ASYMPTOTIC SEARCH FOR GROUND STATES
OF SU(2) MATRIX THEORY

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We introduce a complete set of gauge-invariant variables and a generalized Born–Oppenheimer formulation to search for normalizable zero-energy asymptotic solutions of the Schrödinger equation of SU(2) matrix theory. The asymptotic method gives only ground state candidates, which must be further tested for global stability. Our results include a set of such ground state candidates, including one state which is a singlet under spin (9).

1. Introduction

The $N = 16$ supersymmetric gauge quantum mechanics,[1–3] including its action formulation by dimensional reduction, was first studied in 1984–85. The model was noted again in 1988–89 as a regularization,[4] with a continuous spectrum,[5] of the $D = 11$ supermembrane. In early 1996 the model was identified[6] as the dynamics of interacting $D0$-branes, which led to further study, including a truncated version of the model[7] and the identification by $D0$ scattering[8,9] of the scale of $D0$ physics with the scale of $D = 11$ supergravity.

Interest in the $N = 16$ model exploded in late 1996, when the large n limit of the model, now christened matrix theory, was proposed[10] as a nonperturbative formulation of M theory. Among the many papers since then, we mention only the extension[11] of the conjecture to include finite n and those papers with direct relevance to the ground state of the theory, in particular the study of the Witten index of the theory[12–14] and the ongoing study of the zero supercharge condition for SUSY ground states.[15,16]

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Since the original work of Claudson and Halpern, however, the ground state wave function of the theory has remained elusive. One obstruction to the investigation of such dynamical questions, pointed out in the original paper, is that matrix theory has no conserved fermion number, which blocks the fermion sector analysis applicable to simpler supersymmetric quantum-mechanical systems. As a consequence, one expects that any particular matrix theory eigenstate is spread over a considerable portion of the fermionic Hilbert space. The lore[6,10] is that the theory should have a unique normalizable zero-energy "threshold" bound state, which is a singlet under spin (9).

In this paper we develop an asymptotic method to search for zero-energy ground states of the SU(2) matrix theory. The method has two basic ingredients:

- A complete set of gauge-invariant bosonic and fermionic variables,
- A generalized Born–Oppenheimer formulation,

which allow us to extend some of the ideas of Ref. 7. Moreover, there are strong parallels between our generalized Born–Oppenheimer formulation and the analysis of Ref. 13. The method yields only candidate ground states, which are gauge-invariant asymptotic solutions, near the flat directions of the potential, of the zero-energy Schrödinger equation of the theory. The ground state candidates must be further checked for global stability at nonasymptotic values of the gauge-invariant distance R.

Our results include a set of such candidate ground states, including exactly one state which is a singlet under spin (9) and which, as it turns out, has bosonic angular momentum $l = 2$. The fermionic structure of the ground state candidates is relatively simple in the asymptotic domain, though one expects increasing complexity at higher order in R^{-1}.

Matrix theory

We will follow the original notation[1] for the theory, beginning with the 16 supercharges Q_α,

$$Q_\alpha = (\Gamma^m \Lambda_a)_\alpha \pi_a^m + ig f_{abc} (\Sigma^{mn} \Lambda_a)_\alpha \phi_b^m \phi_c^n \,, \tag{1.1a}$$

$$[\phi_a^m, \pi_b^n] = i\hbar \delta_{ab} \delta_{mn} \,, \qquad \{\Lambda_{a\alpha}, \Lambda_{b\beta}\} = \delta_{ab} \delta_{\alpha\beta} \,, \tag{1.1b}$$

$$\{\Gamma^m, \Gamma^n\} = 2\delta_{mn} \,, \qquad \Sigma^{mn} = -\frac{i}{4}[\Gamma^m, \Gamma^n] \,, \tag{1.1c}$$

$$a = 1 \cdots g \,, \quad m = 1 \cdots 9 \,, \quad \alpha = 1 \cdots 16 \,, \tag{1.1d}$$

where ϕ_a^m are the real bosonic variables and f_{abc} are the Cartesian structure constants of any compact Lie algebra with dimension g. The gamma matrices $(\Gamma^m)_{\alpha\beta}$ are real, symmetric and traceless and the fermions $\Lambda_{a\alpha}$ are real. We will also need the generators G_a of gauge transformations,

$$G_a = f_{abc}\left(\phi_b^m \pi_c^m - \frac{i\hbar}{2}\Lambda_{b\alpha}\Lambda_{c\alpha}\right) \,, \tag{1.2}$$

and the generators J^{mn} of spin (9),

$$J^{mn} = \pi_a^{[m}\phi_a^{n]} - \frac{\hbar}{2}\Lambda_{a\alpha}(\Sigma^{mn})_{\alpha\beta}\Lambda_{a\beta}\,, \tag{1.3}$$

where $[mn]$ means antisymmetrization of indices.

The supercharges satisfy

$$\{Q_\alpha, Q_\beta\} = 2\delta_{\alpha\beta}H + 2g(\Gamma^m)_{\alpha\beta}\phi_a^m G_a\,, \tag{1.4}$$

where H is the Hamiltonian

$$H = H_B + H_F\,, \tag{1.5a}$$

$$H_B = \frac{1}{2}\pi_a^m\pi_a^m + V\,, \qquad V = \frac{g^2}{4}f_{abc}\phi_b^m\phi_c^n f_{ade}\phi_d^m\phi_e^n\,, \tag{1.5b}$$

$$H_F = -\frac{ig\hbar}{2}f_{abc}\Lambda_{a\alpha}(\Gamma^m)_{\alpha\beta}\phi_b^m\Lambda_{c\beta}\,, \tag{1.5c}$$

and the gauge-invariant states $G_a|G.I.\rangle = 0$ form the physical subspace of the theory.

2. Bosonic Preliminaries

In this section we sharpen our tools on some bosonic subproblems, allowing the Lorentz vector index to run over $m = 1 \cdots d$, $d \geq 3$ for generality, although $d = 9$ for matrix theory.

2.1. *Gauge-invariant bosonic variables*

For the gauge group SU(2), with $f_{abc} = \varepsilon_{abc}$, it is useful to define the real symmetric matrix Φ,

$$\Phi_{ab} \equiv \phi_a^m\phi_b^m\,, \qquad a, b = 1, 2, 3\,, \tag{2.1}$$

and the solutions to its eigenvalue problem,

$$\Phi_{ab}\psi_b^i = \lambda_i^2\psi_a^i\,, \qquad i = 1, 2, 3\,, \tag{2.2a}$$

$$\psi_a^i\psi_a^j = \delta_{ij}\,, \qquad \psi_a^i\psi_b^i = \delta_{ab}\,, \tag{2.2b}$$

$$\lambda_3 \geq \lambda_2 \geq \lambda_1 \geq 0\,. \tag{2.2c}$$

The eigenvectors ψ form a real orthogonal matrix and the eigenvalues λ are a complete set of rotation- and gauge-invariant bosonic variables for this case.

A complete set of $3(d-1)$ independent gauge-invariant bosonic variables includes the three eigenvalues λ and the $3(d-2)$ gauge-invariant angular variables

$$\eta_i^m \equiv \frac{\phi_a^m\psi_a^i}{\lambda_i}\,, \qquad \eta_i^m\eta_j^m = \delta_{ij}\,. \tag{2.3}$$

In the first part of this paper, we focus primarily on the gauge- and rotation-invariant λ's, returning to the η's in Sec. 7. For the gauge group SU(3), there are more gauge-invariant variables, including $d_{abc}\phi_a^m\phi_b^n\phi_c^p$.

On functions of $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, the bosonic Hamiltonian takes the form

$$H_B = -\frac{\hbar^2}{2}\Delta + V\,, \qquad \Delta = \partial_a^m\partial_a^m\,, \tag{2.4a}$$

$$\Delta f(\lambda) = \rho^{-1}\frac{\partial}{\partial\lambda_i}\left(\rho\frac{\partial}{\partial\lambda_i}f(\lambda)\right)\,, \tag{2.4b}$$

$$V = \frac{g^2}{2}(\lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2)\,, \tag{2.4c}$$

$$\rho(\lambda) \equiv (\lambda_1\lambda_2\lambda_3)^{d-3}(\lambda_3^2 - \lambda_1^2)(\lambda_3^2 - \lambda_2^2)(\lambda_2^2 - \lambda_1^2) \geq 0\,, \tag{2.4d}$$

where $\partial_a^m = \partial/\partial\phi_a^m$ and H_B is Hermitian in the inner product

$$\int d^3\lambda\rho(\lambda)f^*(\lambda)g(\lambda)\,, \qquad d^3\lambda \equiv d\lambda_1 d\lambda_2 d\lambda_3\,. \tag{2.5}$$

More generally, the full bosonic measure is

$$(d\phi) = d^3\lambda\rho(\lambda)(d\Omega)\,, \tag{2.6a}$$

$$\int (d\Omega) = 1\,, \tag{2.6b}$$

$$\int (d\Omega)f(\phi) = 0 \quad \text{when } f(\phi) = -f(-\phi)\,, \tag{2.6c}$$

where Ω are $3(d-1)$ "angles" [which include the $3(d-2)$ gauge-invariant angles η in (2.3), plus three gauge degrees of freedom]. Through Sec. 6 of this paper, the relations (2.6b) and (2.6c) are all we shall need to know about Ω.

2.2. *Zero-energy Hamilton–Jacobi equation*

It was emphasized by Claudson and Halpern that a SUSY ground state must satisfy the zero-energy Hamilton–Jacobi equation

$$\psi \underset{\hbar\to 0}{\sim} \exp\left(\pm\frac{S_0}{\hbar}\right)\,, \tag{2.7a}$$

$$\frac{1}{2}|\nabla S_0|^2 = V\,, \tag{2.7b}$$

in the extreme semiclassical limit, and this equation takes the d-independent form

$$\left(\frac{\partial S_0(\lambda)}{\partial\lambda_i}\right)\left(\frac{\partial S_0(\lambda)}{\partial\lambda_i}\right) = g^2(\lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2) \tag{2.8}$$

when we restrict ourselves to gauge- and rotation-invariant wave functions. These authors also gave an exact solution of Eq. (2.7b) or (2.8):

$$S_0(\lambda) = \sqrt{W}, \tag{2.9a}$$

$$W = \frac{1}{6}g^2 \varepsilon_{abc}\varepsilon_{def}\phi_a^m\phi_b^n\phi_c^p\phi_d^m\phi_e^n\phi_f^p \tag{2.9b}$$

$$= g^2 \det \Phi \tag{2.9c}$$

$$= g^2 \lambda_1^2 \lambda_2^2 \lambda_3^2. \tag{2.9d}$$

Here, we make a brief systematic study of the solutions of the zero-energy Hamilton–Jacobi equation (2.8).

