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THE ALGEBRAS OF LARGE N MATRIX MECHANICS

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Extending early work, we formulate the large N matrix mechanics of general bosonic, fermionic and supersymmetric matrix models, including Matrix theory: the Hamiltonian framework of large N matrix mechanics provides a natural setting in which to study the algebras of the large N limit, including (reduced) Lie algebras, (reduced) supersymmetry algebras and free algebras. We find in particular a broad array of new free algebras which we call symmetric Cuntz algebras, interacting symmetric Cuntz algebras, symmetric Bose/Fermi/Cuntz algebras and symmetric Cuntz superalgebras, and we discuss the role of these algebras in solving the large N theory. Most important, the interacting Cuntz algebras are associated to a set of new (hidden!) local quantities which are generically conserved only at large N. A number of other new large N phenomena are also observed, including the intrinsic nonlocality of the (reduced) trace class operators of the theory and a closely related large N field identification phenomenon which is associated to another set (this time nonlocal) of new conserved quantities at large N.

1. Introduction

Studies of the large N limit of matrix models have included many intertwined directions, among which we mention:

- Planar diagram summation[1]
- Integration[2–5]
- Schwinger–Dyson methods[6–11]
- Euclidean master fields[7,12]
- Solution of the Schrodinger equation[2,13]
- Phase space master fields[14–20]

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- Large N matrix mechanics[17–19]
- Reduced integral and stochastic formulations[21–25]
- Stochastic master fields[23,26]
- Microcanonical master fields[27]
- Free algebras[7,16–19,8,28,12,11,26,20,29,30]

Other references and approaches can be found in the partial reviews of Refs. 31–34. The independent observations in physics[7,16–19,8] of free or Cuntz algebras in the large N limit are also intertwined chronologically with the development of these algebras in mathematics.[35–39]

In this paper, we focus on large N matrix mechanics,[17–19] which was originally introduced to systematize closely related ongoing work on phase space master fields.[14–16] The approach through matrix mechanics was interrupted, however, in the early 1980's after the solution of the one Hermitian matrix model[17,18] and the (one polygon) unitary matrix model.[19]

Extending this early work, we study here the large N matrix mechanics of general bosonic, fermionic and supersymmetric matrix models, including gauged matrix models such as the $n = 16$ supersymmetric gauge quantum mechanics,[40] now called Matrix theory.[41] Because it is a Hamiltonian approach, large N matrix mechanics is an ideal laboratory for studying the algebras of the large N limit, including (reduced) Lie algebras, (reduced) supersymmetry algebras and free algebras. In particular, we will find a broad array of new free algebras in the large N limit, and we will discuss the role of these algebras in solving the various theories. To aid the reader in making the transition from the early work, we give here a brief review of the approach.

Large N matrix mechanics, which follows Heisenberg's original development,[42] is based on the large N completeness relation[17]

$$\mathbf{1.} \underset{N}{=} |0.\rangle\langle .0| + \sum_{rs,A} |rs, A\rangle\langle rs, A| \,, \qquad r,\ s = 1 \cdots N \tag{1.1}$$

for the states which saturate the traced Wightman functions of the theory. Here $|0.\rangle$ is the ground state of the theory (which dominates the invariant channels by large N factorization) and $\{|rs, A\rangle\}$ is a set of dominant adjoint eigenstates of the Hamiltonian. The large N dynamics is then formulated in terms of reduced states and operators, using Bardakci's reduced matrix elements.[16] For example, the reduced completeness relation reads

$$\mathbf{1} = |0\rangle\langle 0| + \sum_{A} |A\rangle\langle A| \,, \tag{1.2}$$

where $|0\rangle$ is the reduced ground state and the states $\{|A\rangle\}$ are the corresponding reduced adjoint eigenstates.

The master fields[43] of the theory are the set of reduced matrix elements of the reduced fields, and have the translation-covariant form[17]

$$M_{\mu\nu}(x) = \exp(ip_{\mu\nu} \cdot x) M_{\mu\nu}(0)\,, \tag{1.3a}$$

$$p_{\mu\nu} = p_\mu - p_\nu\,, \quad \mu = (0, A)\,, \quad \nu = (0, B)\,, \tag{1.3b}$$

where $\{p_{\mu\nu}\}$ is the set of energy–momentum differences of the various reduced states in (1.2). The master fields also satisfy the large N classical equations of motion and a set of equal-time "constraints,"[15–19] which follow directly by taking matrix elements of the reduced equations of motion and the reduced equal-time algebra of the theory.[16–19] In the one-matrix model, for example, the reduced equal-time algebra takes the "semiclassical" form[7,16–18]

$$[\phi, \pi] = i|0\rangle\langle 0| \tag{1.4}$$

for any potential. More generally, this gives the large N correspondences:

- master fields $\leftrightarrow$ reduced fields;
- large N classical equations of motion $\leftrightarrow$ reduced equations of motion;
- equal-time constraints $\leftrightarrow$ reduced equal-time algebra.

In this paper, we shall prefer the equivalent terminology on the right side of this list.

Here is an overview of our main conclusions:

(1) *Unified formulation.* Large N matrix mechanics provides a unified large N Hamiltonian formulation of bosonic (see Secs. 2–4), fermionic (see Secs. 2, 5 and 6) and supersymmetric matrix models (see Secs. 5 and 6), as well as gauged matrix models (see Secs. 2 and 6): the special case of Matrix theory[40,41] is discussed explicitly in Sec. 6. The algebraic structures discussed below can be straightforwardly generalized to higher-dimensional large N Hamiltonian quantum field theory,[16,17] and we expect that the same algebraic structures can also be found in large N Euclidean quantum field theory.[7,12]

(2) *Generalized free algebras.* The reduced large N theories come equipped with their own reduced equal-time algebras (see Subsec. 2.5), which generalize Eq. (1.4). These equal-time algebras are new free algebras in their own right, and, with the help of the reduced equations of motion, one sees that the equal-time algebras are closely related to the Cuntz algebra[a]

$$a_m a_n^\dagger = \delta_{mn}\,, \quad a_m|0\rangle = 0\,, \quad a_m^\dagger a_m = 1 - |0\rangle\langle 0| \tag{1.5}$$

and generalizations thereof. The Cuntz algebra (1.5) arises in the special case of large N bosonic oscillators (see Subsec. 3.1), while other cases show a broad array of generalizations of the Cuntz algebra, which we call:

[a]More precisely, the algebra (1.5), which appeared independently in Refs. 37 and 7, is called the extended Cuntz algebra in mathematics. A Kronecker-delta realization of the one-dimensional algebra was also seen independently in large N matrix mechanics.[17]

- symmetric Cuntz algebras [Subsec. 3.1; see Eq. (3.10)];
- interacting symmetric Cuntz algebras [Sec. 4; see Eq. (4.36)];
- symmetric Bose/Fermi/Cuntz algebras [Subsec. 5.1; see Eq. (5.4)];
- symmetric Cuntz superalgebras [Subsec. 5.2; see Eqs. (5.5) and (5.6)].

Symmetric Cuntz algebras contain two Cuntz subalgebras ((1.5) and a tilde version of (1.5)) which act respectively at the beginning or the end of large N words [see, for example, Eq. (2.40)]. We remark in particular on the interacting Cuntz algebras,

$$A_m A_n^\dagger = C_{mn}(\phi)\,, \quad A_m|0\rangle = 0\,, \quad A_m^\dagger\big(C^{-1}(\phi)\big)_{mn} A_n = 1 - |0\rangle\langle 0|\,, \tag{1.6}$$

where the reduced operator $C_{mn}(\phi)$, which is a function only of the reduced coordinates ϕ, is determined by the potential.

The interacting Cuntz algebras are a central result of this paper, in part because they imply a number of new local conserved quantities at large N (see Subsec. 4.5), including

$$\mathcal{J} = A_m^\dagger\big(C^{-1}(\phi)\big)_{mn} A_n\,, \qquad \frac{d}{dt}\mathcal{J} = \mathcal{J}|0\rangle = 0 \tag{1.7}$$

which follows directly from (1.6). In the original unreduced theory, these quantities correspond to new (hidden) local but nonpolynomial operators which are generically conserved only at large N.

(3) *Conserved nonlocal reduced operators.* The local conserved trace class operators of the theory, such as the Hamiltonian, the angular momenta and the supercharges, are represented at large N by reduced conserved operators called the reduced Hamiltonian, the reduced angular momenta and the reduced supercharges. These reduced operators satisfy reduced algebras (see Subsecs. 2.3, 2.6, 3.3, 5.3, 5.4 and Sec. 6) which are closely related to the unreduced algebras of the theory. As an example, the reduced Hamiltonian still controls, in the normal fashion, the time evolution of all reduced operators (see Subsec. 2.6), although the form of the reduced supersymmetry algebras can be surprisingly different from their unreduced form in the case of a gauge theory such as Matrix theory [see Eq. (6.14a)].

The explicit composite forms of the reduced trace class operators can in principle be determined by solving their reduced algebraic relations, and this is one of the central problems of large N matrix mechanics. The generalized free algebras above are seen to play an important role in the construction of these reduced operators.

What is most interesting here is that the reduced trace class operators are intrinsically nonlocal (see Subsecs. 3.2, 3.3, 4.6, 5.3 and 5.5). Early examples of this general phenomenon were seen in Refs. 18 and 19.

(4) *Large N field identification.* Because each reduced trace class operator T (corresponding to a conserved local trace class operator T.) is nonlocal, we find that

there exists, universally for each $T_{\bullet}$, another nonlocal operator D_{rs} in the unreduced theory which also corresponds at large N to the same nonlocal reduced operator

$$\left.\begin{array}{r} T_{\bullet}\ \text{(local)} \\ D_{rs}\ \text{(nonlocal)} \end{array}\right\rangle \underset{N}{\to} T\ \text{(nonlocal)}\,, \tag{1.8a}$$

$$\frac{d}{dt}T_{\bullet} = \frac{d}{dt}T = 0\,, \qquad \frac{d}{dt}D_{rs} \underset{N}{=} 0 \tag{1.8b}$$

(see Subsecs. 3.5, 4.6, 5.3 and 5.5). The new large N-conserved nonlocal operators D_{rs} are closely related to the densities of the original local trace class operators $T_{\bullet}$, and provide us in principle with another class of unreduced operators which are generically conserved only at large N.

(5) *Large N fermions and bosons.* Large N fermions and bosons are surprisingly similar, exhibiting some aspects of a Bose–Fermi equivalence (see Subsecs. 5.2, 5.3 and 5.5). This equivalence, which is explicit in the Cuntz superalgebras above, is another example of the classical nature of the large N limit. In particular, large N fermions and bosons both satisfy the same classical or Boltzmann statistics, and the Pauli principle is lost for large N fermions. The equivalence also makes possible certain large N bosonic constructions of supersymmetry (see Subsec. 5.4).

The interacting Cuntz algebras have not yet been extended to matrix models with fermions, although we believe that they can be. Further study in this direction is particularly important for Matrix theory, where the associated new large N-conserved quantities (local and nonlocal) may be related to the question of hidden 11-dimensional symmetry.[41]

2. The Setup

2.1. SU(N)-*invariant Hamiltonian systems*

In this section, we establish our notation for a large class of SU(N)-invariant matrix Hamiltonian systems, where the symmetry can be global or local. In the course of this discussion, we will often loosely refer to the group SU(N) as the gauge group of the theory, whether the symmetry is gauged or not.

We begin with a canonical set of B Hermitian adjoint bosons and f Hermitian adjoint fermions

$$[\phi_a^m, \pi_b^n] = i\delta_{ab}\delta^{mn}\,, \qquad [\Lambda_{\alpha a}, \Lambda_{\beta b}]_+ = \delta_{ab}\delta_{\alpha\beta}\,, \tag{2.1a}$$

$$\phi_a^{m\dagger} = \phi_a^m\,, \qquad \pi_a^{m\dagger} = \pi_a^m\,, \qquad \Lambda_{\alpha a}^\dagger = \Lambda_{\alpha a}\,, \tag{2.1b}$$

$$a = 1 \cdots N^2\,, \qquad m = 1 \cdots B\,, \qquad \alpha = 1 \cdots f\,. \tag{2.1c}$$

The generators of SU(N), sometimes called the gauge generators, are

$$G_a = G_a^\dagger = f_{abc}\left(\phi_b^m \pi_c^m - \frac{i}{2}\Lambda_{\alpha b}\Lambda_{\alpha c}\right). \tag{2.2}$$

The dynamics of the system is described by an invariant Hamiltonian $H.$,[b]

$$\dot{\rho} = i[H., \rho], \qquad \rho = \phi,\ \pi \text{ or } \Lambda, \tag{2.3a}$$

$$[G_a, H.] = 0 \tag{2.3b}$$

which is constructed from the canonical operators.

To go over to a matrix notation, we also introduce a set of $N \times N$ matrices in the fundamental representation of the gauge group[c]

$$[T_a, T_b] = i f_{abc} T_c, \qquad T_a^\dagger = T_a, \qquad \text{Tr}\ T_a T_b = \delta_{ab}, \tag{2.4a}$$

$$\text{Tr}\ T_a = \sqrt{N}\delta_{a,N^2}, \quad (T_a)_{rs}(T_a)_{uv} = \delta_{su}\delta_{rv}, \qquad r,\ s = 1\cdots N \tag{2.4b}$$

and define the adjoint matrix fields as

$$\phi^m = \phi_a^m T_a, \qquad \pi^m = \pi_a^m T_a, \qquad \Lambda_\alpha = \Lambda_{\alpha a} T_a, \tag{2.5a}$$

$$(\phi_{rs}^m)^\dagger = \phi_{sr}^m, \qquad (\pi_{rs}^m)^\dagger = \pi_{sr}^m, \qquad \left((\Lambda_\alpha)_{rs}\right)^\dagger = (\Lambda_\alpha)_{sr}, \tag{2.5b}$$

$$[\phi_{rs}^m, \pi_{uv}^n] = i\delta^{mn}\delta_{su}\delta_{rv}, \qquad [(\Lambda_\alpha)_{rs}, (\Lambda_\beta)_{uv}]_+ = \delta_{\alpha\beta}\delta_{su}\delta_{rv}. \tag{2.5c}$$

The corresponding form of the gauge generators is

$$G_{rs} = G_a(T_a)_{rs} = (-i[\phi^m, \pi^m] - \Lambda_\alpha\Lambda_\alpha + (F - B)N)_{rs}, \tag{2.6a}$$

$$G_{rs}^\dagger = G_{sr}, \qquad \text{Tr}\ G = 0, \qquad F = \frac{f}{2}, \tag{2.6b}$$

$$[G_{rs}, H.] = 0. \tag{2.6c}$$

See Sec. 6 for the corresponding forms when the matrix fields are traceless.

We consider next the traced Wightman functions

$$\langle .0|\text{Tr}\big(\rho_1(t_1)\cdots\rho_n(t_n)\big)|0.\rangle, \qquad \rho = \phi, \quad \pi \text{ or } \Lambda, \tag{2.7}$$

where $|0.\rangle$ is the vacuum or ground state of the theory (see below for the case of degenerate ground states). For gauged matrix models, these are the invariant Wightman functions in the temporal gauge, where the missing factors $T\exp\big(i\int dt A_0(t)\big)$ are unity.

The channels of the traced Wightman functions are defined as

$$\langle .0|\big(\rho_1(t_1)\cdots\rho_i(t_i)\big)_{rs}\big(\rho_{i+1}(t_{i+1})\cdots\rho_n(t_n)\big)_{sr}|0.\rangle \tag{2.8}$$

[b]This $H.$ is the ordinary Hamiltonian of the system in the Heisenberg picture; the purpose of the dot subscript is to distinguish certain objects in the original, unreduced theory from their reduced counterparts.
[c]The normalization in (2.4) corresponds to $\alpha^2 = 2$ for any root α of SU(N).

and the subset of Hamiltonian eigenstates which saturate the channels span a Hilbert space. At finite N, these states are the set of all invariant and adjoint eigenstates. The Hilbert space simplifies, however, at large N because large N factorization tells us that the ground state $|0.\rangle$ dominates among the invariant states. Moreover, a certain dynamically-determined subset of the adjoint states $|rs, A\rangle$ may dominate at large N. This situation is summarized by the completeness relation[17]

$$\mathbf{1.} \underset{N}{=} |0.\rangle\langle .0| + \sum_{rs,A} |rs, A\rangle\langle rs, A| \tag{2.9}$$

for the large N traced Wightman functions. In further detail, we may specify the properties of these time-independent states as

$$H.|0.\rangle = E_0|0.\rangle\,, \quad H.|rs, A\rangle = E_A|rs, A\rangle\,, \tag{2.10a}$$

$$E_0 = O(N^2)\,, \quad E_\mu - E_\nu = O(N^0)\,, \quad \mu = (0, A)\,, \quad \nu = (0, B)\,, \tag{2.10b}$$

$$G_{rs}|0.\rangle = 0\,, \quad G_{rs}|pq, A\rangle = \delta_{rq}|ps, A\rangle - \delta_{sp}|rq, A\rangle\,, \tag{2.10c}$$

$$|rr, A\rangle = 0\,, \quad \langle rs, A|0.\rangle = 0\,, \quad \langle rs, A|pq, B\rangle = P_{sr,pq}\delta_{AB}\,, \tag{2.10d}$$

$$P_{sr,pq} = \delta_{rp}\delta_{sq} - \frac{1}{N}\delta_{sr}\delta_{pq}\,, \quad P_{rs,pq}P_{qp,tu} = P_{rs,tu}\,, \tag{2.10e}$$

where P in (2.10e) is a projector. In the large N matrix mechanics of gauge theories, the ground state $|0.\rangle$ is the only state which satisfies the Gauss law in (2.10c), although the adjoint eigenstates $|rs, A\rangle$ are needed[16,19] as well to saturate the channels of the invariant Wightman functions.

For bosonic systems, one expects that the ground state $|0.\rangle$ is unique, but for systems with fermions the completeness relation (2.9) should generally include the sum over a set of possibly degenerate ground states $\{|0.\rangle_i\}$,

$$\mathbf{1.} \underset{N}{=} |0.\rangle_i\, {}_i\langle .0| + \sum_{rs,A} |rs, A\rangle\langle rs, A|\,, \tag{2.11a}$$

$$G_{rs}|0.\rangle_i = 0\,, \qquad H.|0.\rangle_i = E_0|0.\rangle_i\,, \quad {}_i\langle .0|0.\rangle_j = \delta_{ij} \tag{2.11b}$$

to be dynamically determined as well. For simplicity we will continue to treat the ground state as unique, and the explicit examples of this paper are limited to cases where the ground state is unique or is believed to be unique, as in Matrix theory. Our discussion below goes through as well, however, for degenerate vacua, and the reader can obtain the corresponding results by appending a subscript i to each vacuum state with the summation convention of (2.11a).

2.2. *Reduced formulation*

The large N theory can be reformulated in terms of reduced operators which act in a reduced Hilbert space.[16–19] The reduced ground state $|0\rangle$ and the dominant reduced adjoint eigenstates $|A\rangle$ are in correspondence with the true ground state and dominant adjoint eigenstates $|0.\rangle$ and $|rs, A\rangle$. These time-independent states satisfy the reduced completeness relation

$$\mathbf{1} = \sum_{\mu=(0,A)} |\mu\rangle\langle\mu| = |0\rangle\langle 0| + \sum_A |A\rangle\langle A| \tag{2.12}$$

in the reduced Hilbert space. The reduced quantities are related to the original, unreduced quantities of the theory via reduced matrix elements which we illustrate first for the bosonic fields ϕ:

$$\langle .0|\frac{\phi^m_{rs}(t)}{\sqrt{N}}|0.\rangle = \delta_{rs}\langle 0|\phi_m(t)|0\rangle\,, \tag{2.13a}$$

$$\langle .0|\frac{\phi^m_{rs}}{\sqrt{N}}|pq, A\rangle = f(N)P_{rs,pq}\langle 0|\phi_m|A\rangle\,, \tag{2.13b}$$

$$\langle pq, A|\frac{\phi^m_{rs}}{\sqrt{N}}|0.\rangle = f(N)P_{qp,rs}\langle A|\phi_m|0\rangle\,, \tag{2.13c}$$

$$\langle pq, A|\frac{\phi^m_{rs}}{\sqrt{N}}|st, B\rangle = \langle sq, A|\frac{\phi^m_{sp}}{\sqrt{N}}|rt, B\rangle = P_{qp,rt}\langle A|\phi_m|B\rangle\,, \tag{2.13d}$$

$$\langle pq, A|\frac{\phi^m_{rs}}{\sqrt{N}}|tr, B\rangle = \langle pr, A|\frac{\phi^m_{qr}}{\sqrt{N}}|ts, B\rangle = P_{qp,ts}\langle A|\tilde{\phi}_m|B\rangle\,, \tag{2.13e}$$

$$\tilde{\phi}_m|0\rangle \equiv \phi_m|0\rangle\,, \qquad \langle 0|\tilde{\phi}_m \equiv \langle 0|\phi_m\,, \tag{2.13f}$$

$$f(N) = \left(N - \frac{1}{N}\right)^{-\frac{1}{2}}\,, \tag{2.13g}$$

where P is the projector defined in (2.10e). All the operators above are evaluated at time t, although we have written this explicitly only in (2.13a). The same definitions apply for π (take time derivatives of all definitions in (2.13)) and also for Λ, which defines a map from the original operators to the reduced operators

$$\frac{\phi^m_{rs}}{\sqrt{N}}\,, \quad \frac{\pi^m_{rs}}{\sqrt{N}}\,, \quad \frac{(\Lambda_\alpha)_{rs}}{\sqrt{N}} \to \begin{cases} \phi_m, & \pi_m, \quad \Lambda_\alpha\,, \\ \tilde{\phi}_m, & \tilde{\pi}_m, \quad \tilde{\Lambda}_\alpha\,. \end{cases} \tag{2.14}$$

It follows for example that

$$\rho^\dagger = \rho\,, \qquad \tilde{\rho}^\dagger = \tilde{\rho}\,, \qquad \rho = \phi,\ \pi,\ \text{or}\ \Lambda\,, \tag{2.15a}$$

$$\tilde{\rho}|0\rangle = \rho|0\rangle\,, \qquad \langle 0|\tilde{\rho} = \langle 0|\rho\,, \qquad \dot{\tilde{\rho}}|0\rangle = \dot{\rho}|0\rangle\,. \tag{2.15b}$$

The reduced matrix elements (2.13a)–(2.13d) were studied in Refs. 16–18, but the reduced matrix elements which define the reduced tilde operators $\tilde{\rho}$ in (2.13e), (2.13f) are new and the new tilde operators will play a central role in this paper.

According to (2.13d), (2.13e), the existence of two distinct reduced operators ρ and $\tilde{\rho}$ for each unreduced ρ_{rs} corresponds to the presence of a symmetric and an antisymmetric adjoint representation

$$(\text{adjoint}) \otimes (\text{adjoint}) = (\text{singlet}) \oplus (\text{adjoint}) \oplus (\text{adjoint})' \oplus \cdots \tag{2.16}$$

in the product of two adjoint representations of SU(N). A related interpretation of the tilde operators is noted in App. A.

We consider next the evaluation of matrix elements of matrix products at equal time. Using the matrix elements of the canonical operators and the completeness relations (2.9), (2.12) one finds that

$$\langle .0|\left(\frac{\phi^{m_1}(t)}{\sqrt{N}}\cdots\frac{\phi^{m_n}(t)}{\sqrt{N}}\right)_{rs}|0.\rangle = \delta_{rs}\langle 0|\phi_{m_1}(t)\cdots\phi_{m_n}(t)|0\rangle \tag{2.17a}$$

$$= \delta_{rs}\langle 0|\tilde{\phi}_{m_n}(t)\cdots\tilde{\phi}_{m_1}(t)|0\rangle\,, \tag{2.17b}$$

$$\langle .0|\left(\frac{\phi^{m_1}}{\sqrt{N}}\cdots\frac{\phi^{m_n}}{\sqrt{N}}\right)_{rs}|pq,A\rangle = f(N)P_{rs,pq}\langle 0|\phi_{m_1}\cdots\phi_{m_n}|A\rangle \tag{2.18a}$$

$$= f(N)P_{rs,pq}\langle 0|\tilde{\phi}_{m_n}\cdots\tilde{\phi}_{m_1}|A\rangle\,, \tag{2.18b}$$

$$\langle pq,A|\left(\frac{\phi^{m_1}}{\sqrt{N}}\cdots\frac{\phi^{m_n}}{\sqrt{N}}\right)_{rs}|0.\rangle = f(N)P_{qp,rs}\langle A|\phi_{m_1}\cdots\phi_{m_n}|0\rangle \tag{2.19a}$$

$$= f(N)P_{qp,rs}\langle A|\tilde{\phi}_{m_n}\cdots\tilde{\phi}_{m_1}|0\rangle\,, \tag{2.19b}$$

$$\langle pq,A|\left(\frac{\rho_1}{\sqrt{N}}\frac{\rho_2}{\sqrt{N}}\frac{\rho_3}{\sqrt{N}}\right)_{rs}|st,B\rangle = \langle sq,A|\left(\frac{\rho_1}{\sqrt{N}}\frac{\rho_2}{\sqrt{N}}\frac{\rho_3}{\sqrt{N}}\right)_{sp}|rt,B\rangle$$

$$= P_{qp,rt}\langle A|\rho_1\rho_2\rho_3|B\rangle\,, \tag{2.20a}$$

$$\langle pq,A|\left(\frac{\rho_1}{\sqrt{N}}\frac{\rho_2}{\sqrt{N}}\frac{\rho_3}{\sqrt{N}}\right)_{rs}|tr,B\rangle = \langle pr,A|\left(\frac{\rho_1}{\sqrt{N}}\frac{\rho_2}{\sqrt{N}}\frac{\rho_3}{\sqrt{N}}\right)_{qr}|ts,B\rangle$$

$$= P_{qp,ts}\langle A|\widetilde{\rho_1\rho_2\rho_3}|B\rangle\,, \tag{2.20b}$$

$$\widetilde{\rho_1\rho_2\rho_3}|0\rangle = \rho_1\rho_2\rho_3|0\rangle\,, \tag{2.21}$$

where ρ_1, ρ_2, ρ_3 can be any of the canonical operators. All the operators above are evaluated at time t, although this is explicit only in (2.17).

