (B.1). We mention in particular that

$$a_w {a_{w'}}^\dagger = \begin{cases} \delta_{w,w'} & \text{if } [w] = [w']\,, \\ a_u & \text{if } w = uw'\,, \\ {a_u}^\dagger & \text{if } w' = uw\,, \\ 0 & \text{otherwise} \end{cases} \tag{B.6}$$

is the simple form of the infinite-dimensional free-algebraic relation (1.5d) in this case.

Appendix C. Composite Structure of the Master Constraints

Define

$$Q_{mwn} \equiv \pi_m G_w \pi_n + i\pi_m {F_{n\bar{w}}}^\dagger - iF_{mw}\pi_n + F_{mwn} - F_{mw}F_n\,. \tag{C.1}$$

The master constraints (3.19a) are $Q_{mwn} = 0$, but one can show from (3.5) and (3.9) that

$$Q_{mwnp} = Q_{mwn}B_p^\dagger + B_{mw}Q_{np} \tag{C.2}$$

without using the constraints. (The cubic terms in π on the right simply cancel.)

Starting with the two-index Q's

$$Q_{mn} = B_m B_n^\dagger - E_{mn} \tag{C.3}$$

we may iterate (C.2) to obtain the higher-indexed Q's, for example

$$Q_{mnp} = Q_{mn}B_p^\dagger + B_m Q_{np}\,, \tag{C.4a}$$

$$Q_{mnpq} = (Q_{mn}B_p^\dagger + B_m Q_{np})B_q^\dagger + B_{mn}Q_{pq} \tag{C.4b}$$

and one finds more generally that all the Q's are linear in Q_{mn}. It follows that all the Q's are zero when the first one is set to zero:

$$B_m B_n^\dagger = E_{mn} \to Q_{mwn} = 0 \tag{C.5}$$

and so the set of master constraints (3.19a) contain no new constraints beyond the first.

Appendix D. Identification of $X(\beta)$

Here we will derive, by simple algebra, the functional relation between the generating function

$$X = X(\beta) = \sum_w \beta^w X_w \,, \qquad X_0 = 0 \tag{D.1}$$

and the generating function

$$Z(j) = \sum_w j^w \langle \phi^w \rangle \,, \qquad Z(0) = 1 \tag{D.2}$$

of the ordinary planar parts.

Start by rewriting the generator for the polynomials T_w as follows:

$$\sum_w \beta^w T_w = \frac{1}{1 - \beta_m \phi_m + X(\beta)} \tag{D.3a}$$

$$= (1 + X)^{-1} \frac{1}{1 - \phi_m \beta_m (1 + X)^{-1}}$$

$$= (1 + X)^{-1} \sum_w j^w \phi^w \,, \tag{D.3b}$$

where we have made the identification

$$j_m = \beta_m \big(1 + X(\beta)\big)^{-1} \tag{D.4}$$

between the two sets of free-algebraic sources. Now multiply (D.3) on the left by $\phi_m \beta_m$, take the vev and use the definition $X_{mw} = \langle \phi_m T_w \rangle$ to get

$$\sum_{m,w} \beta^{mw} X_{mw} = \sum_{m,w} j^{mw} \langle \phi^{mw} \rangle \tag{D.5}$$

which is just

$$X(\beta) = Z(j) - 1 \,. \tag{D.6}$$

Combining (D.4) with (D.6) we have

$$Z(j) = 1 + X\big(jZ(j)\big) \tag{D.7}$$

or alternatively

$$X(\beta) = Z\Big(\beta\big(1 + X(\beta)\big)^{-1}\Big) - 1 \,. \tag{D.8}$$

Following Refs. 10 and 8, the relation (D.7) identifies $X(\beta)$ as a generating function of connected planar parts.

Similarly, the relation[8]

$$Z(j) = 1 + X\big(Z(j)j\big) \tag{D.9}$$

is obtained by expanding (D.3) with $(1 + X)^{-1}$ on the right and using (5.28).

Finally, we can establish the similar relation

$$\bar{Z}(j) = 1 + \bar{X}\big(j\bar{Z}(j)\big)\,, \qquad \bar{Z}(j) = \sum_w j^{\bar{w}}\langle\phi^w\rangle \tag{D.10}$$

for the alternate generating functions $\bar{Z}$ and $\bar{X}$. To derive this result, start with the relations

$$\big(1 - \beta_m\tilde{\phi}_m + \bar{X}(\beta)\big)^{-1} = \sum_w \beta^w \widetilde{T_{\bar{w}}(\phi)}\,, \quad \widetilde{T_w(\phi)}|0\rangle = T_w(\phi)|0\rangle\,. \tag{D.11}$$

These can be derived from Ref. 1 and (5.2a), (5.7) and (5.8), and then proceed as earlier in this appendix.

Appendix E. Schwinger–Dyson as Null State Ward Identities

There are many free-algebraic forms of the Schwinger–Dyson equations, some of which are discussed in Sec. 8. In this Appendix, we discuss a form of the Schwinger–Dyson equations which follows from the Ward identities of the infinite-dimensional free algebra.