For this investigation, it is convenient to introduce spherical coordinates

$$\lambda_1 = r\sin\theta\cos\phi, \quad \lambda_2 = r\sin\theta\sin\phi, \quad \lambda_3 = r\cos\theta, \tag{2.10a}$$

$$r^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \tag{2.10b}$$

and to write the solution in the form

$$S_0 = gr^3 F(\theta, \phi). \tag{2.11}$$

Then the Hamilton-Jacobi equation (2.8) reduces to

$$\sin^2\theta(9F^2 + F_\theta^2) + F_\phi^2 = \sin^4\theta(\cos^2\theta + \sin^2\theta\cos^2\phi\sin^2\phi), \tag{2.12}$$

where the subscripts denote partial derivatives. The right side of (2.12) is proportional to V, so the flat directions of V correspond to $\theta = 0$ in these variables.

For small θ, (2.12) admits two solutions which are nonsingular as $\theta \to 0$:

$$S_0 = \frac{gr^3\theta^2}{2}\left\{ \begin{array}{c} \sin 2(\phi - \phi_0) \\ 1 \end{array} \right\} + O(\theta^3). \tag{2.13}$$

The first solution, which we call the Claudson–Halpern (CH) branch, contains the CH solution $S_0(\lambda) = \sqrt{W}$ when $\phi_0 = 0$, and the second solution is the solution $S_0(\lambda) \cong \frac{V}{gr}$, studied later by Itoyama.[17,18] The $\phi_0 \neq 0$ solutions of the CH branch are new.

Because the exponential decrease of $\exp(-\frac{S_0}{\hbar})$ at large r is lost near $\theta = 0$, none of these solutions is normalizable,[a] whether we choose the naive measure $d^3\lambda$ or the quantum measure $d^3\lambda\rho(\lambda)$. More precisely, we find nonnormalizability in the flat directions

[a]We note in passing that the full bosonic Hamiltonian H_B has exact gauge-invariant zero-energy solutions

$$\psi(W) = W^\gamma K_{2|\gamma|}\left(\frac{\sqrt{W}}{\hbar}\right) \underset{\hbar \to 0}{\sim} \exp\left(-\frac{\sqrt{W}}{\hbar}\right), \qquad \gamma = \frac{4-d}{4},$$

and $K \to I$, where K and I are cylinder functions of imaginary argument. These solutions are quantum extensions of the Claudson–Halpern solution, but neither is normalizable.

$$\int d^3\lambda \exp\left(-\frac{2S_0}{\hbar}\right) \propto \int_0 \frac{d\theta}{\theta}, \tag{2.14a}$$

$$\int d^3\lambda \rho(\lambda) \exp\left(-\frac{2S_0}{\hbar}\right) \propto \int_0 \frac{d\theta}{\theta^3} \tag{2.14b}$$

for the Itoyama solution and the CH branch with $0 < \phi_0 < \frac{\pi}{4}$. (For small θ the range of the angle ϕ is $\frac{\pi}{4} \le \phi \le \frac{\pi}{2}$.) The CH solution itself, with $\phi_0 = 0$, has an extra multiplicative divergence in the ϕ integration.

It is clear that the Hamilton–Jacobi equation by itself is unable to choose among solutions or to answer the question of normalizability: any of the solutions might, in principle, be made normalizable by quantum corrections including the fermions, and

$$\psi \sim \exp\left[-\frac{1}{\hbar}(S_0 + \hbar\alpha \ln r)\right] \tag{2.15}$$

is normalizable in $d^3\lambda\rho(\lambda)$ when $\alpha > \frac{3}{2}$ for the Itoyama solution and for the CH branch with $0 < \phi_0 < \frac{\pi}{4}$. For the CH solution itself, normalizability requires that $\alpha > 3$. In what follows, our task is to study such quantum corrections in detail.

It is also important to note that our study of the zero-energy Hamilton–Jacobi equation is incomplete because the full equation (2.7b) allows other (gauge- but not rotation-invariant) solutions, which include the η variables in (2.3) as well (see Sec. 7).

2.3. *Born–Oppenheimer approximation*

Our approach in this paper follows the line of the Born–Oppenheimer approximation,[19] which we illustrate first on the gauge- and rotation-invariant sector of the bosonic Hamiltonian

$$H_B\psi(\lambda) = E\psi(\lambda), \tag{2.16a}$$

$$V = \frac{g^2}{2}[R^2(\lambda_1^2 + \lambda_2^2) + \lambda_1^2\lambda_2^2], \tag{2.16b}$$

where we have set $R = \lambda_3$. Our goal is to study the asymptotic behavior at large R near the classical flat directions, $\lambda_1 = \lambda_2 = 0$, of V. In the language of the Born–Oppenheimer approximation, we integrate out the "fast" variables λ_1, λ_2 to obtain an effective Hamiltonian for the asymptotic behavior in the "slow" variable $R = \lambda_3$.

Toward this end we first decompose the Hamiltonian and the measure as

$$H_B = H_0 + H_1, \tag{2.17a}$$

$$H_0 = -\frac{\hbar^2}{2}\left[\frac{\partial^2}{\partial\lambda_1^2} + \left(\frac{d-3}{\lambda_1} + \frac{2\lambda_1}{\lambda_1^2 - \lambda_2^2}\right)\frac{\partial}{\partial\lambda_1} + \frac{\partial^2}{\partial\lambda_2^2}\right.$$
$$\left. + \left(\frac{d-3}{\lambda_2} + \frac{2\lambda_2}{\lambda_2^2 - \lambda_1^2}\right)\frac{\partial}{\partial\lambda_2}\right] + \frac{g^2}{2}R^2(\lambda_1^2 + \lambda_2^2), \tag{2.17b}$$

$$H_1 = -\frac{\hbar^2}{2}\left[\frac{\partial^2}{\partial R^2} + \left(\frac{d+1}{R} + \frac{2(\lambda_1^2 + \lambda_2^2)}{R^3} + \cdots\right)\frac{\partial}{\partial R}\right.$$

$$\left. -\frac{2}{R^2}\left(\lambda_1\frac{\partial}{\partial\lambda_1} + \lambda_2\frac{\partial}{\partial\lambda_2} + \cdots\right)\right] + \frac{g^2}{2}\lambda_1^2\lambda_2^2\,, \tag{2.17c}$$

$$\rho = R^{d+1}\sigma\,, \tag{2.17d}$$

$$\sigma \equiv (\lambda_1\lambda_2)^{d-3}(\lambda_2^2 - \lambda_1^2)\left(1 - \frac{\lambda_1^2}{R^2}\right)\left(1 - \frac{\lambda_2^2}{R^2}\right), \tag{2.17e}$$

where we will see that H_0 is the dominant part of H_B at large R and the dots in H_1 indicate terms with higher inverse powers of R.

The first term H_0 in (2.17b) describes a rotation- and gauge-invariant two-dimensional oscillator whose frequency is linear in R. The nodeless eigenstate

$$u_R(\lambda_1, \lambda_2) = C(R)R^{\frac{(d-1)}{2}}\exp\left[-\left(\frac{gR}{2\hbar}\right)(\lambda_1^2 + \lambda_2^2)\right], \tag{2.18a}$$

$$E_0(R) = \hbar gR(d-1)\,, \tag{2.18b}$$

$$\int d^2\lambda\,\sigma|u_R|^2 = 1 \tag{2.18c}$$

is almost certainly the unique ground state of H_0 (see App. G), where $E_0(R)$ is the energy and $d^2\lambda = d\lambda_1 d\lambda_2$. The power of R in u_R guarantees that $C(R)$ approaches a constant at large R,

$$|C(R)|^2 = \frac{(d-3)!}{2^{d-1}}[1 + O(R^{-3})]\,, \tag{2.19}$$

because

$$\lambda_1, \lambda_2 = O\left[\left(\frac{\hbar}{g}\right)^{\frac{1}{2}}R^{-\frac{1}{2}}\right] \tag{2.20}$$

when averaged over $|u_R|^2$. The orders of magnitude in (2.20) define the quantum neighborhood of the classical flat directions of the potential.

In this paper we compute only through $O(R^{-2})$, and, for this purpose, $C(R)$ may be treated as a constant. Similarly, the measure σ in (2.17e) can be replaced by its asymptotic form

$$\sigma \to \sigma_\infty = (\lambda_1\lambda_2)^{d-3}(\lambda_2^2 - \lambda_1^2) \tag{2.21}$$

in all computations through $O(R^{-2})$.

More generally, all the eigenfunctions of H_0 can be written as the normalizing power of R in (2.18a) times functions of the scaled variables

$$z_1 = \lambda_1\left(\frac{g}{\hbar}R\right)^{\frac{1}{2}}, \qquad z_2 = \lambda_2\left(\frac{g}{\hbar}R\right)^{\frac{1}{2}}, \tag{2.22}$$

and the corresponding eigenvalues are all proportional to R. At finite values of $z_{1,2}$, it follows that, throughout the Hilbert space of H_0, we may estimate the order of magnitude of λ_1 or λ_2 at large R as $O(R^{-\frac{1}{2}})$ [as recorded in (2.20)], and the derivatives with respect to λ_1 or λ_2 as $O(R^{\frac{1}{2}})$. Using these orders of magnitude, one sees that H_0 is the dominant part of H_B [contains all terms of $O(R)$] in the gauge- and rotation-invariant sector and $H_1 = O(R^{-2})$.

Another way to view the large R expansion of this paper, although we have chosen not to write things out in this way, is to use as independent variables $(z_1, z_2, \lambda_3 = R)$, and then formally expand in powers of R^{-1}.

The conventional Born–Oppenheimer approximation is essentially first order perturbation theory in H_1 around u_R. In variational language, we study a separable trial wave function of the form

$$\psi(\lambda) = u_R(\lambda_1, \lambda_2)\psi(R)\,, \tag{2.23}$$

where $\psi(R)$ may be called the reduced wave function or state vector. Averaging over the fast variables, we obtain an effective Hamiltonian for the slow variable R,

$$H_{\text{eff}}(R)\psi(R) = E\psi(R)\,, \tag{2.24a}$$

$$H_{\text{eff}}(R) = \int d^2\lambda\sigma u_R^* H_B u_R\,, \tag{2.24b}$$

$$\int dR R^{d+1}|\psi(R)|^2 < \infty\,, \tag{2.24c}$$

where the normalization condition on the reduced state vector is given in (2.24c). The effective Hamiltonian (2.24b) can be evaluated exactly but we confine ourselves in this paper to the leading terms [through $O(R^{-2})$] at large R.

Using the integrals given in App. F, we obtain the asymptotic form of the effective Hamiltonian

$$H_{\text{eff}}(R) = g\hbar(d-1)R - \frac{\hbar^2}{2}\left(\frac{d^2}{dR^2} + \frac{d+1}{R}\frac{d}{dR} + \frac{B}{R^2} + \cdots\right) \tag{2.25a}$$

$$B = -\frac{(d-1)(d-9)}{4} \tag{2.25b}$$

whose linear potential is nothing but $E_0(R)$ in (2.18b). The coefficient of the first derivative term in (2.25a) could have been fixed in advance by hermiticity of H_{eff} in the reduced measure R^{d+1} of Eq. (2.24c), and in fact the operator

$$\Delta_R = \frac{d^2}{dR^2} + \frac{d+1}{R}\frac{d}{dR} \tag{2.26}$$

is the natural Laplacian on this measure.