The (a) parts of each of these results can be extended to the product $\rho_1(t_1)\cdots\rho_n(t_n)$ of any number of operators at arbitrary times, for example,

$$\langle .0|\mathrm{Tr}\left(\frac{\rho_1(t_1)}{\sqrt{N}}\cdots\frac{\rho_n(t_n)}{\sqrt{N}}\right)|0.\rangle = N\langle 0|\rho_1(t_1)\cdots\rho_n(t_n)|0\rangle \tag{2.22}$$

so that all the traces of the theory are computable in terms of the reduced quantities. Recall that, for locally invariant theories, these are the invariant Wightman functions in the temporal gauge. For globally invariant theories, a broader class of invariant Wightman functions is discussed in App. B.

The (b) parts of these results and (2.21) can also be extended to define the tilde of the composite operator $\rho_1\cdots\rho_n$: When all the operators in the original, unreduced matrix product commute, one finds that the tilde of the reduced product is just the product of the tilde operators in the opposite order. This applies for example to the case of a general equal-time function R of the ϕ fields,

$$\begin{aligned} R(\phi) &= \sum_{n=0}^{\infty} r^{(n)}_{m_1 \cdots m_n} \phi_{m_1} \cdots \phi_{m_n} \,, \\ \tilde{R}(\tilde{\phi}) &= \sum_{n=0}^{\infty} r^{(n)}_{m_1 \cdots m_n} \tilde{\phi}_{m_n} \cdots \tilde{\phi}_{m_1} \,, \end{aligned} \tag{2.23}$$

where $\{r\}$ are arbitrary coefficients and $\widetilde{R(\phi)} = \tilde{R}(\tilde{\phi})$. Note that this relation is consistent with the (b) parts of (2.17) through (2.20). The same relations (2.23) hold when all the ϕ's are replaced by π's at equal time.

The tilde forms do not, however, extend in such a simple manner when the unreduced operators fail to commute, including, for example, a mixed product of ϕ's and π's at equal time or a product of ϕ's at different times. The tilde of a general equal-time product is determined in Subsec. 2.5 and App. D, but, owing to the complexity of many-time commutators, we will not discuss the tilde of many-time composite operators. Unless specified otherwise, all the operator products below are taken at equal time.

This completes the definition of the general equal-time reduced operators R, $\tilde{R}$ which correspond to the general equal-time matrix products (and sums of products) R_{rs} of the original canonical operators:

$$\langle .0| R_{rs}\left(\frac{\rho}{\sqrt{N}}\right) |0.\rangle = \delta_{rs} \langle 0| R(\rho) |0\rangle \,, \tag{2.24a}$$

$$\langle .0| R_{rs}\left(\frac{\rho}{\sqrt{N}}\right) |pq, A\rangle = f(N) P_{rs,pq} \langle 0| R(\rho) |A\rangle \,, \tag{2.24b}$$

$$\langle pq, A| R_{rs}\left(\frac{\rho}{\sqrt{N}}\right) |0.\rangle = f(N) P_{qp,rs} \langle A| R(\rho) |0\rangle \,, \tag{2.24c}$$

$$\langle pq, A| R_{rs}\left(\frac{\rho}{\sqrt{N}}\right) |st, B\rangle = P_{qp,rt} \langle A| R(\rho) |B\rangle \,, \tag{2.24d}$$

$$\langle pq, A| R_{rs}\left(\frac{\rho}{\sqrt{N}}\right) |tr, B\rangle = P_{qp,ts} \langle A| \tilde{R}(\tilde{\rho}) |B\rangle \,, \tag{2.24e}$$

$$\widetilde{R(\rho)} = \tilde{R}(\tilde{\rho}) \,, \quad \tilde{R}(\tilde{\rho})|0\rangle = R(\rho)|0\rangle \,, \quad \langle 0|\tilde{R}(\tilde{\rho}) = \langle 0|R(\rho) \,, \tag{2.24f}$$

where ρ is the set of canonical variables. In what follows, we shall refer to any operator of this type as a *density class operator* or simply a *density.*

2.3. *Trace class operators*

A *trace class operator* $T.$ is an invariant operator which is the trace of a density, such as the Hamiltonian, the angular momentum generators or the supercharges of the theory. The reduced matrix elements and reduced operators T which correspond to any trace class operator $T.$ are defined as

$$T. = C(N)\mathrm{Tr}\left(t\left(\frac{\phi}{\sqrt{N}}, \frac{\pi}{\sqrt{N}}, \frac{\Lambda}{\sqrt{N}}\right)\right), \tag{2.25a}$$

$$\begin{aligned}\langle .0|T.|0.\rangle &= \langle 0|T|0\rangle = NC(N)\langle 0|t(\phi,\pi,\Lambda)|0\rangle \\ &= NC(N)\langle 0|\tilde{t}(\tilde{\phi},\tilde{\pi},\tilde{\Lambda})|0\rangle\,,\end{aligned} \tag{2.25b}$$

$$\langle A|T|0\rangle \equiv \langle 0|T|A\rangle \equiv 0\,, \tag{2.25c}$$

$$\langle pq, A|T.|rs, B\rangle \equiv P_{qp,rs}\langle A|T|B\rangle\,, \tag{2.25d}$$

where (2.25b) is nothing but the trace of (2.24a) above. The definitions in (2.25c) and (2.25d) are required for the consistency of multiplication of reduced trace class operators, and are also consistent with (2.24). We leave the constant $C(N)$ undetermined here, but we will see below that $C(N) = N$ is selected for the usual Hamiltonian, supercharges and angular momenta.

The unreduced operators $T.$ are important quantities which generate the dynamics and internal symmetries of the theory, and we will see below that their reduced counterparts T still generate the same important transformations in the reduced theory at large N. The reduced Hamiltonian H, which generates the time translations of the reduced theory, was constructed in Refs. 18 and 19 for the one Hermitian and the (one polygon) unitary matrix models.

For all such quantities, we encounter here an "opacity" phenomenon which was seen but not emphasized in the examples of Refs. 18 and 19: the composite structure of the reduced operator T is apparently computable at this level only for the vacuum expectation value, as given in (2.25b), but not for the adjoint matrix elements in (2.25d). Technically, the reason is that the trace class operators are formed from adjoint operators but an adjoint operator acting on an adjoint state generally contains higher representations than singlet and adjoint, so that we cannot straightforwardly saturate the adjoint matrix elements of a trace class operator with the states of our reduced space.

So, the opacity phenomenon means that we do not yet know the composite structure of reduced operators T (only that the vacuum expectation value of T must equal the forms shown in (2.25b)). Nevertheless, it is known from Refs. 18 and 19 that the composite structure of these reduced operators can be found by solving their reduced algebraic relations, and we will place special emphasis on the construction of these operators below (see Subsecs. 3.2, 3.3, 4.6 and Sec. 5). In particular, we will see that the reduced trace class operators are intrinsically nonlocal, in accord with the early examples.[18,19]

2.4. *Derived maps*

In this section, we use the formalism above to infer a number of derived maps into the reduced space. Applications of these maps are given later.

(A) *Gauge generators*

Considering the matrix elements of the gauge generators G_{rs} in (2.6a) we find from (2.10c) that

$$[\phi_m, \pi_m] - i\Lambda_\alpha\Lambda_\alpha = [\tilde{\phi}_m, \tilde{\pi}_m] - i\tilde{\Lambda}_\alpha\tilde{\Lambda}_\alpha = i[(B - F - 1) + |0\rangle\langle 0|]\,. \tag{2.26}$$

These relations are written as part of the reduced equal-time algebra of the theory, according to the original interpretation in Ref. 18. Because G_{rs} is a density, however, these relations can be equivalently understood as the action of the reduced symmetry generators G, $\tilde{G}$ on the states,

$$\begin{aligned} G &\equiv -i[\phi_m, \pi_m] - \Lambda_\alpha\Lambda_\alpha + (F - B)\,, \\ \tilde{G} &\equiv i[\tilde{\phi}_m, \tilde{\pi}_m] + \tilde{\Lambda}_\alpha\tilde{\Lambda}_\alpha + (B - F)\,, \end{aligned} \tag{2.27a}$$

$$G^\dagger = G\,, \qquad \tilde{G}^\dagger = \tilde{G}\,, \tag{2.27b}$$

$$\tilde{G} = -G = 1 - |0\rangle\langle 0|\,, \tag{2.27c}$$

$$\tilde{G}|0\rangle = G|0\rangle = 0\,, \qquad \tilde{G}|A\rangle = -G|A\rangle = |A\rangle\,. \tag{2.27d}$$

(B) *Density maps*

For each density relation $R_{rs} = 0$ we obtain a pair of reduced equations

$$R_{rs}\left(\frac{\rho}{\sqrt{N}}, \frac{\dot{\rho}}{\sqrt{N}}\right) = 0 \to R(\rho, \dot{\rho}) = \tilde{R}(\tilde{\rho}, \dot{\tilde{\rho}}) = 0\,, \tag{2.28}$$

where ρ includes all the canonical variables. Similarly, if $R_{rs}|0.\rangle = 0$, then $R|0\rangle = \tilde{R}|0\rangle = 0$.

This map gives us, for example, an untilde and a tilde version of the reduced equations of motion (see for example Subsec. 2.6).

(C) *Canonical maps*

When ρ and σ are any of the canonical variables ϕ, π or Λ, we find that

$$[\rho_{rs}, \sigma_{pq}]_\mp = ic\delta_{sp}\delta_{rq} \to [\tilde{\rho}, \sigma]_\mp = [\rho, \tilde{\sigma}]_\mp = ic|0\rangle\langle 0| \tag{2.29}$$

which is derived by considering matrix elements such as

$$\langle .0|[\rho_{rs}, \sigma_{sq}]_\mp|0.\rangle\,, \qquad \langle ps, A|[\rho_{rs}, \sigma_{pq}]_\mp|qr, B\rangle\,. \tag{2.30}$$

These contributions to the reduced equal-time algebra of the theory are free algebraic relations because they contain no relations among the untilde operators or among the tilde operators. See Subsec. 2.5 for further discussion of the equal-time algebra.

(D) *A density and a trace class operator*

The form of relations involving the product of a density and a trace class operator are preserved in reduced space. We phrase this in terms of commutators and anticommutators:

$$\left[T_{\boldsymbol{\cdot}}, R_{rs}\left(\frac{\rho}{\sqrt{N}}, \frac{\dot{\rho}}{\sqrt{N}}\right)\right]_{\mp} = S_{rs}\left(\frac{\rho}{\sqrt{N}}, \frac{\dot{\rho}}{\sqrt{N}}\right) \tag{2.31a}$$

$$\to \begin{cases} [T, R(\rho, \dot{\rho})]_{\mp} = S(\rho, \dot{\rho}) \,, \\ [T, \tilde{R}(\tilde{\rho}, \dot{\tilde{\rho}})]_{\mp} = \tilde{S}(\tilde{\rho}, \dot{\tilde{\rho}}) \,, \end{cases} \tag{2.31b}$$

where R_{rs} and S_{rs} are general densities. This map gives us for example the commutator form of the reduced equations of motion in terms of the reduced Hamiltonian H,

$$\dot{\rho} = i[H, \rho] \,, \qquad \dot{\tilde{\rho}} = i[H, \tilde{\rho}] \,, \qquad \rho = \phi, \ \pi, \ \text{or } \Lambda \tag{2.32}$$

which supplements the explicit form of the reduced equations of motion obtained from the map (2.28). Moreover, the reduced images of (2.6c) are

$$\dot{G} = \dot{\tilde{G}} = [G, H] = [\tilde{G}, H] = 0 \,, \tag{2.33}$$

where G and $\tilde{G}$ are the reduced gauge generators in (2.27). The map (2.31) also tells us that the transformation properties of the reduced operators ϕ, π or Λ under the reduced rotation or supersymmetry generators is unchanged by the reduction (see Subsecs. 2.6, 3.2 and 3.3 and Secs. 5 and 6).

(E) *Two trace class operators*

Algebraic relations among trace class operators are preserved in the reduction:

$$[T_{\boldsymbol{\cdot}1}, T_{\boldsymbol{\cdot}2}]_{\mp} = T_{\boldsymbol{\cdot}3} \to [T_1, T_2]_{\mp} = T_3 \,. \tag{2.34}$$

This map tells us, for example, that the angular momentum algebra or the supersymmetry algebra of the theory is preserved in the reduction (See Subsecs. 2.6, 3.3, 5.3, 5.5 and Sec. 6). In the case of a gauged matrix model such as Matrix theory, however, the explicit form of the reduced supersymmetry algebra [see Eq. (6.14a)] can be surprisingly different from that of the original unreduced supersymmetry algebra.

2.5. *Reduced equal-time algebra*

In this section, we familiarize ourselves with the reduced equal-time free algebras of the general matrix model, temporarily deferring the contributions of any conserved quantity.

The explicit form of the reduced equal-time algebras follows from (2.26) and (2.29):

$$[\phi_m, \tilde{\pi}_n] = [\tilde{\phi}_m, \pi_n] = i\delta_{mn}|0\rangle\langle 0| \,, \tag{2.35a}$$

$$[\phi_m, \tilde{\phi}_n] = [\pi_m, \tilde{\pi}_n] = 0 \,, \quad [\Lambda_\alpha, \tilde{\Lambda}_\beta]_+ = \delta_{\alpha\beta}|0\rangle\langle 0| \,, \tag{2.35b}$$

$$[\Lambda_\alpha, \tilde{\phi}_m] = [\tilde{\Lambda}_\alpha, \phi_m] = [\Lambda_\alpha, \tilde{\pi}_m] = [\tilde{\Lambda}_\alpha, \pi_m] = 0 \,, \tag{2.35c}$$

$$[\phi_m, \pi_m] - i\Lambda_\alpha\Lambda_\alpha = [\tilde{\phi}_m, \tilde{\pi}_m] - i\tilde{\Lambda}_\alpha\tilde{\Lambda}_\alpha = i[(B - F - 1) + |0\rangle\langle 0|] \,, \tag{2.35d}$$

$$m,\ n = 1 \cdots B \,, \quad \alpha,\ \beta = 1 \cdots f \,, \quad F = \frac{f}{2} \,, \tag{2.35e}$$

where B and f are integers. The equal-time algebras (2.35) provide our first examples of *symmetric free algebras*, so-called because each algebra is symmetric under the exchange of tilde and untilde operators:

$$\rho \leftrightarrow \tilde{\rho} \,, \qquad \rho = \phi,\ \pi \text{ and } \Lambda \,. \tag{2.36}$$

Historically, a Euclidean operator isomorphic to $\tilde{\pi}$ in (2.35a) was first introduced in Ref. 7, and later as a differential realization in Ref. 12.

In what follows, we discuss a number of properties of the symmetric free algebras (2.35) in combination with the vacuum relations (2.15b) and (2.24f), which we repeat here for reference

$$\tilde{\rho}|0\rangle = \rho|0\rangle \,, \qquad \langle 0|\tilde{\rho} = \langle 0|\rho \,, \tag{2.37a}$$

$$\tilde{R}(\tilde{\rho})|0\rangle = R(\rho)|0\rangle \,, \qquad \langle 0|\tilde{R}(\tilde{\rho}) = \langle 0|R(\rho) \,. \tag{2.37b}$$

(A) *Consistency check*

We note that (2.37) and the vacuum expectation value of (2.35d) imply the relation

$$\langle 0|[\phi_m, \tilde{\pi}_m] - \frac{i}{2}[\Lambda_\alpha, \tilde{\Lambda}_\alpha]_+|0\rangle = i(B - F) \tag{2.38}$$

which is consistent with (2.35a) and (2.35b).

(B) *Tilde operators as right multipliers*

Consider a general state (word) formed by the action of any number of ϕ operators on the vacuum (fermionic operators can be added as well). The action of another ϕ on the state is of course the addition of the operator on the left of the word. On the other hand, (2.35b) and (2.37) tell us that the action of a $\tilde{\phi}$ operator is equivalent to adding a ϕ field on the right:

$$\tilde{\phi}_m\phi_{m_1} \cdots \phi_{m_n}|0\rangle = \phi_{m_1} \cdots \phi_{m_n}\phi_m|0\rangle \,. \tag{2.39}$$

In word notation, the action of ϕ and $\tilde{\phi}$ is

$$|w\rangle = \phi^w|0\rangle \equiv \phi_{m_1}\phi_{m_2} \cdots \phi_{m_n}|0\rangle \,, \quad w \equiv m_1 m_2 \cdots m_n \,, \tag{2.40a}$$

$$\phi_m|w\rangle = |mw\rangle \,, \quad \tilde{\phi}_m|w\rangle = |wm\rangle \,. \tag{2.40b}$$

(C) *Cyclicity of ground state averages*

The equal-time reduced vacuum expectation value of any number of reduced ϕ's is cyclically symmetric. To see this use (2.35b) and (2.37) to follow the steps:

$$\begin{aligned}\langle 0|\phi_{m_1}\phi_{m_2}\cdots\phi_{m_n}|0\rangle &= \langle 0|\tilde{\phi}_{m_1}\phi_{m_2}\cdots\phi_{m_n}|0\rangle \\ &= \langle 0|\phi_{m_2}\cdots\phi_{m_n}\tilde{\phi}_{m_1}|0\rangle \\ &= \langle 0|\phi_{m_2}\cdots\phi_{m_n}\phi_{m_1}|0\rangle\,. \end{aligned} \tag{2.41}$$

According to (2.17a), this result is only the image in reduced space of the cyclic property of the unreduced equal-time traced Wightman functions.[d] We emphasize however the central role of the tilde operators in establishing this property directly in the reduced space.

The same cyclicity is found for the vacuum expectation value of many reduced π's, as expected, but vacuum expectation values of mixed products of ϕ's, π's and Λ's are generally not cyclic. Following steps similar to those in (2.41), however, the corrections to cyclicity can always be computed directly from the equal-time algebra and (2.37). Here are some simple examples

$$\langle 0|\phi_m\pi_n|0\rangle = \langle 0|\pi_n\phi_m|0\rangle + i\delta_{mn}\,, \tag{2.42a}$$

$$\langle 0|\Lambda_\alpha\Lambda_\beta|0\rangle = -\langle 0|\Lambda_\beta\Lambda_\alpha|0\rangle + \delta_{\alpha\beta} \tag{2.42b}$$

which the reader is invited to verify.

(D) *Tilde of general reduced densities*

The $\tilde{R}$ corresponding to a general composite density R is defined in Subsec. 2.2 and satisfies (2.37). The form of $\tilde{R}$ is simple when the operators of the original density commute, as noted in (2.23). We give here a useful algorithm for the form of the general $\tilde{R}$ which nicely packages the results of App. D: One can compute $\tilde{R}$ from R using the equal-time algebra and (2.37), remembering that $\tilde{R}$ is a function only of tilde fields. This means that we eliminate any vacuum projectors $|0\rangle\langle 0|$ which arise by using the identity $|0\rangle\langle 0|0\rangle = |0\rangle$. As a simple example, consider

$$\begin{aligned}\phi_m\pi_n|0\rangle = \phi_m\tilde{\pi}_n|0\rangle &= [\phi_m,\tilde{\pi}_n]|0\rangle + \tilde{\pi}_n\phi_m|0\rangle \\ &= (i\delta_{mn} + \tilde{\pi}_n\tilde{\phi}_m)|0\rangle \end{aligned} \tag{2.43}$$

which tells us that

$$\widetilde{(\phi_m\pi_n)} = \tilde{\pi}_n\tilde{\phi}_m + i\delta_{mn}\,. \tag{2.44}$$

Another example is $R = \Lambda_\alpha\phi_m\Lambda_\beta\pi_n$, for which we find

$$\tilde{R} = -\tilde{\pi}_n\tilde{\Lambda}_\beta\tilde{\phi}_m\tilde{\Lambda}_\alpha + i\delta_{mn}\langle 0|\Lambda_\beta|0\rangle\tilde{\Lambda}_\alpha + \delta_{\alpha\beta}\langle 0|\phi_m|0\rangle\tilde{\pi}_n\,. \tag{2.45}$$

[d] Related identities such as $\langle 0|[R(\phi),S(\phi)]|0\rangle = 0$ also follow in the same way from (2.37b) and (using the results of App. C) we see that this identity is the image of $N^{-1}\,\mathrm{Tr}[R(\phi/\sqrt{N}), S(\phi/\sqrt{N})] = 0$ in the unreduced theory.

The form of the reduced gauge generator $\tilde{G}$ in (2.27a) is also easily computed in this way from the form of the reduced gauge generator G.

(E) *An even number of real fermions*

When the number of real adjoint fermions is even, we may introduce complex reduced fermions as

$$\Lambda_\alpha = \frac{1}{\sqrt{2}}\begin{pmatrix} \psi^\dagger_{\dot\alpha} + \psi_{\dot\alpha} \\ i(\psi^\dagger_{\dot\alpha} - \psi_{\dot\alpha}) \end{pmatrix}, \quad \dot\alpha = 1 \cdots F, \quad F = \frac{f}{2} = \text{integer} \tag{2.46}$$

and similarly for $\tilde{\Lambda}_\alpha \to \tilde{\psi}^\dagger_{\dot\alpha}$, $\tilde{\psi}_{\dot\alpha}$. Then the fermionic part of the reduced equal-time algebra becomes:

$$\begin{aligned} &[\psi_{\dot\alpha}, \tilde{\psi}^\dagger_{\dot\beta}]_+ = [\tilde{\psi}_{\dot\alpha}, \psi^\dagger_{\dot\beta}]_+ = \delta_{\dot\alpha\dot\beta}|0\rangle\langle 0|\,, \\ &[\psi_{\dot\alpha}, \tilde{\psi}_{\dot\beta}]_+ = [\psi^\dagger_{\dot\alpha}, \tilde{\psi}^\dagger_{\dot\beta}]_+ = 0\,, \end{aligned} \tag{2.47a}$$

$$\begin{aligned} [\phi_m, \pi_m] - i[\psi_{\dot\alpha}, \psi^\dagger_{\dot\alpha}]_+ &= [\tilde{\phi}_m, \tilde{\pi}_m] - i[\tilde{\psi}_{\dot\alpha}, \tilde{\psi}^\dagger_{\dot\alpha}]_+ \\ &= i[(B - F - 1) + |0\rangle\langle 0|]\,, \end{aligned} \tag{2.47b}$$

$$\tilde{\psi}_{\dot\alpha}|0\rangle = \psi_{\dot\alpha}|0\rangle\,, \quad \tilde{\psi}^\dagger_{\dot\alpha}|0\rangle = \psi^\dagger_{\dot\alpha}|0\rangle\,. \tag{2.47c}$$

The complex fermionic operators continue to commute with the bosonic tilde operators and vice-versa with respect to the tilde.

This decomposition allows us to see many of the properties discussed above for the bosonic operators. For example the relation

$$\tilde{\psi}^\dagger_{\dot\alpha}\psi^\dagger_{\dot\alpha_1}\cdots\psi^\dagger_{\dot\alpha_n}|0\rangle = (-1)^n \psi^\dagger_{\dot\alpha_1}\cdots\psi^\dagger_{\dot\alpha_n}\psi^\dagger_{\dot\alpha}|0\rangle \tag{2.48}$$

shows that $\tilde{\psi}^\dagger$ is a right multiplication operator with respect to the daggered fermionic words. Similarly, the tilde of the composite fermionic operators

$$R = \psi^\dagger_{\dot\alpha_1}\cdots\psi^\dagger_{\dot\alpha_n}\,, \qquad \tilde{R} = (-1)^{\frac{1}{2}n(n-1)}\tilde{\psi}^\dagger_{\dot\alpha_n}\cdots\tilde{\psi}^\dagger_{\dot\alpha_1} \tag{2.49}$$

is easily computed from (2.37) and (2.47a).

2.6. *Example: general bosonic system*

As an explicit example, we collect here the setup for a general system of B bosons, starting with the Hermitian Hamiltonian

$$H_{\bullet} = \mathrm{Tr}\left(\frac{1}{2}\pi^m\pi^m + NV\left(\frac{\phi}{\sqrt{N}}\right)\right), \tag{2.50a}$$

$$\mathrm{Tr}\, V(\phi) = \mathrm{Tr}\left(v^{(0)} + \sum_{n=1}^{\infty}\frac{1}{n}\, v^{(n)}_{m_1\cdots m_n}\phi^{m_1}\cdots\phi^{m_n}\right), \tag{2.50b}$$

$$v^{(n)}_{m_1 m_2\cdots m_n} = v^{(n)}_{m_2\cdots m_n m_1}\,, \quad (\mathrm{Tr}\, V)^\dagger = \mathrm{Tr}\, V\,, \quad v^{(n)*}_{m_1\cdots m_n} = v^{(n)}_{m_n\cdots m_1}\,, \tag{2.50c}$$

where the numerical coefficients $v^{(n)}$ of the potential are cyclically symmetric in their subscripts. These coefficients are also independent of N to maintain 't Hooft scaling at large N. Comparing (2.50a) with (2.25a) we see that $C(N) = N$ for the Hamiltonian, as noted above.