This development is based on the null states

$$\big(B^{\dagger w} - G_w(\phi)\big)|0\rangle = 0 \tag{E.1}$$

which give the null state Ward identities

$$\big\langle\phi^{\bar{w}'}\big(B^{\dagger w} - G_w(\phi)\big)\big\rangle = 0\,. \tag{E.2}$$

To put these identities in a useful form, we leave the coupling constant-dependent $G_w(\phi)$ terms as they are and evaluate the $B^{\dagger\, w}$ terms as follows:

$$\langle\phi^{\bar{w}'}G_w(\phi)\rangle = \langle\tilde{\phi}^{w'}B^{\dagger w}\rangle\,, \tag{E.3a}$$

$$= \begin{cases} \displaystyle\sum_{w\subset w'}\ \prod_{\{u_i\}=w'/w}\langle\phi^{u_i}\rangle\,, \\ 0 \text{ when } w \text{ is not embedded in } w'\,. \end{cases} \tag{E.3b}$$

The last form is obtained by writing $B^{\dagger\, w}$ as a product of $B^\dagger_m$'s and moving each to the left using

$$[\tilde{\phi}_m, B^\dagger_n] = \delta_{m,n}|0\rangle\langle 0|\,, \qquad \langle 0|B^\dagger_m = 0\,. \tag{E.4}$$

This procedure shows that the average (E.3) vanishes unless the word w is embedded in the word w', which we write as $w \subset w'$. In further detail, w is embedded in w' if the two words can be written as

$$w = m_1 m_2 \cdots m_n\,, \tag{E.5a}$$

$$w \subset w':\ \ w' = u_1 m_1 u_2 m_2 \cdots u_n m_n u_{n+1} \tag{E.5b}$$

which defines the "quotient set" $\{u_i\} = w'/w$ of words u_i uniquely for each embedding.

As examples of (E.3) we list

$$\langle G_w \rangle = \delta_{w,0}\,, \tag{E.6a}$$

$$\begin{aligned} \langle \phi_m G_n \rangle &= \delta_{m,n}\,, \\ \langle \phi_m \phi_n \phi_p G_q \rangle &= \delta_{m,q} \langle \phi_n \phi_p \rangle \\ &\quad + \delta_{n,q} \langle \phi_m \rangle \langle \phi_p \rangle \\ &\quad + \delta_{p,q} \langle \phi_m \phi_n \rangle\,, \\ \langle \phi_m G_{np} \rangle &= 0\,, \end{aligned} \tag{E.6b}$$

where (E.6a) was noted in (4.3b).

Appendix F. Perturbation Theory

We work with the dual basis system (8.8) and assume that some zeroth-order system has already been solved

$$B_m^\dagger + E_{mn}^{(0)}\big(\phi^{(0)}\big)\bar{B}_n - G_m^{(0)}\big(\phi^{(0)}\big) = 0\,, \quad \phi_p^{(0)} = \bar{B}_p\big(1 + \bar{X}^{(0)}(B^\dagger)\big)\,. \tag{F.1}$$

The general perturbation problem is stated as follows. Given

$$G_m(\phi) = G_m^{(0)}(\phi) + \lambda G'_m(\phi)\,, \quad E_{mn}(\phi) = E_{mn}^{(0)}(\phi) + \lambda E'_{mn}(\phi) \tag{F.2}$$

we want to solve for the corrections to the connected parts X_w

$$\bar{X}(B^\dagger) - \bar{X}^{(0)}(B^\dagger) = \sum_{k=1}^{\infty} \lambda^k \bar{X}^{(k)}(B^\dagger) = \sum_{k=1}^{\infty} \lambda^k \sum_w X_{\bar{w}}^{(k)} B^{\dagger w} \tag{F.3}$$

order by order in λ.

We have

$$\phi = \phi^{(0)} + \phi' = \phi^{(0)} + \sum_{k=1} \lambda^k \bar{B}\bar{X}^{(k)}(B^\dagger) \tag{F.4}$$

and we subtract (F.1) from (8.8a) to get the general perturbation equation

$$\begin{aligned} &\big[E_{mn}^{(0)}(\phi) - E_{mn}^{(0)}\big(\phi^{(0)}\big)\big]\bar{B}_n - \big[G_m^{(0)}(\phi) - G_m^{(0)}\big(\phi^{(0)}\big)\big] \\ &\quad = \lambda[G'_m(\phi) - E'_{mn}(\phi)\bar{B}_n]\,. \end{aligned} \tag{F.5}$$

If we have oscillators for the zeroth-order problem, this general equation simplifies somewhat. But we can work from any zeroth-order problem and get the desired results by straightforward algebraic computation with (F.5), remembering that the B operators serve as "dummy" variables, obeying $\bar{B}_m B_n^\dagger = \delta_{m,n}$.

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