For this bosonic case, the positive potential growing linearly with R gives $E > 0$ normalizable bound states which show exponential decrease,

$$\psi(R) \sim \exp\left[-\frac{2}{3}\left(\frac{2g(d-1)}{\hbar}\right)^{\frac{1}{2}} R^{\frac{3}{2}}\right]\,, \tag{2.27}$$

at large R. For the full matrix theory, we expect from Ref. 7 that the fermionic contributions will exactly cancel[b] the bosonic contribution $E_0(R) = 8g\hbar R$, leaving an effective Hamiltonian of the form

$$H_{\text{eff}}(R) = -\frac{\hbar^2}{2}\left(\frac{d^2}{dR^2} + \frac{d+1}{R}\frac{d}{dR} + \frac{B'}{R^2} + \cdots\right),\qquad(2.28)$$

and such a Hamiltonian can have an asymptotic power-law-behaved zero-energy normalizable bound state, provided that

$$B' < \frac{d^2 - 4}{4}\,.\qquad(2.29)$$

The problem here is that the Born–Oppenheimer approximation cannot be trusted to give the true value of the constant B', even approximately, because (unlike molecular physics) matrix theory has no natural small parameters to control the approximation. In what follows, we develop an improved formalism which allows us to compute the necessary coefficient B' exactly in matrix theory.

3. Generalized Perturbation Theory

In order to study the asymptotic behavior of the wave function, we need a procedure which combines the idea of the Born–Oppenheimer approximation with the techniques of perturbation theory. Here is such a general formalism for studying the equation

$$\mathcal{L}|\Psi\rangle = 0\,,\qquad(3.1)$$

in which the linear operator is $\mathcal{L} = H - E$ and $|\Psi\rangle$ is a vector in the Hilbert space of H.

We start by choosing a normalized state $|\cdot\rangle$ in the Hilbert space and its associated projection operators

$$P = P^2 = |\cdot\rangle\langle\cdot|\,,\qquad Q = Q^2 = 1 - P\,,\qquad(3.2)$$

and the action of these projection operators on the state vector will be written as

$$|\Psi_P\rangle = P|\Psi\rangle\,,\qquad |\Psi_Q\rangle = Q|\Psi\rangle\,.\qquad(3.3)$$

The original Schrödinger equation (3.1) is then broken down into two coupled equations. The first equation is

$$P\mathcal{L}P|\Psi_P\rangle + P\mathcal{L}Q|\Psi_Q\rangle = 0\qquad(3.4)$$

or, equivalently,

$$\langle\cdot|\mathcal{L}|\cdot\rangle\langle\cdot|\Psi_P\rangle + \langle\cdot|\mathcal{L}Q|\Psi_Q\rangle = 0\,,\qquad(3.5)$$

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[b]Following Ref. 7, we expect that sectors with uncanceled linear R terms are associated to excited states.

and the second equation is

$$QLP|\Psi_P\rangle + QLQ|\Psi_Q\rangle = 0, \tag{3.6}$$

which can be formally solved as

$$|\Psi_Q\rangle = -(QLQ)^{-1}QLP|\Psi_P\rangle. \tag{3.7}$$

If we substitute (3.7) into (3.4), we get the "reduced" Schrödinger equation,

$$[PLP - PLQ(QLQ)^{-1}QLP]|\Psi_P\rangle = 0 \tag{3.8}$$

or, equivalently,

$$[\langle\cdot|L|\cdot\rangle - \langle\cdot|LQ(QLQ)^{-1}QL|\cdot\rangle]\langle\cdot|\Psi\rangle = 0. \tag{3.9}$$

One can also write a variational principle for the exact solution of (3.6):

$$J[\chi] = \langle\chi|QLQ|\chi\rangle + 2\langle\chi|QLP|\Psi_P\rangle, \tag{3.10}$$

where J is stationary under variations of $|\chi\rangle$ about $|\Psi_Q\rangle$.

This formulation is exact and can be adapted to a number of different applications. For the familiar problem of nondegenerate perturbation theory, where $L = H_0 - E + V$, one chooses P to project onto a particular eigenstate of H_0, and then the introduction of power series expansions into Eqs. (3.7) and (3.8) leads to familiar formulas.

We may illustrate this situation by choosing

$$|\Psi\rangle = |p\rangle, \qquad (H_0 + V)|p\rangle = E_p|p\rangle, \qquad H_0|p\rangle^0 = E_p^0|p\rangle^0, \tag{3.11a}$$

$$|\cdot\rangle = |p\rangle^0, \qquad P = |p\rangle^0\,{}^0\langle p|, \qquad |\Psi_P\rangle = |p\rangle^0\,{}^0\langle p|p\rangle. \tag{3.11b}$$

Equation (3.8) then becomes the energy equation,

$$E_p = E_p^0 + V_{pp} + {\sum_{m,n}}' V_{pm}[1 + Q(H_0 - E_p)^{-1}QVQ]_{mn}^{-1}\frac{V_{np}}{E_p - E_n^0}, \tag{3.12}$$

and Eq. (3.7) becomes the wave function equation,

$${}^0\langle m \neq p|p\rangle = \left\{{\sum_n}'[1 + Q(H_0 - E_p)^{-1}QVQ]_{mn}^{-1}\frac{V_{np}}{E_p - E_n^0}\right\}{}^0\langle p|p\rangle, \tag{3.13}$$

and both are easily iterated to any desired order of the perturbation V. In this example, each choice of projector P is a choice to study a "nearby" exact state $|\Psi\rangle$.

In the case of degenerate perturbation theory, one starts by choosing P as the projector into the degenerate subspace of interest and Eq. (3.8) becomes a matrix equation in that subspace.

For generalized Born–Oppenheimer problems, we proceed as follows. The original problem involves a number of coordinates, which we partition into two groups, called x (the "fast" variables) and y (the "slow" variables):

$$|\Psi\rangle = |\Psi(x,y)\rangle \,, \tag{3.14}$$

and we choose a particular projector P to act only on the x variables:

$$|\cdot\rangle = |\psi_0(x)\rangle_R \,, \tag{3.15a}$$

$$P = |\cdot\rangle\langle\cdot| = |\psi_0(x)\rangle_R \int dx'\,_R\langle\psi_0(x')| \,. \tag{3.15b}$$

In this class of applications, the partition into fast and slow variables and the choice of the projector state and its symmetry determine a preferred sector of the Hilbert space. In practice, our choice of projector state $|\cdot\rangle$ below will be guided by the need to cancel the linear term in R in (2.25). The projected state is

$$|\Psi_P\rangle = |\psi_0(x)\rangle_R |\psi(y)\rangle \,, \tag{3.16a}$$

$$|\psi(y)\rangle = \langle\cdot|\Psi(x,y)\rangle = \int dx'\,_R\langle\psi_0(x')|\Psi(x',y)\rangle \,, \tag{3.16b}$$

where $|\psi(y)\rangle$ will be called the reduced state vector. Note that inner products with this projection operator involve integration over the fast variables x but not over the slow variables y; the symbol R stands for a subset of the y variables, and the subscript R is placed on the projector state $|\psi_0(x)\rangle_R$ to indicate that this vector in the Hilbert space of the x variables may be parametrized by some of the y variables.

In these applications Eqs. (3.4) and (3.8) are reduced Schrödinger equations in the slow variables y, the fast variables x having been integrated out, and in particular Eq. (3.9),

$$\{\langle\cdot|H - E|\cdot\rangle - \langle\cdot|HQ(Q(H - E)Q)^{-1}QH|\cdot\rangle\}|\psi(y)\rangle = 0 \,, \tag{3.17}$$

is the effective Schrödinger equation for the reduced state vector $|\psi(y)\rangle$. The terms of (3.17) are in 1–1 correspondence with the terms of (3.4),

$$\int dx\,_R\langle\psi_0(x)|H - E|\psi_0(x)\rangle_R|\psi(y)\rangle + \int dx\,_R\langle\psi_0(x)|H|\Psi_Q(x,y)\rangle = 0 \,, \tag{3.18}$$

and the first term would give the "first order" Born–Oppenheimer approximation [i.e. Eqs. (2.24a) and (2.24b)] if we were to ignore the second term. The second term can contribute in principle, however, to the effective Hamiltonian for $|\psi(y)\rangle$, and so we must proceed to solve the other Eq. (3.6),

$$QH|\psi_0(x)\rangle_R|\psi(y)\rangle + Q(H - E)|\Psi_Q(x,y)\rangle = 0 \,, \tag{3.19}$$

for the state $|\Psi_Q(x,y)\rangle$. If we have some small quantity, such as $\frac{1}{R}$ at large R, solutions of the system may be carried out in practice to any desired order of the small quantity.

Application of this machinery to matrix theory requires that we first make some transformations from the original variables.

4. Canonical Transformations

We focus now on the fermionic variables $\Lambda_{a\alpha}$ of matrix theory and carry out canonical transformations in order to introduce gauge-invariant fermions (Subsec. 4.1) and to obtain a form of the Hamiltonian (Subsec. 4.2) which is amenable to the computational method of the previous section.

In what follows we scale out $\hbar$ and the coupling constant g, according to the relations

$$Q_\alpha(\hbar, g; \phi) = (g\hbar^2)^{\frac{1}{3}} Q_\alpha(1, 1; \phi'), \tag{4.1a}$$

$$H(\hbar, g; \phi) = (g\hbar^2)^{\frac{2}{3}} H(1, 1; \phi'), \tag{4.1b}$$

$$\phi = \left(\frac{\hbar}{g}\right)^{\frac{1}{3}} \phi', \tag{4.1c}$$

and it is really ϕ' which appears below, although we drop the prime. At any point, the reader may reinstate these parameters with the substitution

$$\phi \rightarrow \left(\frac{g}{\hbar}\right)^{\frac{1}{3}} \phi \tag{4.2}$$

and the rescalings of Q_α and H above.

4.1. *Gauge-invariant fermions*

Our first step involves the introduction of gauge-invariant fermions, using the eigenvectors ψ_a^i, which were introduced in (2.2). The gauge-invariant fermions are defined as

$$\Lambda'_{i\alpha} \equiv \psi_a^i \Lambda_{a\alpha}, \qquad i = 1, 2, 3, \qquad \alpha = 1 \cdots 16, \tag{4.3}$$

and these preserve the anticommutation relations

$$\{\Lambda'_{i\alpha}, \Lambda'_{j\beta}\} = \delta_{ij} \delta_{\alpha\beta}. \tag{4.4}$$

Moreover, the gauge-invariant fermions allow us to write the Yukawa term in the Hamiltonian (1.5) as

$$H_F = -\frac{i}{2} \varepsilon_{ijk} \Lambda'_{i\alpha} (\Gamma_j)_{\alpha\beta} \Lambda'_{k\beta} \lambda_j \equiv -\frac{i}{2} \varepsilon_{ijk} (\Lambda'_i \Gamma_j \Lambda'_k) \lambda_j. \tag{4.5}$$

The real symmetric, traceless and gauge-invariant matrices Γ_i in (4.5) are defined by

$$\Gamma_i \equiv \frac{\Gamma^m \phi_a^m \psi_a^i}{\lambda_i} = \Gamma^m \eta_i^m, \qquad i = 1, 2, 3, \tag{4.6a}$$

$$\{\Gamma_i, \Gamma_j\} = 2\delta_{ij}, \tag{4.6b}$$

and they preserve the Clifford algebra as shown. [The 21 gauge-invariant angles η_i^m are defined in (2.3).] In what follows, Γ_1, Γ_2 and Γ_3 are the components of Γ_i.