Going over now to the reduced formulation at large N, we record first the equal-time free algebra of the system

$$[\phi_m, \tilde{\pi}_n] = [\tilde{\phi}_m, \pi_n] = i\delta_{mn}|0\rangle\langle 0|\,, \tag{2.51a}$$

$$[\phi_m, \tilde{\phi}_n] = [\tilde{\pi}_m, \pi_n] = 0\,, \tag{2.51b}$$

$$[\phi_m, \pi_m] = [\tilde{\phi}_m, \tilde{\pi}_m] = i[B - 1 + |0\rangle\langle 0|]\,, \tag{2.51c}$$

$$\tilde{\rho}|0\rangle = \rho|0\rangle\,, \quad \langle 0|\tilde{\rho} = \langle 0|\rho\,, \quad \rho = \phi \text{ or } \pi \tag{2.51d}$$

and then the reduced equations of motion

$$\dot{\phi}_m = i[H, \phi_m] = \pi_m\,, \qquad \dot{\pi}_m = i[H, \pi_m] = -V'_m(\phi)\,, \tag{2.52a}$$

$$\dot{\tilde{\phi}}_m = i[H, \tilde{\phi}_m] = \tilde{\pi}_m\,, \qquad \dot{\tilde{\pi}}_m = i[H, \tilde{\pi}_m] = -\tilde{V}'_m(\tilde{\phi})\,, \tag{2.52b}$$

$$\begin{aligned} V'_m(\phi) &= \sum_{n=1}^{\infty} v^{(n)}_{mm_2\cdots m_n}\phi_{m_2}\cdots\phi_{m_n}\,, \\ \tilde{V}'_m(\tilde{\phi}) &= \sum_{n=1}^{\infty} v^{(n)}_{mm_2\cdots m_n}\tilde{\phi}_{m_n}\cdots\tilde{\phi}_{m_2}\,, \end{aligned} \tag{2.52c}$$

$$V'^{\dagger}_m = V'_m\,, \qquad \tilde{V}'^{\dagger}_m = \tilde{V}'_m\,, \qquad \tilde{V}'_m|0\rangle = V'_m|0\rangle \tag{2.52d}$$

are obtained from the original equations of motion and maps (B) and (D) of Subsec. 2.4. Here H is the reduced Hamiltonian of the system which satisfies

$$E_0 = \langle .0|H.|0.\rangle = \langle 0|H|0\rangle = N^2\langle 0|\frac{1}{2}\pi_m\pi_m + V(\phi)|0\rangle\,, \tag{2.53a}$$

$$(H - E_0)|0\rangle = 0\,, \quad (H - E_0)|A\rangle = \omega_{A0}|A\rangle\,, \tag{2.53b}$$

$$\omega_{\mu\nu} = E_\mu - E_\nu = O(N^0), \quad \mu = (0, A)\,, \quad \nu = (0, B) \tag{2.53c}$$

and governs the time dependence of the reduced system[17,18] according to

$$\rho(t) = e^{iHt}\rho(0)e^{-iHt}\,, \qquad \tilde{\rho}(t) = e^{iHt}\tilde{\rho}(0)e^{-iHt}\,, \tag{2.54a}$$

$$\rho(t)_{\mu\nu} = e^{i\omega_{\mu\nu}t}\rho(0)_{\mu\nu}\,, \qquad \tilde{\rho}(t)_{\mu\nu} = e^{i\omega_{\mu\nu}t}\tilde{\rho}(0)_{\mu\nu}\,, \tag{2.54b}$$

where $\rho = \phi$ or π. As noted in the Introduction, the matrix elements in (2.54b) are the master fields of the theory, and the results (2.54) hold as well in general matrix models including fermions. As emphasized in Subsec. 2.3, we do not yet know the composite structure of the reduced H.

In what follows, we discuss a number of useful aspects of this system.

(A) *Connection and curvature*

Define a connection on the free Hilbert space as the collection $J \equiv \{J_m(\phi), \tilde{J}_m(\tilde{\phi}), m = 1 \cdots B\}$ where $J_m(\phi)$ is a set of reduced densities whose tildes are $\tilde{J}_m$. Further define the curvature of the connection J as

$$\mathcal{F}_{mn}(J) \equiv [\pi_m, \tilde{J}_n] - [\tilde{\pi}_n, J_m]\,. \tag{2.55}$$

As an example, use the equations of motion to compute

$$\frac{d}{dt}[\pi_m, \tilde{\pi}_n] = 0 \rightarrow [\tilde{\pi}_n, V'_m] - [\pi_m, \tilde{V}'_n] = 0\,. \tag{2.56}$$

This shows that $V' \equiv \{V'_m, \tilde{V}'_m\}$, which is a natural "gradient" associated to the reduced potential V, is a flat connection on the free Hilbert space,

$$\mathcal{F}_{mn}(V') = 0\,. \tag{2.57}$$

We have also checked that the notions of flat connection and cyclic coefficients are equivalent: Any connection J of the form

$$\begin{aligned} J_m(\phi) &= \sum_{n=1}^{\infty} j^{(n)}_{mm_2\cdots m_n} \phi_{m_2} \cdots \phi_{m_n}\,, \\ \tilde{J}_m(\tilde{\phi}) &= \sum_{n=1}^{\infty} j^{(n)}_{mm_2\cdots m_n} \tilde{\phi}_{m_n} \cdots \tilde{\phi}_{m_2}\,, \end{aligned} \tag{2.58a}$$

$$j^{(n)}_{m_1 m_2\cdots m_n} = j^{(n)}_{m_2\cdots m_n m_1} \tag{2.58b}$$

is flat and we have also solved the flatness condition to prove that any flat connection has this form. The notions of flat connection and integrability are equivalent as well, so that every flat connection on the free Hilbert space is associated to a trace class generating function of the form (2.50b) in the unreduced large N theory. Flat connections will play an important role in the development of Sec. 4.

(B) *Rotational invariance*

We consider the special case when the original bosonic theory is rotation invariant with trace class generators $J_{\cdot}^{mn}$:

$$J_{\cdot}^{mn} = \mathrm{Tr}\big(\pi^{[m}\phi^{n]}\big)\,, \quad m,\ n = 1 \cdots B\,, \tag{2.59a}$$

$$\dot{J}_{\cdot}^{mn} = i[H_{\cdot}, J_{\cdot}^{mn}] = 0\,, \quad J_{\cdot}^{mn}|0_{\cdot}\rangle = 0 \tag{2.59b}$$

which satisfy the algebra of spin (B). The operators ϕ and π are in the vector representation of spin (B), and (2.59a) specifies the constant in (2.25a) as $C(N) = N$. At large N, these generators map onto reduced generators J_{mn} which satisfy

$$\langle 0|J_{mn}|0\rangle = \langle 0|\pi_{[m}\phi_{n]}|0\rangle = 0\,, \tag{2.60a}$$

$$\dot{J}_{mn} = i[H, J_{mn}] = 0\,, \quad J_{mn}|0\rangle = 0\,, \tag{2.60b}$$

$$[J_{mn}, B_p] = -i\delta_{p[m}B_{n]}\,, \quad B = \phi,\ \pi,\ \tilde{\phi} \text{ or } \tilde{\pi}\,, \tag{2.60c}$$

$$[J_{mn}, J_{pq}] = i(\delta_{q[m}J_{n]p} - \delta_{p[m}J_{n]q}) \tag{2.60d}$$

including (according to map (E) of Subsec. 2.4) the same algebra of spin (B). As in the case of the reduced Hamiltonian H, we do not yet know the composite structure of the reduced generators J_{mn} (see Subsec. 2.3).

(C) *One Hermitian matrix*

The case $B = 1$ above is called the (Hamiltonian) one Hermitian matrix model, whose solution[17,18] we review here as a special case of our general development.

We have seen above that untilde and tilde operators correspond respectively to left and right multiplication in the word notation [see Eq. (2.40)], but left and right multiplication are indistinguishable when $B = 1$, so that

$$B = 1\colon \tilde{\phi} = \phi\,, \quad \tilde{\pi} = \pi\,. \tag{2.61}$$

The identification (2.61), which does not hold for higher B (or for $B = 1$ and $F \neq 0$), is the essential simplification of the one-matrix model. Then the reduced system reads simply[17,18]

$$\dot{\phi} = i[H, \phi] = \pi\,, \qquad \dot{\pi} = i[H, \pi] = -V'(\phi)\,, \tag{2.62a}$$

$$[\phi, \pi] = i|0\rangle\langle 0|\,, \tag{2.62b}$$

$$\rho(t)_{\mu\nu} = e^{i\omega_{\mu\nu}t}\rho(0)_{\mu\nu}\,, \qquad \rho = \phi \text{ or } \pi\,, \tag{2.62c}$$

where the master fields are given in (2.62c).

We mention two early approaches to the solution of this model. The case of the oscillator was solved in Ref. 17:

$$\begin{aligned} V &= \frac{1}{2}\omega^2\phi^2\,, \quad \phi = \frac{1}{\sqrt{2\omega}}(e^{i\omega t}a^\dagger + e^{-i\omega t}a)\,, \\ \pi &= i\sqrt{\frac{\omega}{2}}(e^{i\omega t}a^\dagger - e^{-i\omega t}a)\,, \end{aligned} \tag{2.63a}$$

$$(a^\dagger)_{\mu\nu} = \delta_{\mu,\nu+1}\,, \quad (a)_{\mu\nu} = \delta_{\mu,\nu-1}\,, \tag{2.63b}$$

$$aa^\dagger = 1\,, \quad a|0\rangle = 0\,, \quad a^\dagger a = 1 - |0\rangle\langle 0|\,. \tag{2.63c}$$

This solution was originally written with Kronecker deltas, as in (2.63b), but we recognize this today as a realization of the one-dimensional Cuntz algebra in (2.63c). We will return to this approach for many oscillators in Sec. 3.

The general system (2.62) was solved[18] in the coordinate basis:

$$\phi|q\rangle = q|q\rangle\,, \quad \pi_{q,q'} = i\frac{\mathcal{P}}{q-q'}\psi_0(q)\psi_0^*(q')\,, \tag{2.64a}$$

$$\begin{aligned}(H-E_0)_{q,q'} &= -\frac{\mathcal{P}}{(q-q')^2}\psi_0(q)\psi_0^*(q') \\ &\quad + \delta(q-q')\int dq''\frac{\mathcal{P}}{(q-q'')^2}\rho(q'')\,,\end{aligned} \tag{2.64b}$$

$$\rho(q) = \psi_0^*(q)\psi_0(q) = \frac{\sqrt{2}}{\pi}\big(\epsilon - V(q)\big)^{\frac{1}{2}}\,, \qquad \int dq\,\rho(q) = 1\,, \tag{2.64c}$$

$$\langle 0|\phi^n|0\rangle = \int dq\,\rho(q)q^n\,, \tag{2.64d}$$

where $\mathcal{P}$ is principal value, ψ_0 is the reduced coordinate space ground state wave function, and sub q, q' denotes matrix elements in the coordinate basis. See Ref. 18 for further details of this solution, including the ground state energy E_0 and the energies of the dominant adjoint states. We note in particular the explicit construction (2.64b) of the reduced Hamiltonian H, whose composite structure is seen to be highly nonlocal. A similarly nonlocal reduced Hamiltonian for the (one polygon) unitary matrix model was obtained in Ref. 19. This approach is also considered for higher B in Subsec. 3.6.

3. Bosonic Oscillators

3.1. *Symmetric Cuntz algebras*

In this section we consider the special case of B bosonic oscillators

$$V = \frac{1}{2}\sum_{m=1}^{B}\omega_m^2\phi_m\phi_m\,, \tag{3.1a}$$

$$\dot\phi_m = i[H,\phi_m] = \pi_m\,, \qquad \dot\pi_m = i[H,\pi_m] = -\omega_m^2\phi_m\,, \tag{3.1b}$$

$$\dot{\tilde\phi}_m = i[H,\tilde\phi_m] = \tilde\pi_m\,, \qquad \dot{\tilde\pi}_m = i[H,\tilde\pi_m] = -\omega_m^2\tilde\phi_m \tag{3.1c}$$

in order to understand the relationship between our equal-time free algebra (2.51) and the Cuntz algebra (1.5).

The solution to the reduced equations of motion (3.1b) and (3.1c) is

$$\begin{aligned}\phi_m &= \frac{1}{\sqrt{2\omega_m}}(e^{i\omega_m t}a_m^\dagger + e^{-i\omega_m t}a_m)\,, \\ \pi_m &= i\sqrt{\frac{\omega_m}{2}}(e^{i\omega_m t}a_m^\dagger - e^{-i\omega_m t}a_m)\,,\end{aligned} \tag{3.2a}$$

$$\begin{aligned}\tilde{\phi}_m &= \frac{1}{\sqrt{2\omega_m}}(e^{i\omega_m t}\tilde{a}_m^\dagger + e^{-i\omega_m t}\tilde{a}_m)\,,\\ \tilde{\pi}_m &= i\sqrt{\frac{\omega_m}{2}}(e^{i\omega_m t}\tilde{a}_m^\dagger - e^{-i\omega_m t}\tilde{a}_m)\end{aligned} \tag{3.2b}$$

and we know that

$$a_m|0\rangle = \tilde{a}_m|0\rangle = \langle 0|a_m^\dagger = \langle 0|\tilde{a}_m^\dagger = 0\,, \tag{3.3a}$$

$$\tilde{a}_m^\dagger|0\rangle = a_m^\dagger|0\rangle, \qquad \langle 0|\tilde{a}_m = \langle 0|a_m\,. \tag{3.3b}$$

The relations (3.3) follow from the maps of Subsec. 2.4 because the corresponding matrix creation and annihilation operators $(a^\dagger)_{rs}$, a_{rs} are densities. We can also define time-dependent creation and annihilation operators

$$a_m(t) = \frac{1}{\sqrt{2\omega_m}}(\omega_m\phi_m + i\pi_m) = e^{-i\omega_m t}a_m\,, \tag{3.4a}$$

$$a_m^\dagger(t) = \frac{1}{\sqrt{2\omega_m}}(\omega_m\phi_m - i\pi_m) = e^{i\omega_m t}a_m^\dagger \tag{3.4b}$$

and similarly for $\tilde{a}_m(t)$, $\tilde{a}_m^\dagger(t)$. The time-dependent creation/annihilation operators also satisfy (3.3) and, similarly, the relations below can be read in terms of either the time-independent or the time-dependent operators.

In terms of these operators, the equal-time algebra (2.35) now reads

$$[a_m, \tilde{a}_n^\dagger] = [\tilde{a}_m, a_n^\dagger] = \delta_{mn}|0\rangle\langle 0|\,, \tag{3.5a}$$

$$[a_m, \tilde{a}_n] = [a_m^\dagger, \tilde{a}_n^\dagger] = 0\,, \tag{3.5b}$$

$$[a_m, a_m^\dagger] = [\tilde{a}_m, \tilde{a}_m^\dagger] = B - 1 + |0\rangle\langle 0|\,. \tag{3.5c}$$

We consider next the construction of complete sets of states, using (3.3) and (3.5). Any state involving mixed untilde and tilde creation operators on the vacuum can be expressed entirely in terms of untilde creation operators, e.g.

$$\begin{aligned}&a_{m_1}^\dagger\cdots a_{m_n}^\dagger\tilde{a}_{n_1}^\dagger\cdots\tilde{a}_{n_m}^\dagger a_{p_1}^\dagger\cdots a_{p_q}^\dagger|0\rangle\\ &\quad = a_{m_1}^\dagger\cdots a_{m_n}^\dagger a_{p_1}^\dagger\cdots a_{p_q}^\dagger a_{n_m}^\dagger\cdots a_{n_1}^\dagger|0\rangle\,.\end{aligned} \tag{3.6}$$

Similarly, mixed states involving a's and $a^\dagger$'s can be expressed entirely in terms of $a^\dagger$'s. To see this follow the steps

$$\begin{aligned}a_m a_{m_1}^\dagger\cdots a_{m_n}^\dagger|0\rangle &= a_m\tilde{a}_{m_n}^\dagger\cdots\tilde{a}_{m_1}^\dagger|0\rangle\\ &= [a_m, \tilde{a}_{m_n}^\dagger\cdots\tilde{a}_{m_1}^\dagger]|0\rangle\\ &= \tilde{a}_{m_n}^\dagger\cdots\tilde{a}_{m_2}^\dagger\delta_{m,m_1}|0\rangle\\ &= \delta_{m,m_1}a_{m_2}^\dagger\cdots a_{m_n}^\dagger|0\rangle\,,\end{aligned} \tag{3.7}$$

where we have used the relation

$$[a_m, \tilde{a}_n^\dagger]\tilde{a}_p^\dagger = \delta_{mn}|0\rangle\langle 0|\tilde{a}_p^\dagger = 0 \tag{3.8}$$

which follows from the equal-time algebra and (3.3). It follows from (3.6) and (3.7) that the $a^\dagger$ states are complete.

The relations (3.6)–(3.8) are also true, however, under exchange of tilde and untilde labels, so that the $\tilde{a}^\dagger$ states are also complete

$$\{a^\dagger_{m_1}\cdots a^\dagger_{m_n}|0\rangle\} = \{\tilde{a}^\dagger_{m_1}\cdots \tilde{a}^\dagger_{m_n}|0\rangle\} = \text{complete set of states}\,, \tag{3.9a}$$

$$\begin{aligned}\mathbf{1} &= \sum_{n=0}^{\infty} a^\dagger_{m_1}\cdots a^\dagger_{m_n}|0\rangle\langle 0|a_{m_n}\cdots a_{m_1} \\ &= \sum_{n=0}^{\infty} \tilde{a}^\dagger_{m_1}\cdots \tilde{a}^\dagger_{m_n}|0\rangle\langle 0|\tilde{a}_{m_n}\cdots \tilde{a}_{m_1}\,,\end{aligned} \tag{3.9b}$$

$$\tilde{a}^\dagger_{m_1}\cdots \tilde{a}^\dagger_{m_n}|0\rangle = a^\dagger_{m_n}\cdots a^\dagger_{m_1}|0\rangle\,. \tag{3.9c}$$

Indeed, the tilde states can be rewritten in terms of the untilde states,[e] as shown explicitly in (3.9c).

Since (3.7) and its tilde $\leftrightarrow$ untilde version are true on complete sets of states, we have established the full equal-time algebra of the reduced creation/annihilation operators

$$a_m a^\dagger_n = \tilde{a}_m \tilde{a}^\dagger_n = \delta_{mn}\,, \quad m,\ n = 1\cdots B\,, \tag{3.10a}$$

$$a^\dagger_m a_m = \tilde{a}^\dagger_m \tilde{a}_m = 1 - |0\rangle\langle 0|\,, \tag{3.10b}$$

$$[a_m, \tilde{a}^\dagger_n] = [\tilde{a}_m, a^\dagger_n] = \delta_{mn}|0\rangle\langle 0|\,, \quad [a_m, \tilde{a}_n] = [a^\dagger_m, \tilde{a}^\dagger_n] = 0\,, \tag{3.10c}$$

$$a_m|0\rangle = \tilde{a}_m|0\rangle = \langle 0|a^\dagger_m = \langle 0|\tilde{a}^\dagger_m = 0\,, \tag{3.10d}$$

$$\tilde{a}^\dagger_m|0\rangle = a^\dagger_m|0\rangle\,, \quad \langle 0|\tilde{a}_m = \langle 0|a_m\,. \tag{3.10e}$$

In particular, the argument (3.7) on all $a^\dagger$ states gives the ordinary Cuntz relation (see (1.5)) in (3.10a), and the tilde $\leftrightarrow$ untilde version of (3.7) gives the tilde Cuntz relation in (3.10a). Then (3.10b) follows from (3.10a) and (3.5c). In what follows, these algebras will be called *symmetric Cuntz algebras*: each algebra is symmetric under the interchange of tilde and untilde operators, and contains two Cuntz subalgebras (untilde and tilde).

3.2. *Reduced Hamiltonian*

Using (3.2) and the symmetric Cuntz algebra (3.10), it is straightforward to compute the large N ground state energy for the oscillators

$$\begin{aligned}E_0 = \langle 0|H|0\rangle &= \frac{N^2}{2}\langle 0|\sum_m(\pi_m\pi_m + \omega_m^2\phi_m\phi_m)|0\rangle \\ &= \frac{N^2}{2}\sum_m \omega_m\,,\end{aligned} \tag{3.11}$$

where H is the reduced Hamiltonian. The reduced Hamiltonian also appears in the reduced equations of motion (3.1b), (3.1c) and, using these relations, it is not difficult to construct the reduced Hamiltonian explicitly in this case.

[e]For $B = 1$, Eq. (3.9c) implies that $\tilde{a}^\dagger = a^\dagger$ and hence $\tilde{a} = a$, in accord with (2.61).

The reduced Hamiltonian has many equivalent forms, beginning with[f]

$$H - E_0 = \sum_{n=0}^{\infty} a^\dagger_{m_1} \cdots a^\dagger_{m_n} (a^\dagger \omega a) a_{m_n} \cdots a_{m_1} \tag{3.12a}$$

$$= \sum_{n=0}^{\infty} \tilde{a}^\dagger_{m_1} \cdots \tilde{a}^\dagger_{m_n} (\tilde{a}^\dagger \omega \tilde{a}) \tilde{a}_{m_n} \cdots \tilde{a}_{m_1} \,, \tag{3.12b}$$

$$(a^\dagger \omega a) = \sum_m a^\dagger_m \omega_m a_m \,. \tag{3.12c}$$

These forms can be used to check the commutators

$$[H, a^\dagger_m] = \omega_m a^\dagger_m \,, \qquad [H, a_m] = -\omega_m a_m \,, \tag{3.13a}$$

$$[H, \tilde{a}^\dagger_m] = \omega_m \tilde{a}^\dagger_m \,, \qquad [H, \tilde{a}_m] = -\omega_m \tilde{a}_m \tag{3.13b}$$

which guarantee the correct equations of motion. Here is a roadmap for checking these commutators, all four of which are true for both forms of H in (3.12): consider first the untilde form of H in (3.12a). In this case the commutators in (3.13a) are easily checked by writing out each term, using (3.10a) and subtracting. The commutators in (3.13b) must be computed directly using the mixed commutators (3.10c), and these come out as

$$[H, \tilde{a}^\dagger_m] = \omega_m \tilde{a}^\dagger_m \left(\sum_{n=0}^{\infty} a^\dagger_{m_1} \cdots a^\dagger_{m_n} |0\rangle\langle 0| a_{m_n} \cdots a_{m_1} \right) = \omega_m \tilde{a}^\dagger_m \,. \tag{3.14}$$

For the tilde form of H in (3.12b), the two types of computation above are reversed but the same results are obtained. The results in (3.13) also show that the two forms of H in (3.12) are equal: the difference Δ of the two forms is zero because Δ annihilates the vacuum and commutes with all the operators of the theory, so that $\Delta = 0$ on any state.

Using (3.4), we see that the reduced Hamiltonian H in (3.12) is a highly nonlocal operator [see (3.25)], and we will see this nonlocality quite generally below for the reduced trace class operators of the various theories. This is the price one must pay in using free algebras (which are not local commutators) to solve reduced algebraic relations such as (3.13).

There are other equivalent forms of H which show its spectrum, e.g.

$$H - E_0 = \sum_{n=0}^{\infty} a^\dagger_{m_1} \cdots a^\dagger_{m_n} |0\rangle E(m_1 \cdots m_n) \langle 0| a_{m_n} \cdots a_{m_1} \,, \tag{3.15a}$$

$$(H - E_0) a^\dagger_{m_1} \cdots a^\dagger_{m_n} |0\rangle = E(m_1 \cdots m_n) a^\dagger_{m_1} \cdots a^\dagger_{m_n} |0\rangle \,, \tag{3.15b}$$

$$E(m_1 \cdots m_n) = \sum_{i=1}^{n} \omega_{m_i} \tag{3.15c}$$

and another form of H is (3.15a) with all operators tilded.

[f] The $B = 1$ form of (3.12a) was given in footnote 8 of Ref. 18 and a number operator of this type (with $\omega = 1$) was later considered for all B in Ref. 12.

3.3. *Isotropic oscillators and angular momentum*

We consider next the special case of B isotropic oscillators ($\omega_m = \omega$),

$$H - E_0 = \omega \sum_{n=0}^{\infty} a^{\dagger}_{m_1} \cdots a^{\dagger}_{m_n} (a^{\dagger} 1 a) a_{m_n} \cdots a_{m_1} \tag{3.16}$$

for which we are also able to find the explicit nonlocal structure of the reduced spin (B) generators discussed in Subsec. 2.6. One form of the generators is

$$J_{mn} = \sum_{n=0}^{\infty} a^{\dagger}_{m_1} \cdots a^{\dagger}_{m_n} (a^{\dagger} L_{mn} a) a_{m_n} \cdots a_{m_1} \,, \tag{3.17a}$$

$$\begin{aligned} (a^{\dagger} L_{mn} a) &= a^{\dagger}_k (L_{mn})_{kl} a_l = i a^{\dagger}_{[m} a_{n]} \,, \\ (L_{mn})_{kl} &= i(\delta_{mk}\delta_{nl} - \delta_{nk}\delta_{ml}) \,, \end{aligned} \tag{3.17b}$$

$$J_{mn}|0\rangle = 0 \,, \qquad \dot{J}_{mn} = i[H, J_{mn}] = 0 \,, \tag{3.17c}$$

$$[J_{mn}, a^{\dagger}_k] = a^{\dagger}_l (L_{mn})_{lk} \,, \quad [J_{mn}, a_k] = a_l (L_{mn})_{lk} \,, \tag{3.17d}$$

$$[J_{mn}, \tilde{a}^{\dagger}_k] = \tilde{a}^{\dagger}_l (L_{mn})_{lk} \,, \quad [J_{mn}, \tilde{a}_k] = \tilde{a}_l (L_{mn})_{lk} \,, \tag{3.17e}$$

$$[J_{mn}, J_{pq}] = i(\delta_{q[m} J_{n]p} - \delta_{p[m} J_{n]q}) \,. \tag{3.17f}$$

The commutators in (3.17d) and (3.17e) are equivalent to (2.60c) and tell us that the reduced fields transform in the vector representation of spin (B). Another form of the reduced generators is obtained by replacing all the operators in (3.17a) by tilde operators, as discussed above for H.

3.4. *Algebraic identities*

We note here some generalizations of the algebraic identities above.

The reduced Hamiltonian (3.12a) and the reduced angular momentum operators (3.17a) are special cases of the family of nonlocal operators

$$\mathcal{M}(M) = \sum_{n=0}^{\infty} a^{\dagger}_{m_1} \cdots a^{\dagger}_{m_n} (a^{\dagger} M a) a_{m_n} \cdots a_{m_1} \tag{3.18a}$$

$$= \sum_{n=0}^{\infty} \tilde{a}^{\dagger}_{m_1} \cdots \tilde{a}^{\dagger}_{m_n} (\tilde{a}^{\dagger} M \tilde{a}) \tilde{a}_{m_n} \cdots \tilde{a}_{m_1} \,, \tag{3.18b}$$

$$(a^{\dagger} M a) = a^{\dagger}_m M_{mn} a_n \,, \tag{3.18c}$$

where M is any constant matrix. For each such M, we find that

$$[\mathcal{M}(M), a^{\dagger}_m] = a^{\dagger}_n M_{nm} \,, \quad [\mathcal{M}(M), a_m] = -M_{mn} a_n \,, \tag{3.19a}$$

$$[\mathcal{M}(M), \tilde{a}^{\dagger}_m] = \tilde{a}^{\dagger}_n M_{nm} \,, \quad [\mathcal{M}(M), \tilde{a}_m] = -M_{mn} \tilde{a}_n \tag{3.19b}$$

and when M and N are any two constant matrices we also find

$$[\mathcal{M}(M), \mathcal{M}(N)] = \mathcal{M}([M, N]) \tag{3.20}$$

so that the algebra of the $\mathcal{M}$'s is faithful to the algebra of the matrices. These identities include the nontrivial commutators among H and J_{mn} above, and allow, for example, the construction of general (reduced) Lie algebras when the reduced fields are in any matrix representation of the algebra.

Using the same constant matrices, we also consider a second family of nonlocal operators

$$\mathcal{M}^{\diamond}(M) = \sum_{n=0}^{\infty} a_{m_1}^{\dagger} \cdots a_{m_n}^{\dagger} (a_m^{\dagger}|0\rangle M_{mn} \langle 0|a_n)\; a_{m_n} \cdots a_{m_1} \tag{3.21}$$

which satisfy

$$\begin{aligned} [\mathcal{M}^{\diamond}(M), a_m^{\dagger}] &= a_n^{\dagger}|0\rangle\langle 0|M_{nm}\,, \\ [\mathcal{M}^{\diamond}(M), a_m] &= -M_{mn}|0\rangle\langle 0|a_n\,, \end{aligned} \tag{3.22a}$$

$$\mathcal{M}^{\diamond}(M)\mathcal{M}^{\diamond}(N) = \mathcal{M}^{\diamond}(MN)\,. \tag{3.22b}$$

We see in (3.22b) that products of the $\mathcal{M}^{\diamond}$ operators follow the matrix products; and moreover we find that the $\mathcal{M}^{\diamond}$ operators transform as

$$[\mathcal{M}(M), \mathcal{M}^{\diamond}(N)] = \mathcal{M}^{\diamond}([M, N]) \tag{3.23}$$

so they form a representation of the algebra of $\mathcal{M}$ operators above.

3.5. *Large N density-trace identifications*

In this section, we use the examples above to point out a new phenomenon at large N which we call large N density-trace identification. This phenomenon involves an unexpected relation between trace class operators (such as the Hamiltonian, the angular momenta and the supercharges) and their densities at large N, and the phenomenon constructs new nonlocal densities in the original unreduced theory, which are generically conserved only at large N.

We have seen that the reduced conserved trace class operators T of the theory have a highly nonlocal composite structure, although they are the images of local conserved trace class operators $T. = C(N)\mathrm{Tr}(t)$ in the original unreduced large N theory. Given the composite structure of any such reduced operator T, however, it is not difficult to work backward to construct a new nonlocal density class operator D_{rs} which also corresponds at large N (via the density maps of Sec. 2) to T in the reduced theory. It follows that D_{rs} is itself conserved at large N, at least in the large N Hilbert space defined by (2.9). Pictorially, we find the 2 to 1 map

$$\left.\begin{matrix} T. \text{ (local)} \\ D_{rs} \text{ (nonlocal)} \end{matrix}\right\rangle \underset{N}{\rightarrow} T \text{ (nonlocal)}\,, \tag{3.24a}$$

$$\dot{T}. = \dot{T} = 0\,, \qquad \dot{D}_{rs} \underset{N}{=} 0 \tag{3.24b}$$

in which both the conserved local trace class operator $T.$ and the large N-conserved nonlocal density D_{rs} correspond to the same conserved reduced operator T at large N. As we will see in the examples, the new nonlocal density D_{rs} can be understood as a nonlocally dressed form of the local density t_{rs} of the original trace class operator $T.$.

To illustrate this field identification phenomenon most simply, we consider the reduced Hamiltonian (3.12a) and reduced angular momentum generators (3.17a) of the B isotropic oscillators at unit frequency ($\omega_m = \omega = 1$)

$$\begin{aligned} H' \equiv H - E_0 = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} (\phi - i\pi)_{m_1} \cdots (\phi - i\pi)_{m_n} \\ \times (\pi_m \pi_m + \phi_m \phi_m + i[\phi_m, \pi_m])(\phi + i\pi)_{m_n} \cdots (\phi + i\pi)_{m_1} \,, \end{aligned} \tag{3.25a}$$

$$\begin{aligned} J_{mn} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} (\phi - i\pi)_{m_1} \cdots (\phi - i\pi)_{m_n} \\ \times (i(\phi - i\pi)_{[m} (\phi + i\pi)_{n]})(\phi + i\pi)_{m_n} \cdots (\phi + i\pi)_{m_1} \,, \end{aligned} \tag{3.25b}$$

$$\dot{H} = \dot{J}_{mn} = 0 \,, \qquad H'|0\rangle = J_{mn}|0\rangle = 0 \,, \tag{3.25c}$$

where we have used (3.4) to reexpress H and J_{mn} in terms of the time dependent reduced fields $\phi(t)$ and $\pi(t)$. These forms are easily pulled back to new unreduced densities H_{rs} and $(J_{mn})_{rs}$

$$\begin{aligned} (H)_{rs} &\equiv \sum_{n=0}^{\infty} \frac{1}{2^n} [(\phi - i\pi)^{m_1} \cdots (\phi - i\pi)^{m_n} \\ &\qquad \times (h^{(0)})(\phi + i\pi)^{m_n} \cdots (\phi + i\pi)^{m_1}]_{rs} \\ &= (h^{(0)})_{rs} + \cdots \,, \end{aligned} \tag{3.26a}$$

$$(h^{(0)})_{rs} = \frac{1}{2} (\pi^m \pi^m + \phi^m \phi^m + i[\phi^m, \pi^m])_{rs} \,, \tag{3.26b}$$

$$\begin{aligned} (J_{mn})_{rs} &\equiv \sum_{n=0}^{\infty} \frac{1}{2^n} [(\phi - i\pi)^{m_1} \cdots (\phi - i\pi)^{m_n} \\ &\qquad \times (j^{(0)}_{mn})(\phi + i\pi)^{m_n} \cdots (\phi + i\pi)^{m_1}]_{rs} \\ &= (j^{(0)}_{mn})_{rs} + \cdots \,, \end{aligned} \tag{3.26c}$$

$$(j^{(0)}_{mn})_{rs} = \frac{i}{2} \big(\phi^{[m} \phi^{n]} + \pi^{[m} \pi^{n]} + i\big(\phi^{[m} \pi^{n]} - \pi^{[m} \phi^{n]}\big)\big)_{rs} \,, \tag{3.26d}$$

$$(\dot{H})_{rs} = (\dot{j}_{mn})_{rs} = 0 \,, \quad H_{rs}|0.\rangle = (J_{mn})_{rs}|0.\rangle = 0 \tag{3.26e}$$

which also correspond at large N (via the density maps of Sec. 2.4) to the same reduced operators H' and J_{mn}. Our construction guarantees that these new densities are conserved at large N, since they map to the conserved reduced trace class operators in this limit. (Oscillator examples are special in that the new densities are conserved at all N.)

To see that these new densities are nonlocally dressed forms of the original energy and angular momentum densities of the theory, note that the first terms of the new densities satisfy

$$\mathrm{Tr}(h^{(0)}) = \frac{1}{2}\mathrm{Tr}(\pi^m\pi^m + \phi^m\phi^m) - E_0 = H_{\boldsymbol{\cdot}} - E_0\,, \tag{3.27a}$$

$$\mathrm{Tr}(j^{(0)}_{mn}) = \mathrm{Tr}(\pi^{[m}\phi^{n]}) = J^{mn}_{\boldsymbol{\cdot}}\,, \tag{3.27b}$$

where $H_{\boldsymbol{\cdot}}$ and $J^{mn}_{\boldsymbol{\cdot}}$ are the original Hamiltonian and angular momenta.

Finding new conserved nonlocal quantities in oscillator theories is never surprising, but these new densities are important quantities in the large N theory since they map onto the important reduced trace class operators. Moreover, this field identification phenomenon is apparently *universal* for each conserved trace class operator in any large N theory, given the explicit composite structure of the reduced trace class operator. For more general matrix models, one expects that these new nonlocal densities are generically conserved only at large N, and only in the large N Hilbert space defined by (2.9). We will return to this phenomenon for oscillator supercharges and supercharge densities in Subsec. 5.3 (see also Subsec. 4.6).

3.6. *Coordinate bases and the rank of the equal-time algebras*

The oscillators also allow us to make some useful comments about the rank of the equal-time algebras and the corresponding coordinate bases.

For $B = 1$, the rank of the equal-time algebra (2.62b) is 1, and we may construct the coordinate eigenstates explicitly for the oscillator:

$$B = 1\,, \qquad \phi|q\rangle = q|q\rangle\,, \tag{3.28a}$$

$$|q\rangle = C_1 \sum_{m=0}^{\infty} \frac{\sin\big((m+1)\theta_q\big)}{\sqrt{\sin\theta_q}}(a^\dagger)^m|0\rangle\,, \tag{3.28b}$$

$$\sqrt{2}\cos\theta_q = q\,, \qquad \mathbf{1} = \int dq|q\rangle\langle q|\,. \tag{3.28c}$$

The coordinate eigenstates are complete in this case, and similarly complete coordinate bases provide the starting point for the solution[18] of the general one-matrix model.

For $B \geq 2$, we find that the rank of the equal-time algebra (2.51a)–(2.51c) is 2. Choosing the commuting set as ϕ_1 and $\tilde{\phi}_2$, we may again construct the coordinate eigenstates explicitly for the oscillators:

$$B \geq 2\,, \qquad [\phi_1, \tilde{\phi}_2] = 0\,, \tag{3.29a}$$

$$\phi_1|xy\rangle = x|xy\rangle\,, \qquad \tilde{\phi}_2|xy\rangle = y|xy\rangle\,, \tag{3.29b}$$

$$|xy\rangle = C_2 \sum_{m,n=0}^{\infty} \frac{\sin\big((m+1)\theta_x\big)}{\sqrt{\sin\theta_x}}\frac{\sin\big((n+1)\theta_y\big)}{\sqrt{\sin\theta_y}}(a_1^\dagger)^m(a_2^\dagger)^n|0\rangle\,, \tag{3.29c}$$

$$\mathbf{1} = \int dx\,dy|xy\rangle\langle xy| + \Delta\,. \tag{3.29d}$$

In this case, the coordinate eigenstates are explicitly not complete, since they have no overlap with more complicated words such as $a_1^\dagger a_2^\dagger a_1^\dagger|0\rangle$. It follows that the coordinate-basis approach of Ref. 18 cannot be extended to matrix models with $B \geq 2$.

4. Interacting Symmetric Cuntz Algebras

The symmetric Cuntz algebras of Sec. 3 arose in the context of large N oscillators, which may be considered to be free theories. In this section, we find the generalization of these algebras for arbitrary interactions, which we call interacting symmetric Cuntz algebras. The final form of these algebras, and their associated new large N-conserved quantities, are found in Eq. (4.36) and Subsec. 4.5 respectively.

4.1. *Generalized creation and annihilation operators*

We begin with the invariant, real and nodeless ground state wave function of the general bosonic system

$$\psi_0(\phi) = \langle\phi|0.\rangle\,, \qquad \phi = \{\phi_{rs}^m\} \tag{4.1}$$

in the coordinate basis of the unreduced theory. The explicit form of the ground state will not be required in this construction. Operating with the matrix momenta defines a set of matrix-valued functions $F^m(\phi)$,

$$i\pi_{rs}^m\psi_0(\phi) = \frac{\partial}{\partial\phi_{sr}^m}\psi_0(\phi) = -F_{rs}^m(\phi)\psi_0(\phi)\,, \qquad \big(F_{rs}^m(\phi)\big)^\dagger = F_{sr}^m(\phi) \tag{4.2}$$

which lead us to generalized matrix creation and annihilation operators

$$A_{rs}^m \equiv \frac{1}{\sqrt{2}}(F_{rs}^m(\phi) + i\pi_{rs}^m)\,, \quad (A^{m\dagger})_{rs} \equiv \frac{1}{\sqrt{2}}(F_{rs}^m(\phi) - i\pi_{rs}^m)\,, \tag{4.3a}$$

$$A_{rs}^m\psi_0(\phi) = 0\,, \qquad\qquad \psi_0(\phi)(A^{m\dagger})_{rs} = 0 \tag{4.3b}$$

for any interaction. A useful property of this system is

$$0 = [A_{pq}^m, A_{rs}^n]\psi_0(\phi) = \frac{i}{2}([\pi_{pq}^m, F_{rs}^n] - [\pi_{rs}^n, F_{pq}^m])\psi_0(\phi)\,, \tag{4.4a}$$

$$[\pi_{pq}^m, F_{rs}^n(\phi)] + [F_{pq}^m(\phi), \pi_{rs}^n] = 0\,, \tag{4.4b}$$

where (4.4b), which may be considered as the ground state integrability condition, follows from (4.4a) because the ground state is nodeless. Further discussion of these operators in the unreduced theory is found in App. E, where it is also shown that F, A and $A^\dagger$ may be considered as densities at large N. Here we go directly to the reduced theory at large N.

Following the line of the canonical maps in Subsec. 2.4, we find first that

$$[\tilde{\pi}_m, F_n(\phi)] - [\pi_n, \tilde{F}_m(\tilde{\phi})] = 0\,, \tag{4.5a}$$

$$F_m^\dagger(\phi) = F_m(\phi)\,, \quad \tilde{F}_m^\dagger(\tilde{\phi}) = \tilde{F}_m(\tilde{\phi})\,, \quad \tilde{F}_m(\tilde{\phi})|0\rangle = F_m(\phi)\,,|0\rangle\,, \tag{4.5b}$$

where the reduced operators F_m, $\tilde{F}_m$ are the images of F^m_{rs}. The result (4.5a), which is the image of (4.4b), tells us that the pair $F = \{F_m, \tilde{F}_m\}$ comprises a flat connection, $\mathcal{F}_{mn}(F) = 0$, on the reduced Hilbert space. It follows that F_m and $\tilde{F}_m$ have the form (see Subsec. 2.6)

$$\begin{aligned} F_m(\phi) &= \sum_{n=1}^{\infty} f^{(n)}_{mm_2\cdots m_n} \phi_{m_2} \cdots \phi_{m_n}\,, \\ \tilde{F}_m(\tilde{\phi}) &= \sum_{n=1}^{\infty} f^{(n)}_{mm_2\cdots m_n} \tilde{\phi}_{m_n} \cdots \tilde{\phi}_{m_2}\,, \end{aligned} \tag{4.6a}$$

$$f^{(n)}_{m_1\cdots m_n} = f^{(n)}_{m_2\cdots m_n m_1}\,, \qquad f^{(n)*}_{m_1\cdots m_n} = f^{(n)}_{m_n\cdots m_1}\,, \tag{4.6b}$$

where the as yet undetermined coefficients f are cyclically symmetric in their lower indices. The reality condition in (4.6b) follows from (4.5b).

We continue with the reduced creation and annihilation operators[g]

$$A_m = \frac{1}{\sqrt{2}}(F_m + i\pi_m)\,, \qquad A^\dagger_m = \frac{1}{\sqrt{2}}(F_m - i\pi_m)\,, \tag{4.7a}$$

$$\tilde{A}_m = \frac{1}{\sqrt{2}}(\tilde{F}_m + i\tilde{\pi}_m)\,, \qquad \tilde{A}^\dagger_m = \frac{1}{\sqrt{2}}(\tilde{F}_m - i\tilde{\pi}_m)\,, \tag{4.7b}$$

$$A_m|0\rangle = \tilde{A}_m|0\rangle = \langle 0|A^\dagger_m = \langle 0|\tilde{A}^\dagger_m = 0\,, \tag{4.7c}$$

$$\tilde{A}^\dagger_m|0\rangle = A^\dagger_m|0\rangle\,, \qquad \langle 0|\tilde{A}_m = \langle 0|A_m \tag{4.7d}$$

which are the images of the matrix creation and annihilation operators in (4.3). The state $|0\rangle$ is the reduced ground state of the interacting system. The equal-time algebra of these operators

$$[A_m, \tilde{A}_n] = [A^\dagger_m, \tilde{A}^\dagger_n] = 0\,, \tag{4.8a}$$

$$[A_m, \tilde{A}^\dagger_n] = [\tilde{A}_n, A^\dagger_m] = i[\tilde{\pi}_n, F_m] = i[\pi_m, \tilde{F}_n] \tag{4.8b}$$

follows directly from the equal-time algebra (2.51) and the flatness condition (4.5a). The relations (4.7c) and (4.8a) tell us that mixed words involving both $A^\dagger$'s and $\tilde{A}^\dagger$'s on the vacuum can be rewritten in terms of only $A^\dagger$'s or only $\tilde{A}^\dagger$'s. We will argue in Subsec. 4.4 that both sets of states are complete, at least for potentials in some neighborhood of the oscillator potential.

The equal-time algebra (2.51) also allows us to compute

$$[\tilde{\phi}_m, A_n] = [\phi_m, \tilde{A}_n] = -\frac{1}{\sqrt{2}}\delta_{mn}|0\rangle\langle 0|\,, \tag{4.9a}$$

$$[\tilde{\phi}_m, A^\dagger_n] = [\phi_m, \tilde{A}^\dagger_n] = \frac{1}{\sqrt{2}}\delta_{mn}|0\rangle\langle 0|\,, \tag{4.9b}$$

$$[\tilde{\phi}_l, A_m A^\dagger_n] = [\phi_l, \tilde{A}_m \tilde{A}^\dagger_n] = 0\,. \tag{4.9c}$$

[g]Using (2.37), the relation $A_m|0\rangle = 0$ can be written as $(\tilde{\pi}_m - iF_m(\phi))|0\rangle = 0$. According to the remark below (2.36), this is the Hamiltonian analogue of Haan's Euclidean equation of motion.[7,12]

Equation (4.9c) strongly suggests[h] that the products $AA^\dagger$ and $\tilde{A}\tilde{A}^\dagger$ are equal to functions of the reduced operators ϕ and $\tilde{\phi}$ respectively, and this intuition is confirmed in App. E. We will call these unknown functions C and $\tilde{D}$:

$$A_m A_n^\dagger = C_{mn}(\phi)\,, \qquad \tilde{A}_m \tilde{A}_n^\dagger = \tilde{D}_{mn}(\tilde{\phi})\,, \tag{4.10a}$$

$$C_{mn}^\dagger = C_{nm}\,, \qquad \tilde{D}_{mn}^\dagger = \tilde{D}_{nm}\,. \tag{4.10b}$$

The relations (4.10a) comprise two copies of a generalized free algebra for arbitrary interaction.

The functions C and $\tilde{D}$ are closely related. To see this, we first evaluate the functions on the ground state using (4.7), (4.8) and (4.10):

$$C_{mn}(\phi)|0\rangle = \tilde{D}_{nm}(\tilde{\phi})|0\rangle = i[\tilde{\pi}_n, F_m]|0\rangle = i[\pi_m, \tilde{F}_n]|0\rangle\,. \tag{4.11}$$

The first relation in (4.11) is easily solved as

$$C_{mn}(\phi) = \sum_{q=2}^{\infty} C^{(q)}_{mnm_3\cdots m_q}\phi_{m_3}\cdots\phi_{m_q}\,, \tag{4.12a}$$

$$\tilde{D}_{mn}(\tilde{\phi}) = \sum_{q=2}^{\infty} C^{(q)}_{nmm_3\cdots m_q}\tilde{\phi}_{m_q}\cdots\tilde{\phi}_{m_3} = \tilde{C}_{nm}(\tilde{\phi})\,, \tag{4.12b}$$

$$C_{mn}(\phi) = D_{nm}(\phi)\,, \tag{4.12c}$$

$$C^{(q)*}_{mnm_3\cdots m_q} = C^{(q)}_{nmm_q\cdots m_3}\,, \tag{4.12d}$$

where the coefficients $C^{(q)}$ are so far undetermined and the reality condition in (4.12d) follows from (4.10b). The other relations in (4.11) will be helpful in computing the explicit forms of C and D below.

We turn now to some explicit computations involving the new operators, returning to formal developments, including completeness, in Subsec. 4.4.

4.2. *Starting from $F_m(\phi)$*

Given the reduced operators F_m one can in principle compute the potential of the system, as well as the commutators (4.8b) and the functions C and D which enter into the generalized free algebras (4.10).

We begin with the relations

$$0 = \dot{A}_m|0\rangle = (\dot{F}_m - iV_m')|0\rangle\,, \tag{4.13a}$$

$$\begin{aligned} F_m &= \sum_{n=1}^{\infty} f^{(n)}_{mm_2\cdots m_n}\phi_{m_2}\cdots\phi_{m_n}\,, \\ V_m' &= \sum_{n=1}^{\infty} v^{(n)}_{mm_2\cdots m_n}\phi_{m_2}\cdots\phi_{m_n}\,, \end{aligned} \tag{4.13b}$$

[h]One may conjecture that any reduced operator X which obeys $[\tilde{\phi}_m, X] = 0, \forall m$ may be expressed as $X = X(\phi)$.

where (4.13a) follows from (4.7c) by the equations of motion (2.52a). Consider the relation obtained by substitution of the forms (4.13b) into (4.13a), remembering that $\dot{\phi} = \pi$. Using the equal-time algebra (2.51), it is not difficult to work out a sequence of relations beginning with

$$\pi_m|0\rangle = iF_m(\phi)|0\rangle\,, \tag{4.14a}$$

$$\begin{aligned}\pi_m\phi_n|0\rangle &= [\pi_m, \tilde{\phi}_n]|0\rangle + i\tilde{\phi}_n F_m(\phi)|0\rangle \\ &= i(-\delta_{mn} + F_m(\phi)\phi_n)|0\rangle\end{aligned} \tag{4.14b}$$

which can be used to eliminate all π operators and rewrite the relation (4.13a) in terms of the ϕ operators alone. The coefficient of each ϕ monomial must vanish separately, allowing us to compute V in terms of F. We record the results of this computation including the f coefficients through $n = 3$:

$$v_m^{(1)} = f_{mn}^{(2)} f_n^{(1)} - f_{mnn}^{(3)} + \cdots\,, \tag{4.15a}$$

$$v_{mn}^{(2)} = f_{mp}^{(2)} f_{pn}^{(2)} + \left(f_{mpn}^{(3)} + f_{npm}^{(3)}\right) f_p^{(1)} + \cdots\,, \tag{4.15b}$$

$$v_{mnp}^{(3)} = f_{mq}^{(2)} f_{qnp}^{(3)} + f_{pq}^{(2)} f_{qmn}^{(3)} + f_{nq}^{(2)} f_{qpm}^{(3)} + \cdots\,, \tag{4.15c}$$

$$v_{mnpq}^{(4)} = f_{mnr}^{(3)} f_{rpq}^{(3)} + f_{qmr}^{(3)} f_{rnp}^{(3)} + \cdots\,, \tag{4.15d}$$

$$v_{mnpqr}^{(5)} = \cdots\,, \tag{4.15e}$$

where the dots indicate the contributions of $f^{(n)}$, $n \geq 4$. The symmetries of the v coefficients in (2.50c) are guaranteed by the symmetries of the f coefficients in (4.6b). This means that $V' = \{V'_m, \tilde{V}'_m\}$ is a flat connection, as it should be, when $F = \{F_m, \tilde{F}_m\}$ is a flat connection. Although we will not present the proof here, we have checked that this statement is true to all orders in the expansions (4.13b).