The eigenvectors ψ_a^i are functions of the bosonic variables ϕ_a^m, so the gauge-invariant fermions Λ' are coordinate-dependent and do not commute with the bosonic derivatives π. We rectify this situation by making an additional canonical transformation to obtain independent bosonic momenta π':

$$\pi_a'^m = \pi_a^m + F_a^m \,, \tag{4.7a}$$

$$F_a^m = \frac{i}{2}(\Lambda_i'(T_a^m)_{ij}\Lambda_j') \,, \qquad (T_a^m)_{ij} = \psi_b^i \partial_a^m \psi_b^j \,, \tag{4.7b}$$

$$[\pi_a'^m, \Lambda_{i\alpha}'] = 0 \,, \tag{4.7c}$$

where π' and ϕ remain canonical. This allows us to specify that

$$\pi_a'^m|\Lambda'\rangle = 0 \,, \tag{4.8a}$$

$$\pi_a'^m[f(\phi)|\Lambda'\rangle] = -i(\partial_a^m f(\phi))|\Lambda'\rangle \,, \tag{4.8b}$$

where $|\Lambda'\rangle$ is any state formed with the gauge-invariant fermions. In what follows, we describe this situation by writing

$$\pi_a'^m = -i\partial_a^m \,, \qquad \partial_a^m \Lambda' = 0 \,. \tag{4.9}$$

Further details of this transformation are given in App. A, which notes that the matrices T_a^m are divergence-free flat connections.

Appendix A also shows that the gauge generators (1.2) become purely bosonic,

$$G_a = \varepsilon_{abc}\left[\phi_b^m \pi_c^m - \frac{i}{2}(\Lambda_b \Lambda_c)\right] = \varepsilon_{abc}\phi_b^m \pi_c'^m \,, \tag{4.10}$$

when written in terms of the independent canonical momenta π'. This result confirms that G_a commutes with Λ' and tells us that states formed with the Λ' fermions,

$$G_a f(\lambda, \eta)|\Lambda'\rangle = 0 \,, \tag{4.11}$$

are gauge-invariant, as expected, when the bosonic coefficient f is separately gauge-invariant.

The rotation generators (1.3) also maintain a simple form,

$$J^{mn} = \pi_a'^{[m}\phi_a^{n]} - \frac{1}{2}(\Lambda_i' \Sigma^{mn} \Lambda_i') \,, \tag{4.12}$$

when expressed in terms of the independent momenta π'. This result shows that, because ψ_a^i is rotation-invariant, the gauge-invariant fermions remain spinors under spin (9).

Other applications of the gauge-invariant fermions include the following. The supercharges Q_α and the Hamiltonian H can be written entirely in terms of gauge-invariant quantities. This gauge-invariant formulation of SU(2) matrix theory is

given in App. B and continued in App. G. Moreover, the complete diagonalization of the Yukawa term

$$H_F = -\sum_{k,\nu} \mu_k a_{k\nu}^+ a_{k\nu} \tag{4.13}$$

is discussed in App. C.

4.2. *Further transformations*

For our consideration below of large $R = \lambda_3$ asymptotic behavior, it is convenient to make another transformation to simplify the leading term in H_F:

$$i\Lambda'_{1\alpha}(\Gamma_3)_{\alpha\beta}\Lambda'_{2\beta}\lambda_3 = i(\Lambda'_1\Gamma_3\Lambda'_2)R. \tag{4.14}$$

We further define

$$\Lambda''_1 = \Gamma_3\Lambda'_1, \qquad \Lambda''_2 = \Lambda'_2, \qquad \Lambda''_3 = \Lambda'_3, \tag{4.15a}$$

$$\{\Lambda''_{i\alpha}, \Lambda''_{j\beta}\} = \delta_{ij}\delta_{\alpha\beta}, \tag{4.15b}$$

along with another canonical transformation,

$$\pi_a''^m = \pi_a'^m + G_a^m, \qquad [\pi_a''^m, \Lambda''_{i\alpha}] = 0, \tag{4.16}$$

for which π'' and ϕ remain canonical. See App. A for further details.

The final form of the gauge generators is

$$G_a = \varepsilon_{abc}\phi_b^m \pi_c''^m, \tag{4.17}$$

because Λ'' are also gauge-invariant fermions. The rotation generators are now

$$J^{mn} = \pi_a''^{[m}\phi_a^{n]} - \frac{1}{2}(\Lambda''_i\Sigma^{mn}\Lambda''_i) \tag{4.18}$$

and it follows that the Λ'' fermions remain spinors under spin (9).

The final form of the Hamiltonian that results from our canonical transformations is the following (we now drop all primes for simplicity):

$$H = H_B + H_F + H_S, \tag{4.19}$$

where

$$H_B = \frac{1}{2}\pi_a^m \pi_a^m + V, \tag{4.20a}$$

$$H_F = i(\Lambda_1\Lambda_2)R + i(\Lambda_2\Gamma_1\Lambda_3)\lambda_1 + i(\Lambda_3\Gamma_2\Gamma_3\Lambda_1)\lambda_2, \tag{4.20b}$$

$$H_S = -(F_a^m + G_a^m)\pi_a^m + \frac{1}{2}F_a^m F_a^m + \frac{1}{2}G_a^m G_a^m + F_a^m G_a^m. \tag{4.20c}$$

Here

$$\pi_a^m = -i\partial_a^m = -\frac{i\partial}{\partial\phi_a^m}\,, \qquad \partial_a^m\Lambda_{i\alpha} = 0\,, \tag{4.21a}$$

$$F_a^m = i(\Lambda_1\Gamma_3\Lambda_2)(T_a^m)_{12} + i(\Lambda_1\Gamma_3\Lambda_3)(T_a^m)_{13} + i(\Lambda_2\Lambda_3)(T_a^m)_{23}\,, \tag{4.21b}$$

$$G_a^m = \frac{i}{2}(\Lambda_1(\Gamma_3\partial_a^m\Gamma_3)\Lambda_1)\,, \tag{4.21c}$$

and the connection T is as defined in (4.7). Note that H_F in (4.20b) is the original Yukawa term, now written in terms of the gauge-invariant fermions, and the shift term H_S, which is quartic in the gauge-invariant fermions, is the result of our canonical transformations.

Our third and final step is to introduce gauge-invariant fermion creation and annihilation operators for Λ_1 and Λ_2:

$$\Lambda_{1\alpha} = \frac{a_\alpha + a_\alpha^+}{\sqrt{2}}\,, \qquad \Lambda_{2\alpha} = \frac{a_\alpha - a_\alpha^+}{i\sqrt{2}}\,, \tag{4.22a}$$

$$\{a_\alpha, a_\beta^+\} = \delta_{\alpha\beta}\,. \tag{4.22b}$$

This gives the first term in H_F as

$$i(\Lambda_1\Lambda_2)R = R\left(\sum_\alpha a_\alpha^+ a_\alpha - 8\right) \tag{4.23}$$

and the gauge-invariant empty state $|0\rangle$, defined by

$$a_\alpha|0\rangle = 0\,, \tag{4.24}$$

gives the lowest value, $-8R$, for this operator.

The final form of the rotation generators is

$$J^{mn} = \pi_a^{[m}\phi_a^{n]} - a_\alpha^+(\Sigma^{mn})_{\alpha\beta}a_\beta - \frac{1}{2}(\Lambda_3\Sigma^{mn}\Lambda_3)\,, \tag{4.25}$$

so that the state $|0\rangle$ is invariant under rotations of the Λ_1, Λ_2 fermions.

5. The First Computation

The Hamiltonian (4.19) acts in the Hilbert space of the following 75 variables:

- 27 bosonic variables ϕ_a^n, which we have packaged into 3 gauge- and rotational-invariant "lengths" — λ_1, λ_2, λ_3 — and 24 remaining "angles" Ω.
- 48 fermionic operators, where 32 have been packaged into the gauge-invariant annihilation and creation operators a_α and a_α^+ and another 16 gauge-invariant fermions $\Lambda_{3\alpha}$.

We begin the computation by choosing a partition into the fast variables,

$$\text{``}x\text{'' variables: } \lambda_1,\ \lambda_2,\ a_\alpha,\ a_\alpha^+,\ \Omega\,, \tag{5.1}$$

and the slow variables,

$$\text{``}y\text{'' variables: } \lambda_3 = R, \Lambda_{3\alpha}\,, \tag{5.2}$$

although we will discuss a slightly different partition in Sec. 7.

Next, we must choose a particular projection operator P and its associated projector state $|\cdot\rangle$. Our choice is

$$|\cdot\rangle = |\psi_0(x)\rangle_R = u_R(\lambda_1, \lambda_2)|0\rangle\,, \tag{5.3a}$$

$$|\Psi_P\rangle = |\cdot\rangle|\psi(R, \Lambda_3)\rangle = |\psi_0(x)\rangle_R|\psi(R, \Lambda_3)\rangle\,, \tag{5.3b}$$

where $u_R(\lambda_1, \lambda_2)$ is as defined in Eq. (2.18) and $|0\rangle$ is the empty fermion state for Λ_1 and Λ_2 defined in (4.24). This state $|\cdot\rangle$ is the gauge-invariant analog of the approximate ground state introduced in Ref. 7, and, as discussed by these authors, it will guarantee the desired cancellation of the term linear in R in the effective Hamiltonian (2.25).

This leaves us to study the reduced state vector $|\psi(R, \Lambda_3)\rangle$ in the "y" variables,

$$|\psi(R, \Lambda_3)\rangle = \langle\cdot|\Psi(x, y)\rangle = \langle\cdot|\Psi_P\rangle = \int d^2\lambda(d\Omega)\sigma u_R^*(\lambda_1, \lambda_2)\langle 0|\Psi(x, y)\rangle\,, \tag{5.4}$$

where σ is as defined in (2.17e) and the normalization integral is

$$\int dR\, R^{10}\langle\psi(R, \Lambda_3)|\psi(R, \Lambda_3)\rangle < \infty\,. \tag{5.5}$$

Consistent with earlier notation, we define

$$\langle\cdot|A|\cdot\rangle \equiv \int d^2\lambda(d\Omega)\sigma u_R^*(\lambda_1, \lambda_2)\langle 0|A|0\rangle u_R(\lambda_1, \lambda_2)\,, \tag{5.6}$$

where A is any operator which may depend upon both the x and y variables. The result of this partial average is an operator that depends only upon the y variables and their derivatives.

We must now go through all the terms in the Hamiltonian (4.19) and answer the following questions for each operator:

(1) What fermionic selection rules apply with respect to the number operator $N_F = \Sigma_\alpha a_\alpha^+ a_\alpha$?