We turn next to the evaluation of the functions C and $\tilde{D}$ in the generalized free algebras (4.10a). First evaluate the commutators (4.8b)

$$\begin{aligned}[A_m, \tilde{A}_n^\dagger] &= [\tilde{A}_n, A_m^\dagger] = i[\tilde{\pi}_n, F_m] \\ &= f_{mn}^{(2)}|0\rangle\langle 0| + f_{mnp}^{(3)}|0\rangle\langle 0|\phi_p + f_{mpn}^{(3)}\phi_p|0\rangle\langle 0| + \cdots\end{aligned} \tag{4.16}$$

through this order in the f coefficients, using (2.35a) and (4.13b). For C and $\tilde{D}$, we use (4.11), (4.12c) and (4.16) to evaluate

$$C_{mn}(\phi)|0\rangle = i[\tilde{\pi}_n, F_m]|0\rangle\,, \tag{4.17a}$$

$$C_{mn}(\phi) = f_{mn}^{(2)} + f_{mnp}^{(3)}\langle 0|\phi_p|0\rangle + f_{mpn}^{(3)}\phi_p + \cdots\,, \tag{4.17b}$$

$$\tilde{D}_{mn}(\tilde{\phi}) = f_{nm}^{(2)} + f_{nmp}^{(3)}\langle 0|\phi_p|0\rangle + f_{npm}^{(3)}\tilde{\phi}_p + \cdots\,, \tag{4.17c}$$

where the result (4.17b) is obtained from (4.17a) by eliminating vacuum projectors to obtain a function of ϕ only on the vacuum. Another form of $\tilde{D}$ is found in Eq. (F.5).

As a simple check on the results above, we note the special case of the anharmonic oscillators

$$F_m = \omega_m \phi_m \,, \quad V'_m = \omega_m^2 \phi_m \,, \quad C_{mn} = \omega_m \delta_{mn} \,, \tag{4.18a}$$
$$\tilde{F}_m = \omega_m \tilde{\phi}_m \,, \quad \tilde{V}'_m = \omega_m^2 \tilde{\phi}_m \,, \quad D_{mn} = \omega_m \delta_{mn} \tag{4.18b}$$

for which the rescaled operators

$$\{a_m, a_m^\dagger, \tilde{a}_m, \tilde{a}_m^\dagger\} \equiv \{A_m, A_m^\dagger, \tilde{A}_m, \tilde{A}_m^\dagger\}/\sqrt{\omega_m} \tag{4.19}$$

are seen to satisfy the symmetric Cuntz algebra (3.10).

4.3. *Basis-independent analysis of the one-matrix model*

We have noted in Subsec. 3.6 that the coordinate-basis approach of Ref. 18 cannot be extended to the case of many matrices. Here, we develop a basis-independent approach to the general one-matrix model which constructs the generalized creation and annihilation operators $A^\dagger$, A as well as the exact form of $C(\phi)$ in their interacting Cuntz algebra

$$AA^\dagger = C(\phi) \,. \tag{4.20}$$

This approach is in principle extendable to many matrices, although we will confine ourselves here to preliminary remarks in this direction. (In this subsection only, we use boldface $\boldsymbol{\pi}$ for the momentum operators, to distinguish them from the number π.)

We begin with the relations

$$\boldsymbol{\pi}|0\rangle = iF|0\rangle \,, \quad \langle 0|\boldsymbol{\pi} = -i\langle 0|F \,, \tag{4.21a}$$
$$\left[i\boldsymbol{\pi}, \frac{1}{z-\phi}\right] = \frac{1}{z-\phi}|0\rangle\langle 0|\frac{1}{z-\phi} \,, \quad \mathrm{Im}\, z > 0 \,, \tag{4.21b}$$
$$\langle 0|\left[F(\phi), \frac{1}{z-\phi}\right]_+ |0\rangle = \langle 0|\frac{1}{z-\phi}|0\rangle^2 \,, \tag{4.21c}$$

where (4.21c) follows from the vacuum properties (4.21a) and the identity (4.21b). We intend to let the complex variable z approach the real axis $z \to q + i\epsilon$, where we will need the following facts ($\mathcal{P}$ is principal value)

$$\frac{1}{q-\phi+i\epsilon} = \frac{\mathcal{P}}{q-\phi} - i\pi\delta(q-\phi) \,, \tag{4.22a}$$
$$\frac{\mathcal{P}}{q-a}\frac{\mathcal{P}}{q-b} = \frac{\mathcal{P}}{a-b}\left(\frac{\mathcal{P}}{q-a} - \frac{\mathcal{P}}{q-b}\right) + \pi^2\delta(q-a)\delta(q-b) \,. \tag{4.22b}$$

We also define the ground state density function $\rho(q)$ and the function $F(q)$

$$\rho(q) \equiv \langle 0|\delta(q-\phi)|0\rangle \geq 0 \,, \quad \int dq\, \rho(q) = 1 \,, \tag{4.23a}$$
$$F(q) \equiv \rho(q)^{-1}\langle 0|F(\phi)\delta(q-\phi)|0\rangle \,, \tag{4.23b}$$

where the latter is just $F(\phi)$ with ϕ replaced by q.

Letting z approach the real axis, we find

$$F(q) = \langle 0|\frac{\mathcal{P}}{q-\phi}|0\rangle = \int dq'\, \frac{\mathcal{P}}{q-q'}\rho(q')\,, \tag{4.24a}$$

$$\langle 0|F(\phi)\frac{\mathcal{P}}{q-\phi}|0\rangle = \frac{1}{2}F^2(q) - \frac{\pi^2}{2}\rho^2(q) \tag{4.24b}$$

from the imaginary and real parts respectively of (4.21c). The result in (4.24a) gives $F(q)$ and the generalized creation and annihilation operators

$$\begin{aligned} A &= \frac{1}{\sqrt{2}}\left(\int dq \frac{\mathcal{P}}{\phi-q}\rho(q) + i\boldsymbol{\pi}\right), \\ A^\dagger &= \frac{1}{\sqrt{2}}\left(\int dq\, \frac{\mathcal{P}}{\phi-q}\rho(q) - i\boldsymbol{\pi}\right) \end{aligned} \tag{4.25}$$

in terms of the ground state density ρ.

The result in (4.24a) can also be used to compute $C(\phi)$ in terms of ρ:

$$\begin{aligned} F(\phi) &= \mathrm{Re}\int dq \frac{1}{\phi-q-i\epsilon}\rho(q)\,, \\ C(\phi)|0\rangle &= i[\boldsymbol{\pi}, F(\phi)]|0\rangle\,, \end{aligned} \tag{4.26a}$$

$$i[\boldsymbol{\pi}, F(\phi)]|0\rangle = -\mathrm{Re}\int dq \frac{1}{\phi-q-i\epsilon}\rho(q)|0\rangle\langle 0|\frac{1}{\phi-q-i\epsilon}|0\rangle\,, \tag{4.26b}$$

$$C(\phi) = -\int dq\, dq'\, \rho(q)\rho(q')\frac{\mathcal{P}}{q-q'}\frac{\mathcal{P}}{q-\phi} + \pi^2\rho^2(\phi)\,. \tag{4.26c}$$

Here (4.26b) follows from (4.26a) and (4.21b), while (4.26c) follows from (4.26a) by rearranging (4.26b) into a function of ϕ on the vacuum. A final form for $C(\phi)$

$$C(\phi) = \frac{1}{2}\big(F^2(\phi) + \pi^2\rho^2(\phi)\big) \geq 0 \tag{4.27}$$

is obtained by symmetrizing the double integral in (4.26c) and using (4.22b).

This completes the first stage of the analysis, in which we have expressed the new operators F, A, $A^\dagger$ and $C(\phi)$ in terms of the ground state density ρ.

In the second stage, we evaluate the ground state density ρ in terms of the reduced potential V of the system. We begin this stage with the identity

$$\begin{aligned} \langle 0|\frac{1}{z-\phi}V'(\phi)|0\rangle &= -\langle 0|\frac{1}{z-\phi}\dot{\boldsymbol{\pi}}|0\rangle = \langle 0|\left(\frac{d}{dt}\frac{1}{z-\phi}\right)\boldsymbol{\pi}|0\rangle \\ &= \langle 0|\frac{1}{z-\phi}\boldsymbol{\pi}\frac{1}{z-\phi}\boldsymbol{\pi}|0\rangle = \langle 0|F\frac{1}{(z-\phi)^2}F|0\rangle \\ &\quad - \langle 0|\frac{1}{z-\phi}|0\rangle\langle 0|\frac{1}{(z-\phi)^2}F|0\rangle\,, \end{aligned} \tag{4.28}$$

where we have used the fact that $\langle 0|\dot{A}|0\rangle = 0$ for any A. Letting z approach the real axis, taking the imaginary part and using $(z-\phi)^{-2} = -\partial_z(z-\phi)^{-1}$ along with previous formulas then allows us to compute ρ as a function of V

$$V'(q) = -\pi^2\rho(q)\rho'(q) \to \rho(q) = \frac{1}{\pi}\sqrt{2\big(\epsilon - V(q)\big)}\,, \tag{4.29}$$

where the constant ϵ is determined by the normalization condition in (4.23a). The ground state density ρ is the same function introduced in Ref. 18.

Although a complete discussion is beyond the scope of this paper, the basis-independent analysis above can be extended to many matrices, beginning with the extension of (4.21b),

$$\begin{aligned} B \geq 2:\ i&\left[\tilde{\boldsymbol{\pi}}_k, \frac{1}{z_{m_1}-\phi_{m_1}}\cdots\frac{1}{z_{m_n}-\phi_{m_n}}\right] \\ &= \sum_{i=1}^{n}\delta_{k,m_i}\frac{1}{z_{m_1}-\phi_{m_1}}\cdots\frac{1}{z_{m_i}-\phi_{m_i}}|0\rangle\langle 0|\frac{1}{z_{m_i}-\phi_{m_i}}\cdots\frac{1}{z_{m_n}-\phi_{m_n}} \end{aligned} \tag{4.30}$$

and the corresponding extension of (4.28).

4.4. *"Ordinary" Cuntz algebras and completeness in interacting theories*

In Subsecs. 4.1–4.3, we have constructed generalized creation and annihilation operators which satisfy generalized free algebras in interacting theories, but we have not yet discussed completeness for these operators. Here we note first that ordinary Cuntz algebras can, under certain technical assumptions, be constructed in the interacting theories as well, and this will help us understand completeness in the case of the generalized operators.

The form of the generalized free algebra in (4.10a) guarantees that C_{mn} and $\tilde{D}_{mn}$ are nonnegative operators. The results of Subsec. 4.2 show that C_{mn} and $\tilde{D}_{mn}$ are in fact positive operators at least where the potential of the interacting theory is in some (say perturbative) neighborhood of the oscillator potential. Moreover (4.27) shows that C is positive for almost all one-matrix models. The following discussion is limited to the broad class of theories for which these operators are strictly positive

$$C(\phi)\,,\quad \tilde{D}(\tilde{\phi}) > 0 \tag{4.31}$$

although we do not yet have a complete characterization of these theories in terms of the potential.

For this class of theories, we can construct the "ordinary" Cuntz operators

$$a_m \equiv \big(C^{-\frac{1}{2}}\big)_{mn}A_n\,,\qquad a_m^\dagger \equiv A_n^\dagger\big(C^{-\frac{1}{2}}\big)_{nm}\,, \tag{4.32a}$$

$$\tilde{a}_m \equiv \big(\tilde{D}^{-\frac{1}{2}}\big)_{mn}\tilde{A}_n\,,\qquad \tilde{a}_m^\dagger \equiv \tilde{A}_n^\dagger\big(\tilde{D}^{-\frac{1}{2}}\big)_{nm} \tag{4.32b}$$

which satisfy the symmetric free algebra

$$a_m a_n^\dagger = \tilde{a}_m \tilde{a}_n^\dagger = \delta_{mn}\,, \tag{4.33a}$$

$$a_m|0\rangle = \tilde{a}_m|0\rangle = \langle 0|a_m^\dagger = \langle 0|\tilde{a}_m^\dagger = 0 \tag{4.33b}$$

as a consequence of (4.7) and (4.10). Using (4.7d), (4.8a) and (4.32), other relations, such as the mixed commutators $[a_m, \tilde{a}_n]$, $[a_m^\dagger, \tilde{a}_n^\dagger]$, can also be computed in terms of C, $\tilde{D}$ and the Cuntz operators.

The states formed by the $a^\dagger$'s or the $\tilde{a}^\dagger$'s on the vacuum should be complete at least in some neighborhood of the oscillator potential, and the expected completeness relations for the Cuntz operators

$$a_m^\dagger a_m = \tilde{a}_m^\dagger \tilde{a}_m = 1 - |0\rangle\langle 0| \tag{4.34}$$

follow with (4.33) by operating on the complete set of $a^\dagger$ or $\tilde{a}^\dagger$ states.

The results (4.32)–(4.34) complete the construction of a symmetric pair of Cuntz algebras in the interacting theories. Our construction is in agreement with the complementary discussion of Ref. 20, which assumed the existence of the (untilde) Cuntz algebra for potentials in a perturbative neighborhood of the oscillator.

Returning to the generalized creation and annihilation operators, we can now show that the completeness of the $a^\dagger$ or $\tilde{a}^\dagger$ states is equivalent to the completeness of the generalized $A^\dagger$ or $\tilde{A}^\dagger$ states: The corresponding completeness relations for the generalized creation and annihilation operators

$$A_m^\dagger (C^{-1})_{mn} A_n = \tilde{A}_m^\dagger (\tilde{D}^{-1})_{mn} \tilde{A}_n = 1 - |0\rangle\langle 0| \tag{4.35}$$

follow immediately from (4.34) and (4.10a). These relations are also obtained by studying the action of the left hand sides of (4.35) on the $A^\dagger$ or $\tilde{A}^\dagger$ states, implying the completeness of these sets of states as well. The argument can easily be run backward, so that all four types $(a^\dagger, \tilde{a}^\dagger, A^\dagger, \tilde{A}^\dagger)$ of completeness are equivalent.

For reference we collect here the final form of our *generalized or interacting symmetric Cuntz algebras*

$$A_m A_n^\dagger = C_{mn}(\phi)\,, \qquad \tilde{A}_m \tilde{A}_n^\dagger = \tilde{D}_{mn}(\tilde{\phi})\,, \tag{4.36a}$$

$$D_{mn}(\phi) = C_{nm}(\phi)\,, \tag{4.36b}$$

$$[A_m, \tilde{A}_n] = [A_m^\dagger, \tilde{A}_n^\dagger] = 0\,, \tag{4.36c}$$

$$[A_m, \tilde{A}_n^\dagger] = [\tilde{A}_n, A_m^\dagger] = i[\tilde{\pi}_n, F_m] = i[\pi_m, \tilde{F}_n]\,, \tag{4.36d}$$

$$A_m^\dagger (C^{-1})_{mn} A_n = \tilde{A}_m^\dagger (\tilde{D}^{-1})_{mn} \tilde{A}_n = 1 - |0\rangle\langle 0|\,, \tag{4.36e}$$

$$A_m|0\rangle = \tilde{A}_m|0\rangle = \langle 0|A_m^\dagger = \langle 0|\tilde{A}_m^\dagger = 0\,, \tag{4.36f}$$

$$\tilde{A}_m^\dagger|0\rangle = A_m^\dagger|0\rangle\,, \qquad \langle 0|\tilde{A}_m = \langle 0|A_m \tag{4.36g}$$

which includes (4.35), the results of Subsec. 4.1 and assumes (4.31). For the special case of the oscillators (see (4.18) and (4.19)), these generalized free algebras reduce to the symmetric Cuntz algebra (3.10).

The interacting symmetric Cuntz algebras (4.36) are a central result of this paper. We turn now to two applications of these algebras.

4.5. *New local conserved quantities at large N*

Our first application is simple but quite remarkable. The interacting symmetric Cuntz algebras (4.36) and their associated "ordinary" Cuntz algebras (4.32)–(4.34) imply *new local conserved quantities at large* N:

$$\dot{\mathcal{J}}_{mn} = \dot{\mathcal{J}} = 0\,, \qquad m,\ n = 1 \cdots B\,, \tag{4.37a}$$

$$\begin{aligned}\mathcal{J}_{mn} &\equiv a_m a_n^\dagger = \left(C^{-\frac{1}{2}}\right)_{mp} A_p A_q^\dagger \left(C^{-\frac{1}{2}}\right)_{qn}\,,\\ &= \frac{1}{2}\left[\left(C^{-\frac{1}{2}}(\phi)\right)_{mp}\left(F_p(\phi) + i\pi_p\right)\left(F_q(\phi) - i\pi_q\right)\left(C^{-\frac{1}{2}}(\phi)\right)_{qn}\right]\,,\end{aligned} \tag{4.37b}$$

$$\begin{aligned}\mathcal{J} &\equiv a_m^\dagger a_m = A_m^\dagger (C^{-1})_{mn} A_n\\ &= \frac{1}{2}\left[\left(F_m(\phi) - i\pi_m\right)\left(C^{-1}(\phi)\right)_{mn}\left(F_n(\phi) + i\pi_n\right)\right]\,,\end{aligned} \tag{4.37c}$$

$$\mathcal{J}|0\rangle = 0\,, \qquad \mathcal{J}^\dagger = \mathcal{J} \tag{4.37d}$$

for all bosonic theories with $C,\ D > 0$. Similarly conserved operators $\tilde{\mathcal{J}}_{mn}$ and $\tilde{\mathcal{J}}$ are constructed by replacing $C \to \tilde{D}$ and each of the other operators by their tilde form.

The conservation of $\mathcal{J}_{mn}$ in (4.37) understates the information we have because we also know that

$$\mathcal{J}_{mn} = \delta_{mn} \leftrightarrow \left(A_m A_n^\dagger - C_{mn}(\phi)\right) = 0 \tag{4.38}$$

which is properly interpreted as a set of B^2 local constraints at large N.

For the one-matrix model, the results of Subsec. 4.3,

$$F(\phi) = \int dq \frac{\mathcal{P}}{\phi - q}\rho(q)\,, \qquad C(\phi) = \frac{1}{2}\left(F^2(\phi) + \pi^2\rho^2(\phi)\right)\,, \tag{4.39a}$$

$$\rho(\phi) = \frac{1}{\pi}\sqrt{2\left(\epsilon - V(\phi)\right)}\,, \qquad \int dq\, \rho(q) = 1 \tag{4.39b}$$

give these new large N-conserved quantities in closed form, and it is possible in principle to evaluate (4.37b) and (4.37c) for higher B to any desired order in the coefficients $v^{(n)}$ of the potential [see Eq. (2.50)]. Explicit forms of these reduced quantities can be pulled back (as in Subsec. 3.5) into hidden local (second order in momenta) but nonpolynomial (in coordinates) unreduced densities $(\mathcal{J}_{mn})_{rs}$ and $(\mathcal{J})_{rs}$ which are conserved only at large N, and only in the large N Hilbert space of (2.9).

These new large N-conserved quantities are another central result of this paper, since they apparently realize an old dream of hidden local conserved quantities in quantum field theory.

4.6. *General reduced Hamiltonian*

The interacting symmetric Cuntz algebras (4.36) and their associated "ordinary" Cuntz algebras (4.32)–(4.34) also allow us in principle to construct the general reduced Hamiltonian for bosonic systems with C, $D > 0$.

Following Subsecs. 3.2 and 3.4, we consider a fitting procedure based on a family of nonlocal reduced Hamiltonians,

$$H' = H - E_0 = \sum_{n=0}^{\infty} a_{m_1}^\dagger \cdots a_{m_n}^\dagger h(a, a^\dagger) a_{m_n} \cdots a_{m_1} \,, \tag{4.40a}$$

$$H'|0\rangle = h(a, a^\dagger)|0\rangle \,, \tag{4.40b}$$

$$\begin{aligned} \dot{a}_m^\dagger &= i[H', a_m^\dagger] = i h(a, a^\dagger) a_m^\dagger \,, \\ \dot{a}_m &= i[H', a_m] = -i a_m h(a, a^\dagger) \,, \end{aligned} \tag{4.40c}$$

$$\pi_m = \frac{i}{\sqrt{2}} \left(a_n^\dagger \left(C^{\frac{1}{2}}\right)_{nm} - \left(C^{\frac{1}{2}}\right)_{mn} a_n \right) , \tag{4.40d}$$

where the arbitrary Hermitian operator $h(a, a^\dagger)$ is to be determined in terms of the potential. According to the Cuntz algebra, a formal solution of (4.40c) is

$$h = h^\dagger = i a_m^\dagger \dot{a}_m = -i \dot{a}_m^\dagger a_m \,, \tag{4.41a}$$

$$H'|0\rangle = h|0\rangle = 0 \tag{4.41b}$$

and we can make contact with the theory in question by using (4.32) to reexpress this system in terms of the interacting Cuntz operators. This gives in particular the useful form of h:

$$\begin{aligned} h &= i\left(A^\dagger C^{-\frac{1}{2}}\right)_m \frac{d}{dt} \left(C^{-\frac{1}{2}} A\right)_m \\ &= \frac{i}{2}(F_m - i\pi_m) \left[\left(C^{-\frac{1}{2}} \frac{d}{dt} C^{-\frac{1}{2}} \right)_{mn} (F_n + i\pi_n) \right. \\ &\quad \left. + (C^{-1})_{mn} \left(\frac{d}{dt} F_n - i V_n' \right) \right] . \end{aligned} \tag{4.42}$$

Using the data of Subsec. 4.3 it is straightforward to evaluate (4.42) and the reduced Hamiltonian H' of the general one-matrix model in closed form. More generally, it is possible in principle to evaluate the reduced Hamiltonian (4.40a) to any order in the coefficients $v^{(n)}$ of the potential [see Eq. (2.50)]. Explicit forms of the general reduced Hamiltonian can be pulled back (see Subsec. 3.5) into new large N-conserved unreduced densities H_{rs}, which also correspond at large N (by the density maps of Sec. 2) to the same reduced H'.

5. Bose, Fermi and SUSY Oscillators

5.1. *Symmetric Bose/Fermi/Cuntz algebras*

We turn now to study the reduced equal-time algebra of a set of B real bosonic and F complex fermionic oscillators. Drawing on the discussion of Subsec. 2.5 and Sec. 3, our goal in this subsection is to find the Bose/Fermi generalization of the symmetric Cuntz algebras (3.10) discussed above.

For the reduced oscillators we may assume that

$$a_m|0\rangle = \tilde{a}_m|0\rangle = 0\,, \qquad \tilde{a}_m^\dagger|0\rangle = a_m^\dagger|0\rangle\,, \tag{5.1a}$$

$$\psi_{\dot\alpha}|0\rangle = \tilde{\psi}_{\dot\alpha}|0\rangle = 0\,, \qquad \tilde{\psi}_{\dot\alpha}^\dagger|0\rangle = \psi_{\dot\alpha}^\dagger|0\rangle\,, \tag{5.1b}$$

$$m = 1\cdots B\,, \quad \dot\alpha = 1\cdots F\,, \quad F = \frac{f}{2} = \text{integer} \tag{5.1c}$$

and the equal-time algebra [see (2.47)] takes the form

$$\left[a_m, \tilde{a}_n^\dagger\right] = \left[\tilde{a}_m, a_n^\dagger\right] = \delta_{mn}|0\rangle\langle 0|\,, \tag{5.2a}$$

$$\left[\psi_{\dot\alpha}, \tilde{\psi}_{\dot\beta}^\dagger\right]_+ = \left[\tilde{\psi}_{\dot\alpha}, \psi_{\dot\beta}^\dagger\right]_+ = \delta_{\dot\alpha\dot\beta}|0\rangle\langle 0|\,, \tag{5.2b}$$

$$\begin{aligned} [a_m, \tilde{a}_n] &= [a_m^\dagger, \tilde{a}_n^\dagger] = 0\,, \\ \left[\psi_{\dot\alpha}, \tilde{\psi}_{\dot\beta}\right]_+ &= \left[\psi_{\dot\alpha}^\dagger, \tilde{\psi}_{\dot\beta}^\dagger\right]_+ = 0\,, \end{aligned} \tag{5.2c}$$

$$\begin{aligned} \left[a_m, a_m^\dagger\right] - \left[\psi_{\dot\alpha}, \psi_{\dot\alpha}^\dagger\right]_+ &= \left[\tilde{a}_m, \tilde{a}_m^\dagger\right] - \left[\tilde{\psi}_{\dot\alpha}, \tilde{\psi}_{\dot\alpha}^\dagger\right]_+ \\ &= B - F - 1 + |0\rangle\langle 0|\,. \end{aligned} \tag{5.2d}$$

The bosonic and fermionic operators also commute with each other when only one is tilded. Following Sec. 3, we see that the set of all untilded words is complete

$$\{\psi_{\dot\alpha_1}^\dagger \cdots a_{m_1}^\dagger \cdots \psi_{\dot\alpha_n}^\dagger \cdots a_{m_p}^\dagger|0\rangle\} = \text{complete} \tag{5.3}$$

as well as the set of all tilded words, and indeed that each word of one set can be reexpressed as a word in the other set.

Studying the action of the annihilation operators on the complete sets of states [see Eq. (3.7)] one finds the *symmetric Bose/Fermi/Cuntz algebra*

$$a_m a_n^\dagger = \tilde{a}_m \tilde{a}_n^\dagger = \delta_{mn}\,, \quad \psi_{\dot\alpha}\psi_{\dot\beta}^\dagger = \tilde{\psi}_{\dot\alpha}\tilde{\psi}_{\dot\beta}^\dagger = \delta_{\dot\alpha,\dot\beta}\,, \tag{5.4a}$$

$$a_m\psi_{\dot\alpha}^\dagger = \psi_{\dot\alpha}a_m^\dagger = \tilde{a}_m\tilde{\psi}_{\dot\alpha}^\dagger = \tilde{\psi}_{\dot\alpha}\tilde{a}_m^\dagger = 0\,, \tag{5.4b}$$

$$a_m^\dagger a_m + \psi_{\dot\alpha}^\dagger\psi_{\dot\alpha} = \tilde{a}_m^\dagger\tilde{a}_m + \tilde{\psi}_{\dot\alpha}^\dagger\tilde{\psi}_{\dot\alpha} = 1 - |0\rangle\langle 0|\,, \tag{5.4c}$$

$$\left[a_m, \tilde{a}_n^\dagger\right] = \left[\tilde{a}_m,\ a_n^\dagger\right] = \delta_{mn}|0\rangle\langle 0|\,, \tag{5.4d}$$

$$\left[\psi_{\dot\alpha}, \tilde{\psi}_{\dot\beta}^\dagger\right]_+ = \left[\tilde{\psi}_{\dot\alpha}, \psi_{\dot\beta}^\dagger\right]_+ = \delta_{\dot\alpha\dot\beta}|0\rangle\langle 0|\,, \tag{5.4e}$$

$$\left[a_m, \tilde{a}_n\right] = \left[a_m^\dagger, \tilde{a}_n^\dagger\right] = \left[\psi_{\dot\alpha}, \tilde{\psi}_{\dot\beta}\right]_+ = \left[\psi_{\dot\alpha}^\dagger, \tilde{\psi}_{\dot\beta}^\dagger\right]_+ = 0\,, \tag{5.4f}$$

$$\left[a_m, \tilde{\psi}_{\dot\alpha}\right] = \left[a_m, \tilde{\psi}_{\dot\alpha}^\dagger\right] = \left[\tilde{a}_m, \psi_{\dot\alpha}\right] = \left[\tilde{a}_m, \psi_{\dot\alpha}^\dagger\right] = 0\,, \tag{5.4g}$$

$$[a_m^\dagger, \tilde{\psi}_{\dot{\alpha}}] = [a_m^\dagger, \tilde{\psi}_{\dot{\alpha}}^\dagger] = [\tilde{a}_m^\dagger, \psi_{\dot{\alpha}}] = [\tilde{a}_m^\dagger, \psi_{\dot{\alpha}}^\dagger] = 0\,, \tag{5.4h}$$

$$\begin{aligned} a_m|0\rangle &= 0\,, \qquad \tilde{a}_m^\dagger|0\rangle = a_m^\dagger|0\rangle\,, \\ \psi_{\dot{\alpha}}|0\rangle &= 0\,, \qquad \tilde{\psi}_{\dot{\alpha}}^\dagger|0\rangle = \psi_{\dot{\alpha}}^\dagger|0\rangle\,, \end{aligned} \tag{5.4i}$$

$$m,\ n = 1\cdots B\,, \qquad \dot{\alpha},\ \dot{\beta} = 1\cdots F\,, \tag{5.4j}$$

where B and F are integers. In particular, the relations (5.4a), (5.4b) come from the analysis of the annihilation operators, while the completeness relations (5.4c) are the result of using these three relations in (5.2d).

The symmetric Bose/Fermi/Cuntz algebra (5.4) is symmetric under interchange of tilde and untilde operators. It may also be considered as a family of algebras which interpolates from the symmetric Bose/Cuntz algebra (3.10) at $F = 0$ to a symmetric Fermi/Cuntz algebra at $B = 0$. In the special case of Fadeev–Popov ghosts, a subalgebra similar to the untilde part of (5.4a), (5.4b) was written down in Ref. 20.

5.2. *Cuntz superalgebras*

A striking feature of the symmetric Bose/Fermi/Cuntz algebra (5.4) is that it contains an important free subalgebra which shows a Bose–Fermi equivalence. To highlight this fact, it is convenient to introduce the oscillator superfields

$$\begin{gathered} A_M = \begin{pmatrix} a_m \\ \psi_{\dot{\alpha}} \end{pmatrix}, \qquad A_M^\dagger = \begin{pmatrix} a_m^\dagger \\ \psi_{\dot{\alpha}}^\dagger \end{pmatrix}, \\ \tilde{A}_M = \begin{pmatrix} \tilde{a}_m \\ \tilde{\psi}_{\dot{\alpha}} \end{pmatrix}, \qquad \tilde{A}_M^\dagger = \begin{pmatrix} \tilde{a}_m^\dagger \\ \tilde{\psi}_{\dot{\alpha}}^\dagger \end{pmatrix}, \\ M = 1\cdots(B+F) \end{gathered} \tag{5.5}$$

in terms of which the free subalgebra (5.4a)–(5.4c) and (5.4i) takes the Bose–Fermi equivalent form

$$A_M A_N^\dagger = \tilde{A}_M \tilde{A}_N^\dagger = \delta_{MN}\,, \tag{5.6a}$$

$$A_M^\dagger A_M = \tilde{A}_M^\dagger \tilde{A}_M = 1 - |0\rangle\langle 0|\,, \tag{5.6b}$$

$$A_M|0\rangle = \tilde{A}_M|0\rangle = 0\,, \quad A_M^\dagger|0\rangle = \tilde{A}_M^\dagger|0\rangle\,. \tag{5.6c}$$

This algebra is a subalgebra of a symmetric Cuntz algebra, but since it contains both Bose and Fermi oscillators, we will refer to it as a *symmetric Cuntz superalgebra*.

It is well known that the Cuntz algebras, being free algebras, dictate classical or Boltzmann statistics for the states. The states of the symmetric Bose/Fermi/Cuntz algebra (5.4) are formed by the application of any number of $A^\dagger$'s (or $\tilde{A}^\dagger$'s) on the vacuum

$$\{A^\dagger_{M_1} \cdots A^\dagger_{M_n}|0\rangle\} = \{\tilde{A}^\dagger_{M_1} \cdots \tilde{A}^\dagger_{M_n}|0\rangle\} = \text{complete} \tag{5.7}$$

so the free superalgebra (5.6) dictates the same Boltzmann statistics for the large N fermions and bosons. In particular, the Pauli Principle is lost for large N fermions. The Bose–Fermi equivalence is also related to the fact that the identification $\tilde{\phi} = \phi$, $\tilde{\pi} = \pi$ (see Subsec. 2.6) is lost for $B = 1$ when $F \neq 0$, just as it is when $F = 0$ and $B \geq 2$.

Finally, we emphasize that the Bose–Fermi equivalence seen in the free subalgebra (5.4a)–(5.4c), (5.4i) is not sustained in the full symmetric Bose/Fermi/Cuntz algebra (5.4), where the mixed relations (5.4d)–(5.4f) distinguish bosons from fermions. See Subsec. 5.5 for the superfield form of the full algebra.

5.3. *One SUSY oscillator*

As a simple example, we consider the large N formulation and solution of the ($\omega = 1$) supersymmetric one-matrix oscillator, with supercharges $Q_{\bullet}$ and $\bar{Q}_{\bullet}$. The unreduced form of this system is

$$Q_{\bullet} = \mathrm{Tr}(\psi\pi^+)\,, \quad \bar{Q}_{\bullet} = \mathrm{Tr}(\psi^\dagger\pi^-)\,, \quad \pi^{\pm}_{rs} = (\pi \pm i\phi)_{rs}\,, \tag{5.8a}$$

$$[\phi_{rs}, \pi_{pq}] = i\delta_{sp}\delta_{rq}\,, \quad [\psi_{rs}, \psi^\dagger_{pq}]_+ = \delta_{sp}\delta_{rq}\,, \tag{5.8b}$$

$$Q_{\bullet}{}^2 = \bar{Q}_{\bullet}{}^2 = 0\,, \quad [Q_{\bullet}, \bar{Q}_{\bullet}]_+ = 2H_{\bullet}\,, \tag{5.8c}$$

$$H_{\bullet} = \frac{1}{2}\mathrm{Tr}(\pi^2 + \phi^2 + [\psi^\dagger, \psi])\,, \tag{5.8d}$$

$$Q_{\bullet}|0_{\bullet}\rangle = \bar{Q}_{\bullet}|0_{\bullet}\rangle = H_{\bullet}|0_{\bullet}\rangle = 0 \tag{5.8e}$$

together with implied relations such as

$$\dot{Q}_{\bullet} = \dot{\bar{Q}}_{\bullet} = \dot{H}_{\bullet} = [Q_{\bullet}, H_{\bullet}] = [\bar{Q}_{\bullet}, H_{\bullet}] = 0 \tag{5.9}$$

and the algebra of the supercharges with the fields.

For the reduced formulation we will temporarily employ a mixed notation, including sometimes the component fields and sometimes the superfield notation (5.5) with $M = 1, 2$:

$$A_M = \begin{pmatrix} a \\ \psi \end{pmatrix}, \qquad A^\dagger_M = \begin{pmatrix} a^\dagger \\ \psi^\dagger \end{pmatrix},$$
$$\tilde{A}_M = \begin{pmatrix} \tilde{a} \\ \tilde{\psi} \end{pmatrix}, \qquad \tilde{A}^\dagger_M = \begin{pmatrix} \tilde{a}^\dagger \\ \tilde{\psi}^\dagger \end{pmatrix}, \tag{5.10a}$$

$$a = \frac{i}{\sqrt{2}}\pi_-\,, \quad a^\dagger = -\frac{i}{\sqrt{2}}\pi_+\,, \quad \pi_\pm = \pi \pm i\phi\,, \tag{5.10b}$$

where the superfields satisfy the Cuntz superalgebra (5.6) with $B + F = 2$. The remaining components of the reduced equal-time algebra are

$$[a, \tilde{a}^\dagger] = [\tilde{a}, a^\dagger] = [\psi, \tilde{\psi}^\dagger]_+ = [\tilde{\psi}, \psi^\dagger]_+ = |0\rangle\langle 0| \,, \tag{5.11a}$$

$$[a, \tilde{a}] = [a^\dagger, \tilde{a}^\dagger] = [\psi, \tilde{\psi}]_+ = [\psi^\dagger, \tilde{\psi}^\dagger]_+ = 0 \,, \tag{5.11b}$$

$$[a, \tilde{\psi}] = [a, \tilde{\psi}^\dagger] = [\tilde{a}, \psi] = [\tilde{a}, \psi^\dagger] = 0 \,, \tag{5.11c}$$

$$[a^\dagger, \tilde{\psi}] = [a^\dagger, \tilde{\psi}^\dagger] = [\tilde{a}^\dagger, \psi] = [\tilde{a}^\dagger, \psi^\dagger] = 0 \tag{5.11d}$$

which can also be written in terms of the superfields (see Subsec. 5.5).

The reduced equations of motion, involving the reduced Hamiltonian H, are

$$\dot{A}_M = i[H, A_M] = -iA_M \,, \qquad \dot{A}_M^\dagger = i[H, A_M^\dagger] = iA_M^\dagger \,, \tag{5.12a}$$

$$\dot{\tilde{A}}_M = i[H, \tilde{A}_M] = -i\tilde{A}_M \,, \qquad \dot{\tilde{A}}_M^\dagger = i[H, \tilde{A}_M^\dagger] = i\tilde{A}_M^\dagger \tag{5.12b}$$

and the ground state energy of the system is evaluated with (5.6c) as

$$\begin{aligned} E_0 &= \langle 0|H|0\rangle \\ &= \frac{N^2}{2}\langle 0|[a^\dagger, a]_+ + [\psi^\dagger, \psi]|0\rangle \\ &= \frac{N^2}{2}\langle 0|[a, \tilde{a}^\dagger] - [\psi, \tilde{\psi}^\dagger]_+|0\rangle = 0 \,. \end{aligned} \tag{5.13}$$

The properties of the reduced supercharges Q and $\bar{Q}$ also follow from the maps of Sec. 2:

$$Q^2 = \bar{Q}^2 = 0 \,, \qquad [\bar{Q}, Q]_+ = 2H \,, \tag{5.14a}$$

$$\dot{Q} = \dot{\bar{Q}} = \dot{H} = [Q, H] = [\bar{Q}, H] = 0 \,, \tag{5.14b}$$

$$Q|0\rangle = \bar{Q}|0\rangle = H|0\rangle = 0 \,, \tag{5.14c}$$

$$\begin{aligned} \langle 0|Q|0\rangle &= \langle 0|i\sqrt{2}\psi a^\dagger|0\rangle = 0 \,, \\ \langle 0|\bar{Q}|0\rangle &= \langle 0| - i\sqrt{2}\psi^\dagger a|0\rangle = 0 \,, \end{aligned} \tag{5.14d}$$

$$[Q, a] = -i\sqrt{2}\psi \,, \qquad [\bar{Q}, a^\dagger] = -i\sqrt{2}\psi^\dagger \,, \tag{5.15a}$$

$$[Q, \psi^\dagger]_+ = i\sqrt{2}a^\dagger \,, \qquad [\bar{Q}, \psi]_+ = -i\sqrt{2}a \,, \tag{5.15b}$$

$$[Q, a^\dagger] = [\bar{Q}, a] = [Q, \psi]_+ = [\bar{Q}, \psi^\dagger]_+ = 0 \,, \tag{5.15c}$$

$$[Q, \tilde{a}] = -i\sqrt{2}\tilde{\psi} \,, \qquad [\bar{Q}, \tilde{a}^\dagger] = -i\sqrt{2}\tilde{\psi}^\dagger \,, \tag{5.16a}$$

$$[Q, \tilde{\psi}^\dagger]_+ = i\sqrt{2}\tilde{a}^\dagger \,, \qquad [\bar{Q}, \tilde{\psi}]_+ = -i\sqrt{2}\tilde{a} \,, \tag{5.16b}$$

$$[Q, \tilde{a}^\dagger] = [\bar{Q}, \tilde{a}] = [Q, \tilde{\psi}]_+ = [\bar{Q}, \tilde{\psi}^\dagger]_+ = 0 \,. \tag{5.16c}$$

Owing to the opacity phenomenon for trace class operators (see Subsec. 2.3), we do not yet know the composite form of the reduced supercharges and the reduced Hamiltonian.

Drawing on experience in earlier sections, we have solved the relations (5.12) and (5.14)–(5.16) to obtain the explicit form of the reduced supercharges and Hamiltonian. The results can be expressed entirely in terms of the superfields

$$Q = i\sum_{n=0}^{\infty}(A^\dagger\tau_3)_{M_1}\cdots(A^\dagger\tau_3)_{M_n}\big(A^\dagger\sqrt{2}\tau_+A\big)A_{M_n}\cdots A_{M_1} \tag{5.17a}$$

$$= i\sum_{n=0}^{\infty}(\tilde{A}^\dagger\tau_3)_{M_1}\cdots\big(\tilde{A}^\dagger\tau_3\big)_{M_n}\big(\tilde{A}^\dagger\sqrt{2}\tau_+\tilde{A}\big)\tilde{A}_{M_n}\cdots\tilde{A}_{M_1}\,, \tag{5.17b}$$

$$\bar{Q} = -i\sum_{n=0}^{\infty}(A^\dagger\tau_3)_{M_1}\cdots\big(A^\dagger\tau_3\big)_{M_n}\big(A^\dagger\sqrt{2}\tau_-A\big)A_{M_n}\cdots A_{M_1} \tag{5.17c}$$

$$= -i\sum_{n=0}^{\infty}(\tilde{A}^\dagger\tau_3)_{M_1}\cdots\big(\tilde{A}^\dagger\tau_3\big)_{M_n}\big(\tilde{A}^\dagger\sqrt{2}\tau_-\tilde{A}\big)\tilde{A}_{M_n}\cdots\tilde{A}_{M_1}\,, \tag{5.17d}$$

$$H = \sum_{n=0}^{\infty}A^\dagger_{M_1}\cdots A^\dagger_{M_n}(A^\dagger A)A_{M_n}\cdots A_{M_1} \tag{5.17e}$$

$$= \sum_{n=0}^{\infty}\tilde{A}^\dagger_{M_1}\cdots\tilde{A}^\dagger_{M_n}(\tilde{A}^\dagger\tilde{A})\tilde{A}_{M_n}\cdots\tilde{A}_{M_1}\,, \tag{5.17f}$$

$$\tau_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \tau_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5.17g}$$

where the Pauli matrices τ operate in the reduced two-dimensional superspace $(\tau A)_M = \tau_{MN}A_N$.

The reduced supercharges and Hamiltonian can be understood as nonlocally dressed forms of their "zeroth order" factors

$$Q = iA^\dagger\sqrt{2}\tau_+A + \cdots = i\sqrt{2}a^\dagger\psi + \cdots\,, \tag{5.18a}$$

$$\bar{Q} = -iA^\dagger\sqrt{2}\tau_-A + \cdots = -i\sqrt{2}\psi^\dagger a + \cdots\,, \tag{5.18b}$$

$$H = A^\dagger A + \cdots = a^\dagger a + \psi^\dagger\psi + \cdots \tag{5.18c}$$

which closely resemble the unreduced supercharge and energy densities in (5.8a) and (5.8d).

Indeed, following the discussion of large N field identification in Subsec. 3.5, we may construct the new nonlocal unreduced densities Q_{rs}, $\bar{Q}_{rs}$ and H_{rs},

$$\dot{Q}_{rs} = \dot{\bar{Q}}_{rs} = \dot{H}_{rs} = 0\,, \quad r,\ s = 1\cdots N\,, \tag{5.19a}$$

$$Q_{rs}|0.\rangle = \bar{Q}_{rs}|0.\rangle = H_{rs}|0.\rangle = 0 \tag{5.19b}$$

which also correspond at large N to the same reduced supercharges and Hamiltonian.[i] These densities have exactly the forms given in (5.17a), (5.17c) and (5.17e), now with all operators interpreted as matrix-valued operators

$$(A_M)_{rs} = \begin{pmatrix} \frac{i}{\sqrt{2}}\pi^-_{rs} \\ \psi_{rs} \end{pmatrix}, \qquad (A^\dagger_M)_{rs} = \begin{pmatrix} -\frac{i}{\sqrt{2}}\pi^+_{rs} \\ \psi^\dagger_{rs} \end{pmatrix} \tag{5.20}$$

and all products as matrix products. The operators $\pi^\pm_{rs}$ are defined in (5.8a). As an example, the first two terms of the new supercharge density are

$$Q_{rs} = i\sum_{n=0}^{\infty}\{(A^\dagger\tau_3)_{M_1}\cdots(A^\dagger\tau_3)_{M_n}(A^\dagger\sqrt{2}\tau_+A)A_{M_n}\cdots A_{M_1}\}_{rs} \tag{5.21a}$$

$$\begin{aligned} &= i(A^\dagger_M)_{rt}\sqrt{2}(\tau_+)_{MN}(A_N)_{ts} \\ &\quad + i\big((A^\dagger\tau_3)_M\big)_{rt}\big(A^\dagger\sqrt{2}\tau_+A\big)_{tu}(A_M)_{us} + \cdots . \end{aligned} \tag{5.21b}$$

As in Sec. 3.5 we find that the new conserved densities are dressed nonlocal forms

$$Q_{rs} = \pi^+_{rt}\psi_{ts} + \cdots, \qquad \bar{Q}_{rs} = \psi^\dagger_{rt}\pi^-_{ts} + \cdots, \tag{5.22a}$$

$$H_{rs} = \frac{1}{2}\pi^+_{rt}\pi^-_{ts} + \psi^\dagger_{rt}\psi_{ts} + \cdots, \tag{5.22b}$$

$$Q_{rr} = Q. + \cdots, \quad \bar{Q}_{rr} = \bar{Q}. + \cdots, \quad H_{rr} = H. + \cdots \tag{5.22c}$$

of the original unreduced supercharge and energy densities.

5.4. *Bosonic construction of supersymmetry*

We note here that the Bose–Fermi equivalence of the free superalgebra (5.6) allows a purely bosonic construction of supersymmetry at large N.

Suppose that the unreduced operator $\psi_{rs} \to b_{rs}$ above was a complex boson. This does not change the free algebra (5.6) of the reduced operators

$$A = \begin{pmatrix} a \\ b \end{pmatrix}, \qquad A^\dagger = \begin{pmatrix} a^\dagger \\ b^\dagger \end{pmatrix} \tag{5.23}$$

so we retain that part of our algebraic construction[j]

$$Q = i\sum_{n=0}^{\infty}(A^\dagger\tau_3)_{M_1}\cdots(A^\dagger\tau_3)_{M_n}(A^\dagger\sqrt{2}\tau_+A)A_{M_n}\cdots A_{M_1}, \tag{5.24a}$$

$$\bar{Q} = -i\sum_{n=0}^{\infty}(A^\dagger\tau_3)_{M_1}\cdots(A^\dagger\tau_3)_{M_n}(A^\dagger\sqrt{2}\tau_-A)A_{M_n}\cdots A_{M_1}, \tag{5.24b}$$

[i] It is again an oscillator artifact that the relations (5.19) are true at finite N.
[j] One can alternately start with the tilde forms of Q, $\bar{Q}$ in (5.17b) and (5.17d), obtaining the SUSY algebra (5.25) and the tilde relations (5.12b), (5.16) and (5.17f).

$$H = \sum_{n=0}^{\infty} A^\dagger_{M_1} \cdots A^\dagger_{M_n} (A^\dagger A) A_{M_n} \cdots A_{M_1} \,, \tag{5.24c}$$

$$\dot{A}_M = i[H, A_M] = -iA_M \,, \quad \dot{A}^\dagger_M = i[H, A^\dagger_M] = iA^\dagger_M \,, \tag{5.24d}$$

$$Q^2 = \bar{Q}^2 = 0 \,, \qquad [\bar{Q}, Q]_+ = 2H \,, \tag{5.25a}$$

$$\dot{Q} = \dot{\bar{Q}} = \dot{H} = [Q, H] = [\bar{Q}, H] = 0 \,, \tag{5.25b}$$

$$Q|0\rangle = \bar{Q}|0\rangle = H|0\rangle = 0 \,, \tag{5.25c}$$

$$[Q, a] = -i\sqrt{2}b \,, \qquad [\bar{Q}, a^\dagger] = -i\sqrt{2}b^\dagger \,, \tag{5.26a}$$

$$[Q, b^\dagger]_+ = i\sqrt{2}a^\dagger \,, \qquad [\bar{Q}, b]_+ = -i\sqrt{2}a \,, \tag{5.26b}$$

$$[Q, a^\dagger] = [\bar{Q}, a] = [Q, b]_+ = [\bar{Q}, b^\dagger]_+ = 0 \tag{5.26c}$$

which follows from the free algebra alone.

The nonlocal forms (5.24a)–(5.24c) provide a purely bosonic construction of supersymmetry at large N, but the construction is apparently not equivalent to two unreduced bosonic oscillators, or indeed to any local unreduced theory: In the first place, the anticommutators persist for b's in (5.26). Moreover, the mixed relations in (5.11a), (5.11b) for b now involve commutators, which loses the tilde forms of Q, $\bar{Q}$, H in (5.17b), (5.17d) and (5.17f) and causes for example a nonlocal equation of motion for $\tilde{b}$.

5.5. *Higher supersymmetry*

In this section, we generalize the construction of Subsec. 5.3 to higher supersymmetry. We will work directly in reduced space, starting with nothing but the Cuntz superalgebra

$$A_M A^\dagger_N = \delta_{MN} \,, \qquad A_M|0\rangle = \langle 0|A^\dagger_M = 0 \,, \tag{5.27a}$$

$$A^\dagger_M A_M = 1 - |0\rangle\langle 0| \,, \qquad M,\ N = 1 \cdots (B + F) \,, \tag{5.27b}$$

where B and F are presently undetermined. Using only (5.27a), we consider the operators

$$Q_i \equiv \sum_{n=0}^{\infty} (A^\dagger\gamma)_{M_1} \cdots (A^\dagger\gamma)_{M_n} (A^\dagger \Gamma_i A) A_{M_n} \cdots A_{M_1} \tag{5.28}$$

which are functions of arbitrary matrices γ and Γ_i. It is straightforward to see that these operators satisfy

$$Q_i|0\rangle = 0 \,, \tag{5.29a}$$

$$\begin{aligned} Q_i A^\dagger_M - (A^\dagger\gamma)_M Q_i &= (A^\dagger\Gamma_i)_M \,, \\ A_M Q_i - Q_i(\gamma A)_M &= (\Gamma_i A)_M \,. \end{aligned} \tag{5.29b}$$

Now choose the matrices to satisfy

$$\gamma^2 = 1\,, \qquad [\gamma, \Gamma_i]_+ = 0\,,\ \forall i \tag{5.30}$$

so that the relations

$$\begin{aligned} Q_i A_M^\dagger &= (A^\dagger \Gamma_i)_M + (A^\dagger \gamma)_M Q_i\,, \\ Q_i A_M &= -(\gamma \Gamma_i A)_M + (\gamma A)_M Q_i \end{aligned} \tag{5.31}$$

follow from (5.29b). These relations can be used in $Q_i Q_j$ to push Q_i through all the A's and $A^\dagger$'s in Q_j, with the result

$$[Q_i, Q_j]_+ = \sum_{n=0}^{\infty} A_{M_1}^\dagger \cdots A_{M_n}^\dagger (A^\dagger [\Gamma_i, \Gamma_j]_+ A) A_{M_n} \cdots A_{M_1}\,. \tag{5.32}$$

To close the algebra (5.32), we now choose the matrices Γ_i to be a Dirac representation of a Clifford algebra in $2d$ Euclidean dimensions

$$[\Gamma_i, \Gamma_j]_+ = 2\delta_{ij}\,, \qquad (\Gamma_i)^\dagger = \Gamma_i\,, \quad i = 1 \cdots 2d\,, \tag{5.33}$$

where $\gamma =$ "γ_5" $= \prod_i \Gamma_i$ and size $(\Gamma_i, \gamma) = 2^d$.