(2) What is the order of magnitude of the operators in powers of R? Here, it is important to remember that λ_1 and λ_2 are of order $R^{-\frac{1}{2}}$ at large R.

The details of this assessment are given in App. D. The results are given below, phrased in the language of "matrix elements," PHP, PHQ, QHP and QHQ, as these appear in the basic equations (3.4) and (3.6). For reference, the first of these equations reads

$$P(H - E)P|\Psi_P\rangle + PHQ|\Psi_Q\rangle = 0 \tag{5.7}$$

or, equivalently,

$$\langle\cdot|H-E|\cdot\rangle|\psi(R,\Lambda_3)\rangle + \int d^2\lambda(d\Omega)\sigma u_R^*(\lambda_1,\lambda_2)\langle 0|H|\Psi_Q(x,y)\rangle = 0\,, \qquad (5.8)$$

and we begin by evaluating the first term of this equation.

Generically, H is dominated by terms of $O(R)$. However, for the "diagonal matrix element" PHP, these leading terms cancel, as anticipated in the discussion of Subsec. 2.3, and we are left with the terms through $O(R^{-2})$,

$$\langle\cdot|H-E|\cdot\rangle = -\frac{1}{2}\frac{d^2}{dR^2} - \frac{5}{R}\frac{d}{dR} + \frac{12}{R^2} - E + \cdots\,, \qquad (5.9)$$

which will act on the reduced state vector $|\psi(R,\Lambda_3)\rangle$. Here, the dots indicate higher order terms in $\frac{1}{R}$. If we were to stop here, the asymptotic effective Hamiltonian would be

$$H_{\text{eff}}^{(1)} = -\frac{1}{2}\frac{d^2}{dR^2} - \frac{5}{R}\frac{d}{dR} + \frac{12}{R^2}\,, \qquad (5.10)$$

which defines the full first order Born–Oppenheimer approximation, now including the fermions. Comparing with the earlier bosonic result (2.25) for H_{eff}, we see that the fermionic contributions have canceled the term linear in R and added the term $+\frac{12}{R^2}$, which comes from the $\frac{F^2+G^2}{2}$ terms in (4.20c). But we cannot stop here, because there are other terms of order $\frac{1}{R^2}$ to be found from the second term of (5.8), and to evaluate this term we need $|\Psi_Q(x,y)\rangle$.

To solve for $|\Psi_Q(x,y)\rangle$ we turn to the other basic equation (3.6), which we write as

$$Q(H-E)Q|\Psi_Q\rangle + QHP|\Psi_P\rangle = 0 \qquad (5.11)$$

or, equivalently,

$$Q(H-E)|\Psi_Q(x,y)\rangle + QH[u_R(\lambda_1,\lambda_2)|0\rangle|\psi(R,\Lambda_3)\rangle] = 0\,. \qquad (5.12)$$

The formal solution of this equation is

$$|\Psi_Q\rangle = -(Q(H-E)Q)^{-1}QHP|\Psi_P\rangle\,. \qquad (5.13)$$

For the term QHP in (5.11) (an "off-diagonal matrix element") the leading contribution is of order $R^{-\frac{1}{2}}$ and comes only from the second and third terms of H_F in (4.20b):

$$QHP|\Psi_P\rangle = QH[u_R(\lambda_1,\lambda_2)|0\rangle|\psi(R,\Lambda_3)\rangle] \qquad (5.14a)$$

$$\simeq \frac{-\lambda_1(a^+\Gamma_1\Lambda_3) + i\lambda_2(\Lambda_3\Gamma_2\Gamma_3 a^+)}{\sqrt{2}} u_R(\lambda_1,\lambda_2)|0\rangle|\psi(R,\Lambda_3)\rangle\,. \qquad (5.14b)$$

The projection operator Q does not appear in this last expression since we can write $Q = 1 - P$ and P annihilates (5.14b) because $\langle 0|a^+|0\rangle = 0$. This state has $N_F = 1$ and so the first term of (5.11) also has $N_F = 1$.

The leading terms in QHQ (the "energy denominator") will have the generic $O(R)$ behavior of H. The $O(R^{-\frac{1}{2}})$ estimate holds for PHQ in Eq. (5.7), the same as for QHP. Then we can see that the formal expression

$$(PHQ)(Q(H-E)Q)^{-1}(QHP) \sim O(R^{-\frac{1}{2}}(R-E+O(R^{-\frac{1}{2}}))^{-1}R^{-\frac{1}{2}}) \qquad (5.15)$$

[substitute (5.13) into (5.7)] will contribute a term of order $\frac{1}{R^2}$ to the (second term of the) reduced wave equation (5.7) and we must determine its numerical coefficient. For this purpose we need to compute only the terms of $O(R)$ in QHQ.

From the details in App. D we find that four terms in H contribute to QHQ at order R and these include differential operators as well as multiplicative operators in the bosonic variables. This makes the explicit inversion of the operator $Q(H-E)Q$ a difficult problem, so we shall go back to Eq. (5.11) and solve it, at large R, as an inhomogeneous differential equation for $|\Psi_Q\rangle$. This procedure is closely related to an early technique[20,21] in atomic physics.

To solve Eq. (5.11) make the asymptotic ansatz

$$|\Psi_Q(x,y)\rangle = \frac{-f_1(\lambda_1,\lambda_2)(a^+\Gamma_1\Lambda_3) + if_2(\lambda_1,\lambda_2)(\Lambda_3\Gamma_2\Gamma_3 a^+)}{\sqrt{2}}$$

$$\times u_R(\lambda_1,\lambda_2)|0\rangle|\psi(R,\Lambda_3)\rangle, \qquad (5.16)$$

which is modeled on Eq. (5.14), with the insertion of two unknown functions, f_1 and f_2. The ansatz is again annihilated by P because $\langle 0|a^+|0\rangle = 0$. Calculating the action of $Q(H-E)Q$ on this $|\Psi_Q\rangle$, we find (see App. D) that the asymptotic form of $Q(H-E)|\Psi_Q(x,y)\rangle$ has the same form as (5.14), involving the same fermion bilinears, $(a^+\Gamma_1\Lambda_3)$ and $(\Lambda_3\Gamma_2\Gamma_3 a^+)$. Setting the total coefficients of these fermion bilinears to zero in (5.11) gives the coupled inhomogeneous differential equations

$$\left[-\frac{1}{2}(\Delta_1 + \Delta_2) + \mathrm{R}D + \frac{3}{\lambda_1^2} + \mathrm{U} + \mathrm{R} - \mathrm{E}\right]f_1 - Zf_2 = -\lambda_1, \qquad (5.17a)$$

$$\left[-\frac{1}{2}(\Delta_1 + \Delta_2) + \mathrm{R}D + \frac{3}{\lambda_2^2} + \mathrm{U} + \mathrm{R} - \mathrm{E}\right]f_2 - Zf_1 = -\lambda_2, \qquad (5.17b)$$

$$\Delta_1 \equiv \left(\frac{\partial}{\partial\lambda_1}\right)^2 + \left(\frac{6}{\lambda_1} + 2\frac{\lambda_1}{\lambda_1^2 - \lambda_2^2}\right)\frac{\partial}{\partial\lambda_1}, \qquad (5.17c)$$

$$\Delta_2 \equiv \left(\frac{\partial}{\partial\lambda_2}\right)^2 + \left(\frac{6}{\lambda_2} + 2\frac{\lambda_2}{\lambda_2^2 - \lambda_1^2}\right)\frac{\partial}{\partial\lambda_2}, \qquad (5.17d)$$

$$D \equiv \lambda_1\frac{\partial}{\partial\lambda_1} + \lambda_2\frac{\partial}{\partial\lambda_2}, \qquad (5.17e)$$

$$U \equiv \frac{\lambda_1^2 + \lambda_2^2}{(\lambda_1^2 - \lambda_2^2)^2}, \qquad (5.17f)$$

$$Z \equiv \frac{2\lambda_1\lambda_2}{(\lambda_1^2 - \lambda_2^2)^2}, \qquad (5.17g)$$

for the unknown functions f_1 and f_2. As planned, we have kept only terms of $O(R)$ multiplying f_1 and f_2 on the left of these equations, and the inhomogeneous terms on the right come from the λ_1 and λ_2 factors in (5.14b).

In fact, we have found a simple exact particular solution of these equations:

$$f_1 = -\frac{\lambda_1}{2R - E}, \qquad f_2 = -\frac{\lambda_2}{2R - E}. \tag{5.18}$$

(Barring such luck, we would have carried out numerical computations, using, for example, the variational principle mentioned earlier.)

The general solution to (5.17) can also include any solution to the homogeneous version of the equations, in addition to this particular solution. Because of the nonvanishing linear terms in R in these equations [which represent $Q(H - E)Q$], any solutions of the homogeneous equations will decay exponentially at large R, as in Eq. (2.27), and can thus be consistently ignored compared to the asymptotic power law behavior expected for the reduced state vector $|\psi(R, \Lambda_3)\rangle$.

Now that we know $|\Psi_Q(x, y)\rangle$ in (5.16), we can compute the large R contribution to the second term of (5.7):

$$\langle \cdot | HQ | \Psi_Q \rangle = \langle \cdot | \frac{\lambda_1(a\Gamma_1\Lambda_3) + i\lambda_2(\Lambda_3\Gamma_2\Gamma_3 a)}{\sqrt{2}} | \Psi_Q(x, y)\rangle \tag{5.19a}$$

$$= -\frac{1}{2(2R - E)} \langle \cdot | (\Lambda_3(\lambda_1^2 + \lambda_2^2 + 2\lambda_1\lambda_2\Theta)\Lambda_3) | \cdot \rangle$$
$$\times |\psi(R, \Lambda_3)\rangle \tag{5.19b}$$

$$= -\frac{1}{2(2R - E)} \int d^2\lambda(d\Omega)\sigma|u_R(\lambda_1, \lambda_2)|^2$$
$$\times (\Lambda_3(\lambda_1^2 + \lambda_2^2 + 2\lambda_1\lambda_2\Theta)\Lambda_3)|\psi(R, \Lambda_3)\rangle. \tag{5.19c}$$

The gauge-invariant matrix Θ in (5.19) is defined as

$$\Theta \equiv -i\Gamma_1\Gamma_2\Gamma_3 \tag{5.20}$$

and when we take the average over angles,

$$\int (d\Omega)\Theta = 0, \tag{5.21}$$

because Θ is odd under reflection of all the ϕ variables [see (E.18)]. From App. F we find that the average value of $\lambda_1^2 + \lambda_2^2$ is $\frac{8}{R}$; and from the anticommutation relations we have $(\Lambda_3\Lambda_3) = 8$, so that our result is independent of any representation we might choose for the Λ_3 variables. Then the result of this "second order" calculation,

$$\langle \cdot | HQ | \Psi_Q \rangle = -\frac{16}{R^2} |\psi(R, \Lambda_3)\rangle, \tag{5.22}$$

is exact through order $\frac{1}{R^2}$.