We have therefore constructed a reduced system with $n = 2d$ supersymmetries

$$[Q_i, Q_j]_+ = 2\delta_{ij} H\,, \quad i,\ j = 1 \cdots 2d\,, \tag{5.34a}$$

$$H = \sum_{n=0}^{\infty} A_{M_1}^\dagger \cdots A_{M_n}^\dagger (A^\dagger A) A_{M_n} \cdots A_{M_1}\,, \tag{5.34b}$$

$$\dot{A}_M = i[H, A_M] = -iA_M\,, \quad \dot{A}_M^\dagger = i[H, A_M^\dagger] = iA_M^\dagger\,, \tag{5.34c}$$

$$Q_i|0\rangle = H|0\rangle = 0 \tag{5.34d}$$

which includes the construction of Subsec. 5.3 as the special case with $d = 1$ and size $(\Gamma_i, \gamma) = 2$.

In a Weyl representation (with blocks of size 2^{d-1})

$$\gamma = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \qquad \Gamma_i = \begin{pmatrix} 0 & \gamma_i \\ \gamma_i^\dagger & 0 \end{pmatrix} \tag{5.35}$$

we may identify $B = F = 2^{d-1}$ bosons and fermions in the superfields as

$$A_M = \begin{pmatrix} a_m \\ \psi_{\dot\alpha} \end{pmatrix}, \quad A_M^\dagger = \begin{pmatrix} a_m^\dagger \\ \psi_{\dot\alpha}^\dagger \end{pmatrix}, \quad m,\ \dot\alpha = 1 \cdots 2^{d-1}\,, \quad M = 1 \cdots 2^d\,. \tag{5.36}$$

This assignment follows because the relations (5.31) read

$$[Q_i, a_m^\dagger] = \psi_{\dot\alpha}^\dagger (\gamma_i^\dagger)_{\dot\alpha m}\,, \qquad [Q_i, \psi_{\dot\alpha}^\dagger]_+ = a_m^\dagger (\gamma_i)_{m\dot\alpha}\,, \tag{5.37a}$$

$$[Q_i, a_m] = -(\gamma_i)_{m\dot\alpha} \psi_{\dot\alpha}\,, \quad [Q_i, \psi_{\dot\alpha}]_+ = (\gamma_i^\dagger)_{\dot\alpha m} a_m \tag{5.37b}$$

under the identification (5.36). Similarly, one has for the reduced supercharges and Hamiltonian

$$Q_i = A^\dagger \Gamma_i A + \cdots = a_m^\dagger (\gamma_i)_{m\dot\alpha} \psi_{\dot\alpha} + \psi_{\dot\alpha}^\dagger (\gamma_i^\dagger)_{\dot\alpha m} a_m + \cdots\,, \tag{5.38a}$$

$$H = A^\dagger A + \cdots = a_m^\dagger a_m + \psi_{\dot\alpha}^\dagger \psi_{\dot\alpha} + \cdots \tag{5.38b}$$

and these operators can be pulled back straightforwardly into unreduced supercharge and energy densities $(Q_i)_{rs}$, H_{rs}

$$(\dot{Q}_i)_{rs} = \dot{H}_{rs} = 0\,, \qquad (Q_i)_{rs}|0.\rangle = H_{rs}|0.\rangle = 0 \tag{5.39}$$

which are nonlocally dressed forms of the original densities of the theory (see Subsecs. 3.5 and 5.3).

In this case, we have also the reduced generators of spin $(2d)$

$$J_{ij} = \sum_{n=0}^{\infty} A^\dagger_{M_1} \cdots A^\dagger_{M_n} \left(-\frac{i}{4} A^\dagger [\Gamma_i, \Gamma_j] A \right) A_{M_n} \cdots A_{M_1}\,, \tag{5.40a}$$

$$[H, J_{ij}] = 0\,, \qquad J_{ij}|0\rangle = 0\,. \tag{5.40b}$$

Using (5.40a), we have checked that the reduced fields a_m and $\psi_{\dot\alpha}$ both transform as Weyl spinors of spin $(2d)$, while the reduced superfield A_M transforms as a Dirac spinor. The reduced supercharges themselves transform in the vector representation of spin $(2d)$.

In the discussion above, we used only the free superalgebra (5.27a) of the untilde superfields to study only the untilde forms of the reduced operators and fields.[k] With the inclusion of the full $B = F = 2^{d-1}$ Bose/Fermi/Cuntz algebra (5.4), one finds also that all the equations of this section hold when every field is also tilded (including the tilde form of the equation of motion and the tilde forms of Q_i and H). The form of the corresponding unreduced generators $Q_{\bullet i}$ and $H_\bullet$, as a function of the unreduced SUSY oscillators, is left as an exercise for the reader.

In this connection we note that when $B = F = 2^{d-1}$ the full symmetric Bose/Fermi/Cuntz algebra (5.4) can be written entirely in terms of the reduced superfields as

$$A_M A^\dagger_N = \tilde{A}_M \tilde{A}^\dagger_N = \delta_{MN}\,, \quad M,\ N = 1 \cdots 2^d\,, \tag{5.41a}$$

$$A^\dagger_M A_M = \tilde{A}^\dagger_M \tilde{A}_M = 1 - |0\rangle\langle 0|\,, \tag{5.41b}$$

$$A_M|0\rangle = \tilde{A}_M|0\rangle = 0\,, \quad A^\dagger_M|0\rangle = \tilde{A}^\dagger_M|0\rangle\,, \tag{5.41c}$$

$$A_M \tilde{A}^\dagger_N = \tilde{A}^\dagger_{N'} R_{MN',M'N} A_{M'} + \delta_{MN}|0\rangle\langle 0|\,, \tag{5.41d}$$

$$\begin{aligned} A_M \tilde{A}_N &= R_{MN,M'N'} \tilde{A}_{N'} A_{M'}\,, \\ A^\dagger_M \tilde{A}^\dagger_N &= \tilde{A}^\dagger_{N'} A^\dagger_{M'} R_{M'N',MN}\,, \end{aligned} \tag{5.41e}$$

$$\begin{aligned} R_{MN,M'N'} &= (\gamma_+)_{MM'} \delta_{NN'} + (\gamma_-)_{MM'} \gamma_{NN'} \\ &= \delta_{MM'} (\gamma_+)_{NN'} + (\gamma)_{MM'} (\gamma_-)_{NN'}\,, \end{aligned} \tag{5.41f}$$

$$\gamma_\pm = \frac{1}{2}(1 \pm \gamma)\,. \tag{5.41g}$$

[k] Up to this point, the construction above can also be interpreted, as in Subsec. 5.4, as a bosonic construction of $n = 2d$ reduced supersymmetries.

Here (5.41a)–(5.41c) is the symmetric free superalgebra (5.6), which collects (5.4a)–(5.4c) and (5.4i), and the matrix R in (5.41d)–(5.41f) correctly distinguishes between fermions and bosons in the mixed relations (5.4d)–(5.4h). The matrices $\gamma_\pm$ in (5.41g) are the natural projection operators onto $B = F = 2^{d-1}$ bosons and fermions, but the full algebra (5.4) for unrestricted B and F can always be put in superfield form by choosing the R matrix as

$$\begin{aligned} R_{MN,M'N'} &= (P_b)_{MM'}(P_b)_{NN'} + (P_b)_{MM'}(P_f)_{NN'} \\ &\quad + (P_f)_{MM'}(P_b)_{NN'} - (P_f)_{MM'}(P_f)_{NN'}\,, \end{aligned} \tag{5.42}$$

where P_b and P_f are projectors onto the bosons and fermions.

Finally, we remark on a more general algebraic identity, with arbitrary matrix γ and operator q,

$$Q = Q(\gamma; q) \equiv \sum_{n=0}^{\infty} (A^\dagger\gamma)_{M_1} \cdots (A^\dagger\gamma)_{M_n}\, q\, A_{M_n} \cdots A_{M_1}\,, \tag{5.43a}$$

$$q = q(A, A^\dagger)\,, \qquad Q(\gamma; q)|0\rangle = q|0\rangle\,, \tag{5.43b}$$

$$(A^\dagger\gamma)_M Q = (Q - q)A^\dagger_M\,, \qquad Q(\gamma A)_M = A_M(Q - q)\,, \tag{5.43c}$$

$$Q_1 Q_2 = Q(\gamma_1\gamma_2; Q_1 q_2 + q_1 Q_2 - q_1 q_2)\,, \tag{5.43d}$$

where $Q_1 = Q(\gamma_1; q_1)$, $Q_2 = Q(\gamma_2; q_2)$. The relations above follow from the free superalgebra (5.27a), and this result includes the identity used in the construction above.

6. Matrix Theory

In this section we give the large N formulation of the $n = 16$ supersymmetric gauge quantum mechanics,[40] now called Matrix theory,[41] in the temporal gauge (see Sec. 2). We will take the matrix fields of the theory to be traceless, so that we may follow the lore[44,41] in assuming that the ground state is unique and hence supersymmetric and rotationally invariant. Since it costs no further effort, we will include in our treatment the entire minimal sequence[40,45,46] of matrix models with 16, 8, 4 and 2 supersymmetries (which is obtained by dimensional reduction of pure super Yang–Mills theories in 10, 6, 4 and 3 space–time dimensions.)

The Hermitian field form of Matrix theory was given for any gauge symmetry in Ref. 40. Our notation for the traceless field formulation is changed only slightly from that of Sec. 2,

$$(T_a)_{rs}(T_a)_{uv} = P_{rs,uv}\,, \quad \mathrm{Tr}(T_a T_b) = \delta_{ab}\,, \quad \mathrm{Tr}(T_a) = 0\,, \tag{6.1a}$$

$$\rho_{rs} = \rho_a (T_a)_{rs}\,, \quad \mathrm{Tr}(\rho) = 0\,, \quad \rho = \phi,\ \pi,\ \text{or}\ \Lambda\,, \tag{6.1b}$$

$$[\phi^m_{rs}, \pi^n_{uv}] = i\delta^{mn} P_{rs,uv}\,, \quad [(\Lambda_\alpha)_{rs}, (\Lambda_\beta)_{uv}]_+ = \delta_{\alpha\beta} P_{rs,uv}\,, \tag{6.1c}$$

$$r,\ s = 1 \cdots N\,, \quad a,\ b = 1 \cdots N^2 - 1\,, \tag{6.1d}$$

$$m = 1 \cdots B\,, \quad \alpha = 1 \cdots f\,, \quad B = F + 1\,, \quad F = \frac{f}{2}\,, \quad f = 16, 8, 4, 2\,, \tag{6.1e}$$

where the projector P is defined in (2.10e). The numbers of real fermions f in (6.1e) are also the numbers of real supersymmetries in the minimal sequence.

The matrix field form of the theory is defined by

$$Q_{\bullet\alpha} = \mathrm{Tr}\big((\Gamma^m\Lambda)_\alpha \pi^m + g(\Sigma^{mn}\Lambda)_\alpha[\phi^m,\phi^n]\big)\,, \tag{6.2a}$$

$$[Q_{\bullet\alpha}, Q_{\bullet\beta}]_+ = 2\delta_{\alpha\beta}H_\bullet + 2g\Gamma^m_{\alpha\beta}\,\mathrm{Tr}(\phi^m G)\,, \tag{6.2b}$$

$$H_\bullet = \frac{1}{2}\,\mathrm{Tr}\left(\pi^m\pi^m - \frac{g^2}{2}[\phi^m,\phi^n][\phi^m,\phi^n] + g[\Lambda_\alpha,\Lambda_\beta]_+\Gamma^m_{\alpha\beta}\phi^m\right), \tag{6.2c}$$

$$G_{rs} = -i[\phi^m,\pi^m]_{rs} - (\Lambda_\alpha\Lambda_\alpha)_{rs} - \left(N - \frac{1}{N}\right)\delta_{rs}\,, \tag{6.2d}$$

$$J_\bullet^{mn} = \mathrm{Tr}\left(\pi^{[m}\phi^{n]} - \frac{1}{2}\Lambda\Sigma^{mn}\Lambda\right), \tag{6.2e}$$

$$[\Gamma^m,\Gamma^n]_+ = 2\delta^{mn}\,, \qquad \Sigma^{mn} = -\frac{i}{4}[\Gamma^m,\Gamma^n]\,, \tag{6.2f}$$

where g is the coupling constant and the matrices Γ^m are real, symmetric and traceless. At large N, the SU(N) gauge generators in (6.2d) are in agreement with the SU(N) generators (2.6) at $B = F + 1$. The generators of spin (B), with $B = 9, 5, 3$ and 2, are given in (6.2e). Derived relations include

$$[Q_{\bullet\alpha}, H_\bullet] = ig\,\mathrm{Tr}(\Lambda_\alpha G)\,, \qquad H_\bullet = \frac{1}{f}\sum_{\alpha=1}^{f} Q_{\bullet\alpha}^2\,, \tag{6.3a}$$

$$[J_\bullet^{mn}, H_\bullet] = 0\,, \qquad [J_\bullet^{mn}, Q_{\bullet\alpha}] = (\Sigma^{mn}Q_\bullet)_\alpha\,, \tag{6.3b}$$

$$[J_\bullet^{mn}, J_\bullet^{pq}] = i(\delta^{q[m}J_\bullet^{n]p} - \delta^{p[m}J_\bullet^{n]q}) \tag{6.3c}$$

and the algebra of $Q_{\bullet\alpha}$, $H_\bullet$ and $J_\bullet^{mn}$ with the fields.

As noted above, we assume here that the gauge-invariant ground state

$$G_{rs}|0_\bullet\rangle = 0 \tag{6.4}$$

of the theory is unique.[1] This means that the ground state is also supersymmetric and rotationally invariant

$$Q_{\bullet\alpha}|0_\bullet\rangle = H_\bullet|0_\bullet\rangle = J_\bullet^{mn}|0_\bullet\rangle = 0 \tag{6.5}$$

and that the large N completeness relation has the same form

$$\mathbf{1}_\bullet \underset{N}{=} |0_\bullet\rangle\langle{}_\bullet 0| + \sum_{rs,A}|rs,A\rangle\langle rs,A| \tag{6.6}$$

which we have been studying throughout this paper.

[1]To study the possibility of degenerate vacua in large N theories with fermions, see the discussion around Eq. (2.11).

For the reduced large N theory, we maintain 't Hooft scaling by choosing

$$g = \frac{\lambda}{\sqrt{N}}\,, \tag{6.7}$$

where the rescaled coupling λ is independent of N. The definitions of reduced matrix elements and operators in Sec. 2 are unchanged, except for the vacuum expectation values [see (2.13a)], which now read

$$\langle 0|\phi_m|0\rangle = \langle 0|\pi_m|0\rangle = \langle 0|\Lambda_\alpha|0\rangle = 0 \tag{6.8}$$

for the reduced fields, because the unreduced fields are traceless. Then, except for a few corrections which are negligible at large N, we find that no other changes are necessary, and we may take over all the reduced results derived in previous sections.

The equal-time free algebra of reduced Matrix theory is a copy of (2.35) and (2.15b)

$$[\phi_m, \tilde{\pi}_n] = [\tilde{\phi}_m, \pi_n] = i\delta_{mn}|0\rangle\langle 0|\,, \tag{6.9a}$$

$$[\phi_m, \tilde{\phi}_n] = [\pi_m, \tilde{\pi}_n] = 0\,, \tag{6.9b}$$

$$[\Lambda_\alpha, \tilde{\Lambda}_\beta]_+ = \delta_{\alpha\beta}|0\rangle\langle 0|\,, \tag{6.9c}$$

$$[\Lambda_\alpha, \tilde{\phi}_m] = [\tilde{\Lambda}_\alpha, \phi_m] = [\Lambda_\alpha, \tilde{\pi}_m] = [\tilde{\Lambda}_\alpha, \pi_m] = 0\,, \tag{6.9d}$$

$$[\phi_m, \pi_m] - i\Lambda_\alpha\Lambda_\alpha = [\tilde{\phi}_m, \tilde{\pi}_m] - i\tilde{\Lambda}_\alpha\tilde{\Lambda}_\alpha = i|0\rangle\langle 0|\,, \tag{6.9e}$$

$$(\tilde{\phi}_m - \phi_m)|0\rangle = (\tilde{\pi}_m - \pi_m)|0\rangle = (\tilde{\Lambda}_\alpha - \Lambda_\alpha)|0\rangle = 0\,, \tag{6.9f}$$

$$(\dot{\tilde{\phi}}_m - \dot{\phi}_m)|0\rangle = (\dot{\tilde{\pi}}_m - \dot{\pi}_m)|0\rangle = (\dot{\tilde{\Lambda}}_\alpha - \dot{\Lambda}_\alpha)|0\rangle = 0 \tag{6.9g}$$

with $B = F + 1$. Recall that (6.9e) summarizes the action of the reduced gauge generators G, $\tilde{G}$ on the reduced states (see Eq. (2.27)).

The reduced equations of motion are

$$\dot{\phi}_m = i[H, \phi_m] = \pi_m\,, \tag{6.10a}$$

$$\dot{\pi}_m = i[H, \pi_m] = \lambda^2\big[\phi_n, [\phi_m, \phi_n]\big] - \lambda\Lambda_\alpha\Gamma^m_{\alpha\beta}\Lambda_\beta\,, \tag{6.10b}$$

$$\dot{\Lambda}_\alpha = i[H, \Lambda_\alpha] = i\lambda\Gamma^m_{\alpha\beta}[\phi_m, \Lambda_\beta]\,, \tag{6.10c}$$

$$\dot{\tilde{\phi}}_m = i[H, \tilde{\phi}_m] = \tilde{\pi}_m\,, \tag{6.10d}$$

$$\dot{\tilde{\pi}}_m = i[H, \tilde{\pi}_m] = \lambda^2\big[\tilde{\phi}_n, [\tilde{\phi}_m, \tilde{\phi}_n]\big] + \lambda\tilde{\Lambda}_\alpha\Gamma^m_{\alpha\beta}\tilde{\Lambda}_\beta\,, \tag{6.10e}$$

$$\dot{\tilde{\Lambda}}_\alpha = i[H, \tilde{\Lambda}_\alpha] = -i\lambda\Gamma^m_{\alpha\beta}[\tilde{\phi}_m, \tilde{\Lambda}_\beta]\,, \tag{6.10f}$$

where H is the reduced Hamiltonian. Notice the sign changes in the tilde equations of motion. These follow from the unreduced equations of motion, and one can check with (6.9a)–(6.9f) that these signs are consistent with (6.9g).

Supersymmetry and rotational invariance tell us that the corresponding reduced operators annihilate the ground state

$$Q_\alpha|0\rangle = H|0\rangle = J_{mn}|0\rangle = 0 \tag{6.11}$$

and therefore that

$$\begin{aligned} 0 &= \langle 0|Q_\alpha|0\rangle \\ &= \langle 0|(\Gamma^m\Lambda)_\alpha\pi_m + \lambda(\Sigma^{mn}\Lambda)_\alpha[\phi_m,\phi_n]|0\rangle\,, \end{aligned} \tag{6.12a}$$

$$\begin{aligned} 0 &= \langle 0|H|0\rangle \\ &= \langle 0|\pi_m\pi_m - \frac{\lambda^2}{2}[\phi_m,\phi_n][\phi_m,\phi_n] + \lambda[\Lambda_\alpha,\Lambda_\beta]_+\Gamma^m_{\alpha\beta}\phi_m|0\rangle\,, \end{aligned} \tag{6.12b}$$

$$0 = \langle 0|J_{mn}|0\rangle = \langle 0|\pi_{[m}\,\phi_{n]} - \frac{1}{2}\Lambda\Sigma^{mn}\Lambda|0\rangle\,, \tag{6.12c}$$

where the explicit forms of the unreduced operators have been used to evaluate the last form in each relation.

For the reduced angular momenta, the maps of Sec. 2 tell us also that

$$\dot J_{mn} = i[H, J_{mn}] = 0\,, \tag{6.13a}$$

$$[J_{mn}, b_p] = -i\delta_{p[m}b_{n]}\,, \qquad b = \phi,\ \pi,\ \tilde\phi,\ \text{or } \tilde\pi\,, \tag{6.13b}$$

$$[J_{mn}, f_\alpha] = (\Sigma^{mn} f)_\alpha\,, \qquad f = \Lambda,\ \tilde\Lambda \text{ or } Q\,, \tag{6.13c}$$

$$[J_{mn}, J_{pq}] = i(\delta_{q[m}J_{n]p} - \delta_{p[m}J_{n]q}) \tag{6.13d}$$

so that the rotational properties of the reduced fields are the same as in the unreduced theory.

For the reduced supercharges, we find

$$[Q_\alpha, Q_\beta]_+ = 2\big(\delta_{\alpha\beta}H + \lambda\Gamma^m_{\alpha\beta}(\tilde\phi_m - \phi_m)\big)\,, \tag{6.14a}$$

$$\dot Q_\alpha = i[H, Q_\alpha] = \lambda(\tilde\Lambda_\alpha - \Lambda_\alpha)\,, \quad H = \frac{1}{f}\sum_{\alpha=1}^{f} Q_\alpha^2\,, \tag{6.14b}$$

$$[Q_\alpha, \phi_m] = -i(\Gamma^m\Lambda)_\alpha\,, \tag{6.15a}$$

$$[Q_\alpha, \pi_m] = 2i\lambda[\phi_n, (\Sigma^{mn}\Lambda)_\alpha]\,, \tag{6.15b}$$

$$[Q_\alpha, \Lambda_\beta]_+ = \Gamma^m_{\alpha\beta}\pi_m + \lambda\Sigma^{mn}_{\alpha\beta}[\phi_m,\phi_n]\,, \tag{6.15c}$$

$$[Q_\alpha, \tilde\phi_m] = -i(\Gamma^m\tilde\Lambda)_\alpha\,, \tag{6.16a}$$

$$[Q_\alpha, \tilde\pi_m] = -2i\lambda[\tilde\phi_n, (\Sigma^{mn}\tilde\Lambda)_\alpha]\,, \tag{6.16b}$$

$$[Q_\alpha, \tilde\Lambda_\beta]_+ = \Gamma^m_{\alpha\beta}\tilde\pi_m - \lambda\Sigma^{mn}_{\alpha\beta}[\tilde\phi_m,\tilde\phi_n]\,. \tag{6.16c}$$

Note in particular the unexpected form of the extra term on the right side of the reduced supersymmetry algebra (6.14a) and the term on the right side of $\dot Q_\alpha$ in (6.14b), both of which are proportional to the difference of a reduced operator and its tilde. These terms are the reduced analogues of the gauge terms in (6.2b) and (6.3a) respectively. Using (6.9f), we see that these terms are consistent with

$$Q_\alpha|0\rangle = \dot{Q}_\alpha|0\rangle = 0 \tag{6.17}$$

as they should be. Although we have derived this system from the unreduced relations and the maps of Sec. 2, we have checked for example that the reduced equations of motion (6.10) also follow from the form of the reduced Hamiltonian in (6.14b) and the algebra (6.15) and (6.16) of the reduced supercharges with the reduced fields.

A next step toward the solution of Matrix theory would be to find explicit nonlocal forms of the reduced operators Q_α, H and J_{mn}, as we have done for simpler systems in previous sections. Towards this, it will be helpful to look for interacting bosonic Cuntz operators, possibly of the form

$$\begin{aligned} A_m A_n^\dagger &= C_{mn}(\phi, \Lambda)\,, \\ A_m|0\rangle &= 0\,, \\ A_m &= \frac{1}{\sqrt{2}}\big(F_m(\phi,\Lambda) + i\pi_m\big) \end{aligned} \tag{6.18}$$

following the line of our construction for bosonic theories in Sec. 4 and App. E. Owing to the complexity of the Matrix theory ground state,[47,48] however, we have not yet been able to prove the existence of such operators in this case. We note, however, that if such bosonic operators exist, then one also obtains a set of generalized fermionic creation and annihilation operators

$$A_{m\alpha} = i[Q_\alpha, A_m]\,, \qquad A_{m\alpha}|0\rangle = 0 \tag{6.19}$$

whose local composite form can be evaluated with (6.15). These two types of operators correspond to $A \sim a$ and $A_{\dot\alpha} \sim \psi_{\dot\alpha}$ in the simpler supersymmetric models above.

Further study of these generalized free algebras is particularly important for Matrix theory, where the associated new large N-conserved quantities (local and nonlocal) may be related to the question of hidden 11-dimensional symmetry.[41]

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Note Added

After submission of this manuscript, Ref. 49 was called to our attention: In an application to "infinite statistics," this reference contains another independent discovery of the Cuntz algebra in physics and gives our Eq. (3.19a).

Appendix A. Time Reversal and the Tilde Operators

For time-reversal invariant theories, the reduced tilde fields of the text can be related to the time-reversed form of the reduced untilde fields. This is simplest to see for bosonic theories, as follows.

In the coordinate representation, where $\phi_a^m(t=0)$ is real, the usual antiunitary time-reversal operator Θ gives

$$\langle\Theta\alpha|\phi_a^m(t)|\Theta\beta\rangle = \langle\beta|\phi_a^m(-t)|\alpha\rangle\,, \tag{A.1a}$$

$$\langle\Theta\alpha|\pi_a^m(t)|\Theta\beta\rangle = -\langle\beta|\pi_a^m(-t)|\alpha\rangle \tag{A.1b}$$

for general time-independent states α, β in the unreduced theory. Because the Hamiltonian and the gauge generators obey

$$\Theta H_{\bullet} = H_{\bullet}\Theta\,, \qquad \Theta G_{rs} = -G_{sr}\Theta \tag{A.2}$$

we can choose a basis of the unreduced singlet and adjoint states such that

$$\Theta|0_{\bullet}\rangle = |0_{\bullet}\rangle\,, \qquad \Theta|rs, A\rangle = |sr, A\rangle\,. \tag{A.3}$$

Then the definitions (2.13) give the relations between the tilde and untilde reduced fields:

$$\big(\tilde\phi_m(t)\big)_{\mu\nu} = \big(\phi_m(-t)\big)_{\nu\mu}\,, \qquad \big(\tilde\pi_m(t)\big)_{\mu\nu} = -\big(\pi_m(-t)\big)_{\nu\mu}\,, \tag{A.4}$$

where $\mu = (0, A)$ and $\nu = (0, B)$ label the reduced matrix elements of the reduced fields [see, for example, (2.10), (2.12) and (2.54)].

Appendix B. More on Many-Time Wightman Functions

For globally invariant theories, the set of all invariant Wightman functions is the set of averages of all fully-contracted products, each contraction being a summation of a left matrix index with a right matrix index. We find that the equal-time traces

$$\langle{}_{\bullet}0|\,\mathrm{Tr}\left(\frac{\rho_1(t)}{\sqrt{N}}\cdots\frac{\rho_n(t)}{\sqrt{N}}\right)|0_{\bullet}\rangle = N\langle 0|\rho_1(t)\cdots\rho_n(t)|0\rangle\,, \quad \rho = \phi,\ \pi \text{ or } \Lambda \tag{B.1}$$

are the most general invariant equal-time Wightman functions, but the many-time traces

$$\langle{}_{\bullet}0|\,\mathrm{Tr}\left(\frac{\rho_1(t_1)}{\sqrt{N}}\cdots\frac{\rho_n(t_n)}{\sqrt{N}}\right)|0_{\bullet}\rangle = N\langle 0|\rho_1(t_1)\cdots\rho_n(t_n)|0\rangle \tag{B.2}$$

are not the most general invariant Wightman functions.

In particular, there are also a large number of invariant "twisted traces," such as

$$\langle .0|\rho_{rs}(t_1)\rho_{tr}(t_2)\rho_{st}(t_3)|0.\rangle = N^{\frac{5}{2}}\langle 0|\rho(t_1)\tilde{\rho}(t_2)\rho(t_3)|0\rangle \tag{B.3}$$

which can be expressed in terms of our reduced operators ρ and $\tilde{\rho}$ (subscripts are suppressed here for simplicity). Moreover, there are an even larger number of invariant twisted traces, such as

$$\langle .0|\rho_{rs}(t_1)\rho_{tu}(t_2)\rho_{st}(t_3)\rho_{ur}(t_4)|0.\rangle \tag{B.4}$$

which cannot be expressed in terms of our reduced operators.

A rule for deciding whether any such invariant product will have its average expressible in terms of our reduced operators is as follows. Draw a set of index lines joining each pair of contracted indices in the ordered invariant operator product. Then draw a vertical line between each neighboring pair of operators in the product and count how many index lines each vertical line will cut. This operation represents inserting a complete set of intermediate states in the channel defined by each vertical line. If no more than two index lines are cut in a given channel, then this channel is saturated by singlet and adjoint states, and the large N completeness relation (2.9) may be inserted in this channel. If no more than two index lines are cut in any channel of the twisted trace, then the corresponding average can be expressed in terms of our reduced operators. However, if more than two index lines are cut in any channel, then higher representations are necessary to saturate that channel, and the average cannot be written in terms of our reduced operators.

Appendix C. More on Trace Class Operators

In this appendix we prove a theorem about any bosonic (including even fermion number) trace class operator $T.$ and its corresponding reduced operator T. In particular, we will show that, to leading order at large N in the large N Hilbert space (2.9), both $T.$ and T are proportional to the unit operator in their respective spaces.

We begin in the unreduced theory, where the large N completeness statement (2.9) tells us that

$$T.|0.\rangle = |0.\rangle\langle .0|T.|0.\rangle \times \left(1 + O(N^{-1})\right). \tag{C.1}$$

We want to extend this by examining the action of $T.$ on adjoint states. For this, we assume that all the operators and states of interest can be built from products of canonical variables. For example,

$$T. = \mathrm{Tr}\left(t^{(w)}\right), \quad t^{(w)}_{rs} = (\chi^{i_1}\chi^{i_2}\cdots\chi^{i_k})_{rs}, \tag{C.2a}$$

$$\chi^i_{rs} = \left(\frac{\phi^m}{\sqrt{N}}\right)_{rs}, \quad \left(\frac{\pi^m}{\sqrt{N}}\right)_{rs} \text{ or } \left(\frac{\Lambda_\alpha}{\sqrt{N}}\right)_{rs}, \quad \text{word } w = i_1 i_2 \cdots i_k, \tag{C.2b}$$

where t is $O(1)$ and $T.$ is $O(N)$. A basis for the adjoint states is

$$\{|rs, u\rangle = t^{(u)}_{rs}|0.\rangle\}, \tag{C.3}$$

where the basis vectors are labeled by the set of all words u. Finally, the energy eigenstates of the text may be expressed as an expansion in this basis

$$|rs, A\rangle = \sum_u K_u(A)|rs, u\rangle\,. \tag{C.4}$$

Now consider

$$\begin{aligned} T.|rs, u\rangle &= t_{rs}^{(u)} T.|0.\rangle + [T., t_{rs}^{(u)}]|0.\rangle \\ &= |rs, u\rangle\langle .0|T.|0.\rangle\big(1 + O(N^{-1})\big) + \text{Leftover}\,, \end{aligned} \tag{C.5}$$

where the "Leftover" terms come from the commutator. The leading term in (C.5) is $O(N)$, and we will show that the Leftovers are $O(N^{-1})$, so that

$$T.|rs, u\rangle = |rs, u\rangle\langle .0|T.|0.\rangle \times \big(1 + O(N^{-1}) + O(N^{-2})\big)\,. \tag{C.6}$$

To see this, we need the canonical commutation relations

$$\begin{aligned} [\chi_{pq}^i, \chi_{rs}^j]_\mp &= \frac{1}{N}\, c_{ij}\, \delta_{ps}\, \delta_{qr}\,, \\ c_{ij} &= \begin{cases} \delta_{\alpha\beta} & \text{if } i,\ j \text{ denote } \Lambda_\alpha,\ \Lambda_\beta\,, \\ \pm i\delta_{mn} & \text{if } i,\ j \text{ denote } \phi^m,\ \pi^n\,, \\ 0 & \text{otherwise}\,. \end{cases} \end{aligned} \tag{C.7}$$

This allows us to check that

$$\big[T., t_{rs}^{(u)}\big] = \frac{1}{N}\sum_v t_{rs}^{(v)} \sim O(N^{-1})\,, \tag{C.8}$$

where the words v are made up from various parts of the words w and u.

Collecting (C.1) and (C.6), we see that the leading term of $T.$ is proportional to the unreduced unit operator $\mathbf{1.}$

$$T. = \mathbf{1.}\langle .0|T.|0.\rangle \times \big(1 + O(N^{-1}) + O(N^{-2})\big)\,. \tag{C.9}$$

Similarly, we find for the reduced operator

$$T = \mathbf{1}\langle 0|T|0\rangle\big(1 + O(N^{-1}) + O(N^{-2})\big)\,, \tag{C.10}$$

where $\mathbf{1}$ is the reduced unit operator. In the case when $T.$ contains an odd number of fermions, a generalization of this result can be obtained involving the phases $(-1)^F$. We emphasize that the result (C.9) and (C.10) is independent of the scale of $T.$ and T, since a factor like $C(N)$ in (2.25a) can be included if desired on both sides of the result.

As an example, consider the Hamiltonian $H.$, for which

$$H. = E_0\mathbf{1.} + H'_.\,, \qquad E_0 = O(N^2)\,, \qquad H'_. = O(N^0) \tag{C.11}$$

tells us that there are no $O(N^{-1})$ corrections to (C.1) in this case. We then see that

$$H.|rs, A\rangle = E_A|rs, A\rangle = E_0|rs, A\rangle \times \big(1 + O(N^{-2})\big) \tag{C.12}$$

is consistent with

$$\omega_{A0} = E_A - E_0 = O(N^0)\,. \tag{C.13}$$

For the reduced Hamiltonian H, the result above implies

$$H = E_0 \mathbf{1} + H'\,, \qquad H' = O(N^0)\,. \tag{C.14}$$

Many explicit examples of this are worked out in the text. The commutator equations of motion involve only the operator H'. In cases where the leading term is zero, such as the Hamiltonian of SUSY systems or the angular momentum operators, the theorem (C.10) tells us nothing useful.

These results imply that pairs of trace class operators commute to leading order at large N (see also (2.34)). This is easy to see in examples, such as

$$\big[\mathrm{Tr}(\pi\phi), \mathrm{Tr}(\phi^2)\big] = -2i\,\mathrm{Tr}(\phi^2)\,. \tag{C.15}$$

Here each trace is $O(N^2)$, so the commutator is naively of $O(N^4)$. But the leading terms contribute zero in the commutator and the result is $O(N^2)$.

Appendix D. More on the Tilde of a General Density

An algorithm for the computation of the tilde of a general reduced density was given in Subsec. 2.5. Here we provide an alternate derivation which gives a recursive form of the result. Let X_{rs} be any operator that transforms in the adjoint

$$[G_{rs}, X_{pq}] = \delta_{rq}\,X_{ps} - \delta_{sp}\,X_{rq}\,. \tag{D.1}$$

Then, using the definitions of Sec. 2, we get the reduced operators X and $\tilde{X}$.

For any two such operators X and Y, we get another adjoint operator U by composing X and Y as a regular matrix product

$$U_{rs} = X_{rt}Y_{ts} \tag{D.2}$$

and we quickly find that the reduced operators satisfy $U = XY$. The form of the reduced matrix $\tilde{U}$, however, has no simple expression in general.

We can further define an irregular product

$$V_{rs} = X_{ts}Y_{rt} \tag{D.3}$$

which also transforms in the adjoint. The results for the reduced operators in this case are reversed: $\tilde{V} = \tilde{X}\tilde{Y}$; but there is no simple form for V in the general case.

One important special case is when all the matrix elements of X_{rs} and Y_{pq} commute (or anticommute) with one another. Then the regular and irregular composites are equivalent and we have $\widetilde{(XY)} = \pm\tilde{Y}\tilde{X}$.

A broader case is the general density, defined in the text as any composite operator built from repeated regular matrix products of the canonical variables. With the result of App. C we can now give an alternate derivation of the general algorithm given in the text for expressing $\widetilde{U^{(w)}}$ where $U^{(w)} = \chi_{i_1}\chi_{i_2}\cdots\chi_{i_n}$. Using the word notation, with the decomposition $w = ui_n$, we have that

$$U^{(w)}_{rs} = U^{(u)}_{rt}\chi^{i_n}_{ts} = \pm\chi^{i_n}_{ts}U^{(u)}_{rt} + \left[U^{(u)}_{rt}, \chi^{i_n}_{ts}\right]_{\mp}, \tag{D.4a}$$

$$\langle pq, A|U^{(w)}_{rs}|tr, B\rangle \equiv P_{qp,ts}\langle A|\widetilde{U^{(w)}}|B\rangle\,, \tag{D.4b}$$

where (D.4b) is taken from the discussion of the text. From the first term on the right in (D.4a) we get simply $\tilde{\chi}_{i_n}\widetilde{U^{(u)}}$. To evaluate the commutator we use (C.7) repeatedly for each χ factor in $U^{(u)}$ and find a series of terms

$$\sum_k c_{i_k i_n} U^{(x)}_{rs}\frac{1}{N}\,\mathrm{Tr}(U^{(y)})\,, \qquad u = xi_k y\,, \tag{D.5}$$

where k counts the letters in the word u. The matrix element of $U^{(x)}_{rs}$ gives the reduced matrix $\widetilde{U^{(x)}}$ while the trace factor gives $\langle 0|U^{(y)}|0\rangle$, using the result of App. C. Thus we have the following recursion relation for the tilde of a regular operator product:

$$\widetilde{U^{(w)}} = \pm\tilde{\chi}_{i_n}\widetilde{U^{(u)}} + \sum_k (\pm)c_{i_k i_n}\widetilde{U^{(x)}}\langle 0|U^{(y)}|0\rangle\,, \quad w = ui_n = xi_k y i_n\,. \tag{D.6}$$

This result can be iterated to decompose any $\tilde{U}$ into simple $\tilde{\chi}$ factors. The $\pm$ signs are determined by counting the number of times a fermion in χ_{i_n} is moved passed other fermions within U. This result is equivalent to the algorithm given in Subsec. 2.5 for the tilde of a general density.

Appendix E. More on the Bosonic Ground State

In this appendix we supplement the discussion of Sec. 4, showing that the matrix $F_{rs}(\phi)$ can be considered a density at large N. At the same time we will provide a closely related discussion of the quantities $C_{mn} = A_m A^\dagger_n$ which emphasizes their unreduced form. Our starting point is the unreduced large N bosonic ground state wave function

$$\langle\phi|0.\rangle = \psi_0(\phi) = e^{-N^2 S(\phi)} \tag{E.1}$$

whose "action" $S(\phi)$ is a general invariant function which is $O(N^0)$ in the Hilbert space of (2.9). This is the form which results from 't Hooft-scaled potentials, as in (2.50). In the following discussion we will work up to the general case by considering a sequence of simpler special cases.

We begin with the special case of a single matrix ϕ and the special action

$$S(\phi) = \sum_{n=1} s_n\zeta_n(\phi)\,, \qquad \zeta_n(\phi) \equiv \frac{1}{N}\mathrm{Tr}\left[\left(\frac{\phi}{\sqrt{N}}\right)^n\right], \tag{E.2}$$

where s_n are numbers. The trace class quantities ζ_n are $O(N^0)$ in the Hilbert space of (2.9) and $\zeta_0 = 1$. In this case we find directly that F_{rs} is a density

$$\pi_{rs}\psi_o = iF_{rs}\psi_o \,, \qquad F_{rs} = N^{\frac{1}{2}} \sum_{n=1} ns_n \left[\left(\frac{\phi}{\sqrt{N}} \right)^{n-1} \right]_{rs} \tag{E.3}$$

and this translates immediately into the reduced form

$$F = \sum_{n=1} ns_n (\phi)^{n-1} \,. \tag{E.4}$$

Consider next the operator C, defined by

$$C_{rs} \equiv [A_{rt}, (A^\dagger)_{ts}] = i[\pi_{rt}, F_{ts}] \,. \tag{E.5}$$

Because A annihilates the ground state, we can show, using a ϕ basis, that the $A^\dagger A$ term in (E.5) contributes to matrix elements at $O(N^{-2})$ compared to the $AA^\dagger$ term, and so can be neglected

$$C_{rs} = i[\pi_{rt}, F_{ts}] = A_{rt}(A^\dagger)_{ts} \times \left(1 + O(N^{-2})\right) . \tag{E.6}$$

This is the basic step, in the unreduced formulation, which leads to Cuntz-like algebras in the reduced formulation. The following is a sketch of the proof, in which the adjoint basis $|su, n\rangle$ has a norm of $O(N^0)$:

$$\begin{aligned} A_{rt}|su, n\rangle &= A_{rt} N^{\frac{1-n}{2}} (\phi)^n_{su} |0.\rangle \\ &= \frac{1}{\sqrt{2}} N^{\frac{1-n}{2}} \sum_m (\phi)^m_{st} (\phi)^{n-m-1}_{ru} |0.\rangle \end{aligned} \tag{E.7}$$

and we use this formula twice to calculate the appropriate matrix element

$$\begin{aligned} &\langle pq, n'| \frac{(A^\dagger)_{ts}}{\sqrt{N}} \frac{A_{rt}}{\sqrt{N}} |su, n\rangle \\ &\quad = \frac{1}{2} N^{-\frac{n+n'}{2}} \sum_{m,m'} \langle .0|(\phi)^{n+n'-m-2}_{qp} (\phi)^m_{ru} |0.\rangle = P_{qp,ru} \times O(N^{-2}) \,. \end{aligned} \tag{E.8}$$

Using (E.6) and (E.3), we now get C in terms of the action parameters as

$$C_{rs} = i[\pi_{rt}, F_{ts}] = N \sum_{n=1} ns_n \sum_{m=0}^{n-2} \zeta_m(\phi) \left[\left(\frac{\phi}{\sqrt{N}} \right)^{n-m-2} \right]_{rs} . \tag{E.9}$$

This exact expression involves a combination of trace class operators and adjoint operators. To leading order at large N, however, we can use the theorem of App. C to replace the trace class operators by their vev's, showing that C is also a density at large N. Consequently, we obtain the expression

$$C = \sum_{n=1} ns_n \sum_{m=0}^{n-2} \phi^{n-m-2} \langle 0|\phi^m|0\rangle \tag{E.10}$$

for the reduced quantity at large N.

Next, we generalize the action $S(\phi)$ to include any function of the variables $\zeta_n(\phi)$. The evaluation of F proceeds as before

$$F_{rs} = N^{\frac{1}{2}} \sum_{n=1} nS_n(\phi) \left[\left(\frac{\phi}{\sqrt{N}} \right)^{n-1} \right]_{rs}, \qquad S_n(\phi) \equiv \frac{\partial S(\phi)}{\partial \zeta_n} \tag{E.11}$$

so that F_{rs} now also involves a combination of trace class and adjoint operators. To leading order at large N, we may again replace the trace class operators by their vev's,

$$F_{rs} = N^{\frac{1}{2}} \sum_{n=1} n\langle .0|S_n(\phi)|0.\rangle \left(\left(\frac{\phi}{\sqrt{N}} \right)^{n-1} \right)_{rs} \tag{E.12}$$

so that F_{rs} is a density at large N. The reduced operator takes the form

$$F = \sum_{n=1} n\langle 0|S_n(\phi)|0\rangle (\phi)^{n-1} \tag{E.13}$$

and we see that the former constant s_n is simply replaced by another constant $\langle 0|\frac{\partial S(\phi)}{\partial \zeta_n}|0\rangle$ in the formula for F or F_{rs}. In this case, the computation of C_{rs} involves a new type of term:

$$\begin{aligned} C_{rs} &= i[\pi_{rt}, F_{ts}] \\ &= N \sum_{n=1} nS_n(\phi) \sum_{m=0}^{n-2} \zeta_m(\phi) \left[\left(\frac{\phi}{\sqrt{N}} \right)^{n-m-2} \right]_{rs} \\ &\quad + N^{-1} \sum_n \sum_m nmS_{nm}(\phi) \left[\left(\frac{\phi}{\sqrt{N}} \right)^{n+m-2} \right]_{rs}, \end{aligned} \tag{E.14a}$$

$$S_{nm}(\phi) = \frac{\partial^2 S(\phi)}{\partial \zeta_n \partial \zeta_m} \tag{E.14b}$$

but the new (second) term is two powers of N smaller than the first term and can be neglected. The reduced matrix for C, as with F, appears exactly as before, with the same reinterpretation of the numbers s_n.

We may redefine the constants in F and C to find the reduced results

$$\begin{aligned} F(\phi) &= \sum_n f_n \phi^{n-1}, \\ C(\phi) &= \sum_n f_n \sum_{m=0}^{n-m-2} \phi^{n-m-2} \langle 0|\phi^m|0\rangle \end{aligned} \tag{E.15}$$

and the relation

$$C(q) = \langle 0|\frac{F(\phi) - F(q)}{\phi - q}|0\rangle \tag{E.16}$$

is implied by these forms.

We turn finally to the case of many matrices, where it is convenient to use the word notation

$$(\phi^w)_{rs} = (\phi^{m_1}\phi^{m_2}\cdots\phi^{m_n})_{rs}\,, \qquad \phi^w = \phi_{m_1}\phi_{m_2}\cdots\phi_{m_n}\,, \tag{E.17a}$$

$$w = m_1 m_2 \cdots m_n\,, \qquad [w] = n\,. \tag{E.17b}$$

As shown, we will write $[w]$ for the length of the word, and we will write $w_1 \sim w_2$ if the words w_1 and w_2 differ only by a cyclic permutation of their letters. The $O(N^0)$ trace class variables ζ are now defined as

$$\zeta_w(\phi) = \mathrm{Tr}(\phi^w)N^{-1-\frac{1}{2}[w]}\,, \qquad \zeta_0 = 1\,, \tag{E.18}$$

where 0 denotes the null word and we have picked the normalizing constant $C(N) = 1/N$ [see (2.25a)]. The action $S(\phi)$ is a general function of the set of cyclically inequivalent ζ_w's (i.e. the set of all ζ_w's, modded out by the $\sim$ operation).

As in the one-matrix case, the $A^\dagger A$ terms are down

$$C_{rs}^{mn} = i[\pi_{rt}^m, F_{ts}^n] = i[\pi_{ts}^n, F_{rt}^m] = [A_{rt}^m, (A^{n\dagger})_{ts}] \underset{N}{=} A_{rt}^m (A^{n\dagger})_{ts} \tag{E.19}$$

in the definition of C.

Following the one-matrix discussion above, we now find the following expressions for F and C:

$$F_{rs}^m = N^{\frac{1}{2}} \sum_w S_w(\phi) \sum_{w\sim mu} \left(\left(\frac{\phi}{\sqrt{N}}\right)^u\right)_{rs}\,, \tag{E.20a}$$

$$C_{rs}^{mn} \underset{N}{=} N \sum_w S_w(\phi) \sum_{w\sim munv} \zeta_v(\phi)\left(\left(\frac{\phi}{\sqrt{N}}\right)^u\right)_{rs}\,, \tag{E.20b}$$

$$S_w(\phi) = \frac{\partial S(\phi)}{\partial \zeta_w}\,. \tag{E.20c}$$

Then using the theorem of App. C, we find that both sets of quantities are densities at large N, with the reduced forms:

$$F_m \underset{N}{=} \sum_w \langle 0|S_w(\phi)|0\rangle \sum_{w\sim mu} \phi^u\,, \tag{E.21a}$$

$$C_{mn} \underset{N}{=} \sum_w \langle 0|S_w(\phi)|0\rangle \sum_{w\sim munv} \langle 0|\zeta_v(\phi)|0\rangle \phi^u\,. \tag{E.21b}$$

These relations generalize Eqs. (4.13b) and (4.17b) of the text.

Appendix F. More on the Operators C and D

In unreduced form, the operators C and D of Sec. 4 are defined by

$$A_{rt}^m (A^{n\dagger})_{ts} = C_{rs}^{mn} + (A^{n\dagger})_{ts} A_{rt}^m\,, \tag{F.1a}$$

$$A_{ts}^m (A^{n\dagger})_{rt} = D_{rs}^{mn} + (A^{n\dagger})_{rt} A_{ts}^m\,, \tag{F.1b}$$

$$\begin{aligned} A^m_{rs} &= \frac{1}{\sqrt{2}}(F^m_{rs} + i\pi^m_{rs})\,, \\ (A^{m\dagger})_{rs} &= \frac{1}{\sqrt{2}}(F^m_{rs} - i\pi^m_{rs}) \end{aligned} \tag{F.1c}$$

and so we see that C^{mn}_{rs} and D^{mn}_{rs} are each a mixture of regular and irregular composites (see App. D).

In the previous appendix, we noted that the last terms in (F.1a), (F.1b) are down by $O(N^{-2})$, and this gives the reduced expressions

$$A_m A^\dagger_n = C_{mn}\,, \qquad \tilde{A}_m \tilde{A}^\dagger_n = \tilde{D}_{mn} \tag{F.2}$$

in the large N limit.

Here we will find a relation between C and D using their exact definitions as commutators in the unreduced theory

$$C^{mn}_{rs} = [A^m_{rt}, (A^{n\dagger})_{ts}] = -\frac{i}{2}[F^m_{rt}, \pi^n_{ts}] + \frac{i}{2}[\pi^m_{rt}, F^n_{ts}]\,, \tag{F.3a}$$

$$D^{mn}_{rs} = [A^m_{ts}, (A^{n\dagger})_{rt}] = -\frac{i}{2}[F^m_{ts}, \pi^n_{rt}] + \frac{i}{2}[\pi^m_{ts}, F^n_{rt}]\,. \tag{F.3b}$$

By examining these formulas, we find that

$$D^{mn}_{rs} = C^{nm}_{rs} \tag{F.4}$$

the reduced form of which was derived by other means in (4.12c).

Using the flatness relations (4.4b) with (F.3b) we also obtain

$$\tilde{D}_{mn} = i\tilde{\pi}_m \tilde{F}_n - i(\widetilde{F_n \pi_m}) = -i\tilde{F}_m \tilde{\pi}_n + i(\widetilde{\pi_n F_m}) \tag{F.5}$$

for the reduced operator $\tilde{D}$; and from this follows the surprising formula

$$(\widetilde{A^\dagger_m A_n}) = 0\,. \tag{F.6}$$

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