6. The First Set of Candidate Ground States

Adding the result in Eq. (5.22) to the "first order" terms in Eq. (5.10), we find the asymptotic form of Eq. (5.7), exact through order $\frac{1}{R^2}$:

$$H_{\text{eff}}|\psi(R, \Lambda_3)\rangle = E|\psi(R, \Lambda_3)\rangle \,, \tag{6.1a}$$

$$H_{\text{eff}} = -\frac{1}{2}\frac{d^2}{dR^2} - \frac{5}{R}\frac{d}{dR} - \frac{4}{R^2} \,. \tag{6.1b}$$

This reduced Schrödinger equation has two solutions at $E = 0$: R^{-1} or R^{-8} times any state $|\Lambda_3\rangle$ formed with the gauge-invariant fermions Λ_3. The second solution,

$$|\psi(R, \Lambda_3)\rangle \simeq R^{-8}|\Lambda_3\rangle \,, \tag{6.2}$$

allows the normalization integral (5.5) to converge at large R. [In the language of Eq. (2.28), we have found that $B' = 8 < \frac{77}{4}$.] The result (6.2) is our first asymptotic set of ground state candidates, which must be tested further for global stability at nonasymptotic values of R.

These solutions also confirm[5] a continuous spectrum for $E > 0$. With $E = \frac{k^2}{2}$, the effective Hamiltonian (6.1) yields plane wave normalizable solutions which behave as

$$|\psi(R, \Lambda_3)\rangle_{\pm} \simeq R^{-5}e^{\pm ikR}|\Lambda_3\rangle \tag{6.3}$$

at large R.

We can also follow the computation backward to reconstruct the asymptotic form of the candidate ground states $|\Psi\rangle$ near the flat directions of the potential V. Using Eqs. (5.3), (5.16) and (6.2), we find the asymptotic forms

$$|\Psi_P\rangle = R^{-8}u_R(\lambda_1, \lambda_2)|0\rangle|\Lambda_3\rangle \,, \tag{6.4a}$$

$$|\Psi_Q\rangle = \frac{1}{2\sqrt{2}R}[\lambda_1(a^+\Gamma_1\Lambda_3) - i\lambda_2(\Lambda_3\Gamma_2\Gamma_3 a^+)]|\Psi_P\rangle \,, \tag{6.4b}$$

where $|0\rangle$ is the ground state of the gauge-invariant fermions $\Lambda_{1,2}$ and u_R is as given in Eq. (2.18).

Adding these results, we obtain the full asymptotic form of the candidate ground states, up to an overall normalization constant,

$$|\Psi\rangle \simeq \left\{1 + \frac{R^{-\frac{3}{2}}}{2\sqrt{2}}[z_1(a^+\Gamma_1\Lambda_3) - iz_2(\Lambda_3\Gamma_2\Gamma_3 a^+)]\right\}$$
$$\times R^{-4}\exp\left(-\frac{z_1^2 + z_1^2}{2}\right)|0\rangle|\Lambda_3\rangle \,, \tag{6.5a}$$

$$z_1 = \lambda_1 R^{\frac{1}{2}} \,, \qquad z_2 = \lambda_2 R^{\frac{1}{2}} \,, \tag{6.5b}$$

where the scaled variables z_1 and z_2 are those defined in Eq. (2.22). In this form of the candidate ground states, the variables $z_{1,2}$ are finite and only R is large.

This result can also be written through this order in $\frac{1}{R}$ as

$$|\Psi\rangle \simeq \exp\left(-\frac{S}{\hbar}\right)|0\rangle|\Lambda_3\rangle \,, \tag{6.6a}$$

$$S = \frac{V}{gr} + \left(\frac{H_F}{2gr} + 4\hbar\ln r\right) \,, \tag{6.6b}$$

$$r = (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)^{\frac{1}{2}} = (\phi_a^m \phi_a^m)^{\frac{1}{2}} = R + O(R^{-2}) \,, \tag{6.6c}$$

$$\lambda_{1,2} = O\left[\left(\frac{\hbar}{g}\right)^{\frac{1}{2}} R^{-\frac{1}{2}}\right] = O\left[\left(\frac{\hbar}{g}\right)^{\frac{1}{2}} r^{-\frac{1}{2}}\right] \,, \tag{6.6d}$$

where V is the bosonic potential, H_F is the Yukawa term and we have reinstated $\hbar$ and g following the rule (4.2). The range of validity in (6.6d) (finite $z_{1,2}$ at large r) defines the quantum neighborhood of the classical flat direction $\lambda_1 = \lambda_2 = 0$. This form of the result shows that these candidate ground states are quantum extensions of Itoyama's solution of the zero-energy Hamilton–Jacobi equation, now made normalizable by the quantum correction $4\hbar\ln r$ (this corresponds to $\alpha = 4 > \frac{3}{2}$ in the discussion of Subsec. 2.2).

This set of candidate ground states does not include a singlet under spin (9). To see this explicitly, we note that the bosonic prefactor $\exp(-\frac{S}{\hbar})$ in (6.6) is rotation-invariant while the rotation generators (4.25) give

$$J^{mn}|0\rangle|\Lambda_3\rangle = |0\rangle\left(-\frac{1}{2}(\Lambda_3\Sigma^{mn}\Lambda_3)\right)|\Lambda_3\rangle \,, \tag{6.7a}$$

$$\frac{1}{2}J^{mn}J^{mn}|0\rangle|\Lambda_3\rangle = 18|0\rangle|\Lambda_3\rangle \tag{6.7b}$$

on the fermion states. The evaluation of the Casimir operator in (6.7b) follows from Fierz transformations and properties of the Γ matrices. This shows that all three irreducible representations of spin (9) in $|\Lambda_3\rangle$ (and hence in the candidate ground states),

$$|\Lambda_3\rangle = |256\rangle = |44\rangle \oplus |84\rangle \oplus |128\rangle \,, \tag{6.8}$$

have the same value of the Casimir. These irreps correspond respectively to the spin (9) irreps of the 11-dimensional supergraviton:

(1) A symmetric, traceless second rank tensor (g_{mn}),
(2) A totally antisymmetric third rank tensor (H_{mnp}),
(3) A "gravitino" or Rarita–Schwinger irrep (B_α^m);

the first two irreps are bosonic and the last is fermionic.

We also note that the set (6.6) of 256 candidate ground states forms a "zero-index unit" whose presence cannot violate the index theorem[13] for SU(2) matrix theory. In this connection, it is clear that there are strong parallels between our generalized Born–Oppenheimer formulation and the computational method of Ref. 13. It is

difficult to make a quantitative comparison, however, because we are computing different quantities.

7. A More General Set of Candidates

Having obtained our first set, (6.6), of ground state candidates, we are now in a position to obtain a more general set of candidates.

One crucial observation is that the angular integration $(d\Omega)$ played a very limited role in the computation of Sec. 5: because our projector state $|\cdot\rangle$ was independent of the angular variables Ω, we needed only $\int (d\Omega) = 1$ in every stage except for Eq. (5.19), where $\int (d\Omega)\Theta = 0$ eliminated the term proportional to the operator Θ. This opens the possibility of broadening our perspective by partitioning the variables Ω into fast and slow variables, while maintaining the requirement that no linear terms in R should appear in the effective Hamiltonian. In what follows, we ignore the three gauge degrees of freedom in Ω, keeping only the 21 gauge-invariant angular variables η, which we give again here for reference:

$$\eta_i^m = \frac{\phi_a^m \psi_a^i}{\lambda_i}\,, \qquad \eta_i^m \eta_j^m = \delta_{ij}\,. \tag{7.1}$$

Recall that these 21 variables plus the 3 λ's are a complete set of $27 - 3 = 24$ gauge-invariant bosonic variables for SU(2).

More precisely, we begin our second computation by choosing the partition

$$\text{fast } (x) \text{ variables } : \lambda_1, \lambda_2, \Lambda_1, \Lambda_2, \eta_1, \eta_2 \tag{7.2a}$$

$$\text{slow } (y) \text{ variables } : \lambda_3, \Lambda_3, \eta_3 \tag{7.2b}$$

because, as demonstrated below, this will allow us to avoid R terms in the effective Hamiltonian for the reduced state vector $|\psi(R, \Lambda_3, \eta_3)\rangle$. Moreover, we choose the same η_1, η_2-independent projector state,

$$|\cdot\rangle = u_R(\lambda_1, \lambda_2)|0\rangle\,, \tag{7.3}$$

used in the first computation, but now we must specify the decomposition of the η measure in order to integrate out the fast variables η_1 and η_2. The full gauge-invariant measure can be written as

$$(d\phi) = d^3\lambda \rho(\lambda)(d^3\eta)\,, \tag{7.4a}$$

$$(d^3\eta) = (d^2\eta)(d\eta_3)\,, \tag{7.4b}$$

$$(d^2\eta) = \left[\prod_{i=1,2}\left(\prod_{m=1}^{9} d\eta_i^m\right)\delta(\eta_i^n \eta_i^n - 1)\delta(\eta_i^p \eta_3^p)\right]\delta(\eta_1^q \eta_2^q)\,, \tag{7.4c}$$

$$(d\eta_3) = \left(\prod_{m=1}^{9} d\eta_3^m\right)\delta(\eta_3^n \eta_3^n - 1)\,, \tag{7.4d}$$

and, in what follows, we will need only the following two properties of $(d^2\eta)$:

$$\int (d^2\eta) = 1 \,, \tag{7.5a}$$

$$\int (d^2\eta)\Theta = 0 \,. \tag{7.5b}$$

It is straightforward to see that $\int (d^2\eta)$ is independent of η_3 and (7.5a) is a convenient convention. The property in (7.5b) follows because the matrix Θ,

$$\Theta = -i\Gamma_1\Gamma_2\Gamma_3 = -i\Gamma^m\Gamma^n\Gamma^p\eta_1^m\eta_2^n\eta_3^p \,, \tag{7.6}$$

is odd in each of the three η's while $(d^2\eta)$ is even in η_1 and/or η_2.

Relative to our first computation, we now have the replacement

$$(d\Omega) \to (d^2\eta) \tag{7.7}$$

in all averages over fast variables. This includes, for example, the new form of Eq. (5.6)

$$\langle \cdot |A| \cdot \rangle = \int d^2\lambda (d^2\eta)\sigma u_R^*(\lambda_1,\lambda_2)\langle 0|A|0\rangle u_R(\lambda_1,\lambda_2) \,. \tag{7.8}$$

Correspondingly, the integration over η_3 appears only in the new normalization condition

$$\int dR R^{10}(d\eta_3)\langle \psi(R,\Lambda_3,\eta_3)|\psi(R,\Lambda_3,\eta_3)\rangle < \infty \tag{7.9}$$

for the reduced state vector.

The second computation (see App. D) then proceeds exactly as did the first computation, using the same ansatz (5.16) for $|\Psi_Q\rangle$ now with

$$|\psi(R,\Lambda_3)\rangle \to |\psi(R,\Lambda_3,\eta_3)\rangle \tag{7.10}$$

for the reduced state vector. The same contributions are obtained [now by $\int (d^2\eta) = 1$] from each term, including the elimination of the Θ term [now by $\int (d^2\eta)\Theta = 0$] in the new version of Eq. (5.19). There is, however, one new contribution to PHP from the action of the Laplacian Δ on the angular variables η_3 of the reduced state vector $|\psi(R,\Lambda_3,\eta_3)\rangle$. We sketch here only the asymptotic results that we need for this computation, referring the reader to App. G for the full structure of the Laplacian on general gauge-invariant functions $f(\lambda,\eta)$.

To study the new contribution of the Laplacian, we begin with the identity

$$(\partial_a^m\eta_i^n)(\partial_a^m\lambda_j) = 0 \,, \tag{7.11}$$

which is, in fact, equivalent to Eq. (E.14). It follows that the Laplacian is separable in the form

$$\Delta = \Delta_\lambda + \Delta_\eta \,, \tag{7.12}$$

where Δ_λ, which contains the λ derivatives, is defined in (2.4b) and Δ_η contains only derivatives with respect to the η variables. With the chain rule and the asymptotic identity

$$\partial_a^m\eta_3^n = \frac{1}{R}(\delta^{mn} - \eta_3^m\eta_3^n)\psi_a^3 + O(R^{-\frac{5}{2}}) \,, \tag{7.13}$$

we can easily compute the extra asymptotic contribution to PHP as

$$\langle\cdot| - \frac{1}{2}\Delta_\eta|\cdot\rangle|\psi(R,\Lambda_3,\eta_3)\rangle \tag{7.14a}$$

$$= \int d^2\lambda(d^2\eta)\sigma u_R^*(\lambda_1,\lambda_2)\left(-\frac{1}{2}\Delta_\eta\right)u_R(\lambda_1,\lambda_2)|\psi(R,\Lambda_3,\eta_3)\rangle \tag{7.14b}$$

$$= \left[\frac{L_3^2}{2R^2} + O(R^{-3})\right]|\psi(R,\Lambda_3,\eta_3)\rangle . \tag{7.14c}$$

Derivatives with respect to η_1 and η_2 do not contribute in this computation because there is no dependence on these variables in $u_R(\lambda_1,\lambda_2)$ or the reduced state vector $|\psi(R,\Lambda_3,\eta_3)\rangle$. Moreover, we have organized the result into the angular momentum operators of η_3,

$$L_3^{mn} = -i\eta_3^{[m}\partial_3^{n]}, \qquad L_3^2 = \frac{1}{2}L_3^{mn}L_3^{mn} , \tag{7.15}$$

where $\partial_3^m = \frac{\partial}{\partial\eta_3^m}$. In this result, the η derivatives may be taken as the naive derivative

$$\partial_3^m \eta_3^n = \delta^{mn} \tag{7.16}$$

or the constrained derivative

$$\partial_3^m \eta_3^n = \delta^{mn} - \eta_3^m \eta_3^n , \tag{7.17}$$

which respects the constraint $\eta_3^m \eta_3^m = 1$: the two give the same operators L_3^{mn}, which generate a bosonic SO(9).

Adding this extra term then, we find the new asymptotic effective Hamiltonian

$$H_{\text{eff}}|\psi(R,\Lambda_3,\eta_3)\rangle = E|\psi(R,\Lambda_3,\eta_3)\rangle , \tag{7.18a}$$

$$H_{\text{eff}} = -\frac{1}{2}\frac{d^2}{dR^2} - \frac{5}{R}\frac{d}{dR} + \frac{L_3^2 - 8}{2R^2} , \tag{7.18b}$$

which is exact through $O(R^{-2})$. This reduced system becomes a simple radial wave equation when we introduce the spherical harmonics $Y_l(\eta_3)$ of SO(9),

$$L_3^2 Y_l(\eta_3) = l(l+7)Y_l(\eta_3) , \qquad l = 0, 1, 2, \dots , \tag{7.19}$$

with "magnetic" degeneracy

$$\deg(l) = \frac{(2l+7)(l+6)!}{l!7!} . \tag{7.20}$$

Using the spherical harmonics, we can immediately write down the normalizable asymptotic solutions

$$|\psi(R,\Lambda_3,\eta_3)\rangle \simeq R^{-(l+8)}Y_l(\eta_3)|\Lambda_3\rangle , \tag{7.21a}$$

$$|\Lambda_3\rangle = |256\rangle = |44\rangle \oplus |84\rangle \oplus |128\rangle , \tag{7.21b}$$

for the reduced state vector.

This set of solutions contains our first solution, (6.2), as the special case with $l = 0$, and the set contains exactly one state,

$$|\psi(R, \Lambda_3, \eta_3)\rangle_{l=2} \simeq R^{-10} Y_2^{mn}(\eta_3)|44; mn\rangle\,, \tag{7.22}$$

which is a singlet under spin (9). Here we have used an explicit form of the 44-dimensional $Y_2(\eta_3)$,

$$Y_2^{mn}(\eta_3) = \eta_3^m \eta_3^n - \frac{1}{9}\delta^{mn}\,, \tag{7.23}$$

to perform the invariant sum over Y_2 times the 44-dimensional irrep in $|\Lambda_3\rangle$.

The new effective Hamiltonian (7.18) also exhibits plane wave normalizable solutions

$$|\psi(R, \Lambda_3, \eta_3)\rangle_{\pm} \simeq \frac{Y_l(\eta_3)e^{\pm ikR}}{R^5}|\Lambda_3\rangle \tag{7.24}$$

and hence a continuous spectrum for $E = \frac{k^2}{2} > 0$. The earlier result (6.3) is included in (7.24) when $l = 0$.

Finally, we may follow the new computation backward to obtain the full asymptotic form of our general set of candidate SUSY ground states. One obtains the generalization of (6.5),

$$|\Psi\rangle \simeq \left\{ 1 + \frac{R^{-\frac{3}{2}}}{2\sqrt{2}}[z_1(a^+\Gamma_1\Lambda_3) - iz_2(\Lambda_3\Gamma_2\Gamma_3 a^+)] \right\}$$

$$\times R^{-l-4} Y_l(\eta_3) \exp\left(-\frac{z_1^2 + z_1^2}{2} \right)|0\rangle|\Lambda_3\rangle\,, \tag{7.25a}$$

$$z_1 = \lambda_1 R^{\frac{1}{2}}\,, \qquad z_2 = \lambda_2 R^{\frac{1}{2}}\,, \tag{7.25b}$$

and the generalization of (6.6),

$$|\Psi\rangle \simeq \exp\left(-\frac{S_l}{\hbar} \right) Y_l(\eta_3)|0\rangle|\Lambda_3\rangle\,, \tag{7.26a}$$

$$S_l = \frac{V}{gr} + \left\{ \frac{H_F}{2gr} + (l+4)\hbar \ln r \right\}\,, \tag{7.26b}$$

$$r = (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)^{\frac{1}{2}} = (\phi_a^m \phi_a^m)^{\frac{1}{2}}\,, \tag{7.26c}$$

$$\lambda_{1,2} = O\left[\left(\frac{\hbar}{g}\right)^{\frac{1}{2}} r^{-\frac{1}{2}} \right]\,, \tag{7.26d}$$

$$|\Psi\rangle_{l=2} \simeq \exp\left(-\frac{S_2}{\hbar} \right) Y_2^{mn}(\eta_3)|0\rangle|44; mn\rangle\,, \tag{7.26e}$$

$$J^{mn}|\Psi\rangle_{l=2} = 0\,, \tag{7.26f}$$

where we have recorded in (7.26e) the unique candidate which is a singlet under spin (9). The extreme semiclassical limit of the $l \neq 0$ solutions in (7.26) are gauge- but not rotation-invariant solutions of the zero-energy Hamilton–Jacobi equation.

The apparent simplicity of the ground state candidates (7.26a and 7.26b) suggests that there may be a more elegant path to this result.

Each of these candidates is a normalizable zero-energy asymptotic solution of the Schrödinger equation of SU(2) matrix theory, but each must be tested further for stability at nonasymptotic values of R. The high l solutions are particularly suspect because they are associated with the growing centrifugal barrier $\frac{L_3^2}{2R^2}$. Since the singlet state has $l = 2$, this leaves the states with $l \leq 2$ as the most auspicious candidates.

Appendix G also outlines a strategy for a proof of a conjecture which, if true, would tell us that our projector state in (7.3) is the only state in the Hilbert space of the fast variables (7.2a) that avoids linear terms in R in the effective Hamiltonian. In this case, our set of candidate ground states would be a complete list for the partition (7.2) of the variables of SU(2) matrix theory.

Acknowledgments

Appendix A. Canonical Transformations

In further detail, the shift in (4.7) is

$$\pi_a^{'m} = \pi_a^m + F_a^m, \tag{A.1a}$$

$$F_a^m = \frac{1}{2i}\Lambda_{b\alpha}(S_a^m)_{bc}\Lambda_{c\alpha} = \frac{i}{2}\Lambda_{i\alpha}'(T_a^m)_{ij}\Lambda_{j\alpha}', \tag{A.1b}$$

$$(S_a^m)_{bc} = \psi_b^i \partial_a^m \psi_c^i, \qquad (T_a^m)_{ij} = \psi_b^i \partial_a^m \psi_b^j, \tag{A.1c}$$

where Λ' are the gauge-invariant fermions (4.3) and ψ_a^i are the eigenvectors of Φ introduced in (2.2). Further properties of T_a^m are found in App. E. The quantities S_a^m and T_a^m are real antisymmetric matrices, which we call connections.

Using the orthonormality and completeness of ψ_a^i in (2.2), one verifies that ϕ and π' are canonical variables which are independent of Λ':

$$[\pi_a^{'m}, \Lambda_{i\alpha}'] = 0, \tag{A.2a}$$

$$[\phi_a^m, \pi_b^{'n}] = i\delta_{mn}\delta_{ab}, \tag{A.2b}$$

$$[\pi_a^{'m}, \pi_b^{'n}] = 0, \tag{A.2c}$$

and this tells us that

$$\pi_a^{'m} = -i\partial_a^m = -i\frac{\partial}{\partial \phi_a^m}, \qquad \partial_a^m \Lambda_{i\alpha}' = 0 \tag{A.3}$$

in coordinate representation. These derivatives could be written more precisely as $(\partial_a^m)_{\Lambda'}$ to show that they act on the bosons as usual, but at fixed Λ'.

The statement (A.2c) is equivalent to the fact that S_a^m and T_a^m are flat connections,

$$\partial_a^m S_b^n - \partial_b^n S_a^m + [S_a^m, S_b^n] = 0\,, \tag{A.4a}$$

$$\partial_a^m T_b^n - \partial_b^n T_a^m + [T_a^m, T_b^n] = 0\,, \tag{A.4b}$$

which follows directly from the properties (2.2) of ψ_a^i. Moreover, we find that both connections are divergence-free:

$$\partial_a^m S_a^m = \partial_a^m T_a^m = 0\,. \tag{A.5}$$

To see this for T_a^m, follow the steps

$$\partial_a^m (T_a^m)_{ij} = \frac{1}{2}\partial_a^m [\psi_b^i \partial_a^m \psi_b^j - (i \leftrightarrow j)] \tag{A.6a}$$

$$= \frac{1}{2}[\psi_b^i \Delta \psi_b^j - (i \leftrightarrow j)] \tag{A.6b}$$

$$= 0\,, \tag{A.6c}$$

where we have used (E.5) in the last step, and similarly for S_a^m. It also follows that

$$\partial_a^m F_b^n - \partial_b^n F_a^m + i[F_a^m, F_b^n] = 0\,, \qquad \partial_a^m F_a^m = 0\,, \tag{A.7}$$

and so the current F_a^m in Eq. (A.1b) is a flat divergenceless connection too.

The canonical transformation (4.7) or (A.1) can also be understood in terms of a unitary (but not gauge-invariant) transformation $K(\phi, \Lambda'(\phi))$:

$$\pi_a'^m = K^{-1}\pi_a^m K\,, \tag{A.8a}$$

$$\Lambda_{i\alpha}'(\phi) = K^{-1}\Lambda_{i\alpha}'(\phi_0)K\,, \tag{A.8b}$$

$$\Lambda_{i\alpha}'(\phi_0) = \psi_a^i(\phi_0)\Lambda_{a\alpha}\,, \tag{A.8c}$$

$$\partial_a^m K = iKF_a^m\,, \qquad K(\phi_0) = 1\,, \tag{A.8d}$$

$$K = P\exp\left[i\int_{\phi_0}^{\phi} d\phi' \cdot F(\phi', \Lambda'(\phi'))\right]\,, \tag{A.8e}$$

where ϕ_0 is a reference point of ϕ and $\Lambda_{a\alpha}$ are the original constant but not gauge-invariant fermions. The path-ordered operator K is well defined because the current F_a^m is a (divergenceless) flat connection.

The gauge-invariant states $|\Lambda'\rangle$ satisfy

$$\pi_a'^m|\Lambda'(\phi)\rangle = (-i\partial_a^m + F_a^m)|\Lambda'(\phi)\rangle = 0 \tag{A.9}$$

and so may be written in terms of the original fermions as

$$|\{\Lambda_{i\alpha}'(\phi)\}\rangle = K^{-1}|\{\Lambda_{i\alpha}'(\phi_0) = \psi_a^i(\phi_0)\Lambda_{a\alpha}\}\rangle\,, \tag{A.10}$$

although neither factor on the right is separately gauge-invariant.

To see the cancellation (4.10) of fermionic terms in the gauge generator G_a, use Eq. (E.2) for $\partial_a^m \psi_b^i$ to verify the intermediate steps

$$\varepsilon_{abc}\phi_b^m \partial_c^m \psi_d^i = \varepsilon_{abd}\psi_b^i \,, \tag{A.11a}$$

$$\varepsilon_{abc}\phi_b^m (S_c^m)_{df} = \varepsilon_{adf} \,, \tag{A.11b}$$

where S_c^m is the flat connection in (A.1). The result in (A.11a) says that ψ_a^i transforms in the adjoint of the gauge group.

Similarly, the form (4.12) of the rotation generators follows with the steps

$$\phi_a^{[m} \partial_a^{n]} \psi_b^j = 0 \,, \tag{A.12a}$$

$$\phi_a^{[m}(T_a^{n]})_{ij} = \psi_b^i \phi_a^{[m} \partial_a^{n]} \psi_b^j = 0 \,, \tag{A.12b}$$

where (A.12a) says that ψ_a^i are singlets under spin (9).

When we make the substitution (A.1) in the Hamiltonian, we encounter

$$\frac{1}{2}\pi_a^m \pi_a^m = \frac{1}{2}\pi_a'^m \pi_a'^m - F_a^m \pi_a'^m + \frac{1}{2}F_a^m F_a^m - \frac{1}{2}[\pi_a'^m, F_a^m] \,, \tag{A.13}$$

where the F^2 terms from the shift are quartic in the gauge-invariant fermions. The last term in (A.13) vanishes, however, because the flat connection T_a^m has zero divergence.

The explicit form of the shift term in the second canonical transformation (4.16) is

$$G_a^m = \frac{1}{2i}(\Lambda_1'(\Gamma_3 \partial_a^m \Gamma_3)\Lambda_1') = \frac{i}{2}(\Lambda_1''(\Gamma_3 \partial_a^m \Gamma_3)\Lambda_1'') \,, \tag{A.14}$$

where Λ_i'' are the final gauge-invariant fermions. Further details of the derivatives of the matrices Γ_i can be found in App. E. Here again we find that π'' and ϕ are canonical variables independent of Λ''. Moreover, as above, the statement $[\pi_a''^m, \pi_b''^n] = 0$ is equivalent to the fact that $\Gamma_3 \partial_a^m \Gamma_3$ is a flat connection, and using Eq. (E.13) we find that this connection is also divergence-free.

The final form (4.17) of the gauge generators G_a is obtained because the shift term G_a^m does not contribute to G_a. To see this, use Eq. (E.11) to verify explicitly that Γ_i is gauge-invariant,

$$\varepsilon_{abc}\phi_b^m \partial_c^m \Gamma_i = 0 \,, \tag{A.15}$$

and hence that $\varepsilon_{abc}\phi_b^m G_c^m = 0$.

We found the identities

$$\Gamma_3 \phi_a^{[m} \partial_a^{n]} \Gamma_3 = -\Gamma_3 \Gamma^{[m} \eta_3^{n]} \,, \tag{A.16a}$$

$$\Gamma_3 \Sigma^{mn} \Gamma_3 + i\Gamma_3 \Gamma^{[m} \eta_3^{n]} = \Sigma^{mn} \tag{A.16b}$$

helpful in obtaining the form (4.18) of the rotation generators. Here η_i^m are the gauge-invariant angular variables defined in (2.3).

Appendix B. Gauge-Invariant Formulation

Given the gauge-invariant fermions Λ' of Subsec. 4.1 and the additional gauge-invariant (but not canonical) coordinates and momenta

$$\phi_i^m \equiv \psi_a^i \phi_a^m\,, \qquad \pi_i'^m \equiv \psi_a^i \pi_a'^m = -i\hbar\psi_a^i \partial_a^m = -i\hbar D_i^m\,, \qquad \text{(B.1a)}$$

$$[\pi_i'^m, \Lambda'_{j\alpha}] = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(B.1b)}$$

[$\pi_a'^m$ is the independent momentum in (4.7)], we can rewrite the supercharges and the Hamiltonian of SU(2) matrix theory entirely in terms of gauge-invariant quantities.

Using the derivative formulas of App. E, the results are

$$Q_\alpha = (\Gamma^m \Lambda'_i)_\alpha \pi_i'^m - i\hbar \sum_{i\neq j} (\lambda_i \Gamma_i \Lambda'_j)_\alpha \frac{(\Lambda'_i \Lambda'_j)}{\lambda_i^2 - \lambda_j^2}$$

$$+ \frac{g}{2}\varepsilon_{ijk}(\lambda_i \Gamma_i \lambda_j \Gamma_j \Lambda'_k)_\alpha\,, \qquad\qquad\qquad \text{(B.2a)}$$

$$H = \frac{1}{2}\pi_i'^m \pi_i'^m + \frac{g^2}{2}(\lambda_1^2 \lambda_2^2 + \lambda_1^2 \lambda_3^2 + \lambda_2^2 \lambda_3^2)$$

$$+ \frac{i\hbar}{2}\sum_{i\neq j}\frac{1}{\lambda_i^2 - \lambda_j^2}\{\phi_j^m \pi_i'^m + ([\Lambda'_i, \Lambda'_j])\phi_i^m \pi_j'^m\}$$

$$- \frac{\hbar^2}{4}\sum_{i\neq j}(\Lambda'_i \Lambda'_j)^2 \frac{\lambda_i^2 + \lambda_j^2}{(\lambda_i^2 - \lambda_j^2)^2} + \frac{ig\hbar}{2}\varepsilon_{ijk}(\Lambda'_i \lambda_k \Gamma_k \Lambda'_j)\,, \qquad \text{(B.2b)}$$

where $(\Lambda'_i B \Lambda'_j) = \Lambda'_{i\alpha} B_{\alpha\beta} \Lambda'_{j\beta}$ and $([\Lambda'_i, \Lambda'_j]) = \Lambda'_{i\alpha}\Lambda'_{j\alpha} - \Lambda'_{j\alpha}\Lambda'_{i\alpha}$. As expected, Q and H are respectively cubic and quartic in the gauge-invariant fermions, and the quartic term in the Hamiltonian is just the F^2 term of the shift. The last term in the Hamiltonian is the Yukawa term H_F, whose diagonalization is discussed in App. C.

Using chain rules, the gauge-invariant momenta $\pi_i'^m$ can be evaluated explicitly when operating on general gauge-invariant bosonic functions $f(\lambda, \eta)$, where the η variables are given in (2.3). The results for $\pi_i'^m$ and the bosonic Laplacian on $f(\lambda, \eta)$,

$$-\hbar^2\Delta = \pi_i'^m \pi_i'^m + i\hbar\sum_{i\neq j}\frac{1}{\lambda_i^2 - \lambda_j^2}\phi_j^m \pi_j'^m \qquad\qquad \text{(B.3a)}$$

$$= -\hbar^2(\Delta_\lambda + \Delta_\eta)\,, \qquad\qquad\qquad\qquad\qquad \text{(B.3b)}$$

are given explicitly in App. G. Here Δ_λ, which contains the λ derivatives, is as given in (2.4b) and Δ_η contains the η derivatives.

We also mention some alternate gauge-invariant forms for the supercharges:

$$Q_\alpha = (\Gamma^m(\pi_a^m + i\Theta\partial_a^m\sqrt{W})\Lambda_a)_\alpha \tag{B.4a}$$

$$= (\Gamma^m\Lambda_i')_\alpha\pi_i'^m - i\hbar\sum_{i\neq j}(\lambda_i\Gamma_i\Lambda_j')_\alpha\frac{(\Lambda_i'\Lambda_j')}{\lambda_i^2 - \lambda_j^2}$$

$$+ i(\Gamma^m\Theta\Lambda_i')_\alpha D_i^m\sqrt{W}. \tag{B.4b}$$

Here, $W = g^2\det\Phi$ is the Claudson–Halpern variable and Θ is the gauge-invariant matrix

$$\Theta = -i\Gamma_1\Gamma_2\Gamma_3, \tag{B.5}$$

which satisfies $\Theta^2 = 1$. Still another form is

$$Q_\alpha = \left(\Gamma^m\frac{\phi_i^m}{\lambda_i^2}\Lambda_i'\right)_\beta C_{i\beta\alpha} - i\hbar\sum_{i\neq j}(\lambda_i\Gamma_i\Lambda_j')_\alpha\frac{(\Lambda_i'\Lambda_j')}{\lambda_i^2 - \lambda_j^2}, \tag{B.6a}$$

$$C_{i\alpha\beta} = \left[-i\hbar\lambda_i\frac{\partial}{\partial\lambda_i} + i\Sigma^{mn}M_i^{mn} - i\Theta\sqrt{W}\right]_{\alpha\beta}, \tag{B.6b}$$

$$M_i^{mn} = \phi_i^{[m}\pi_i'^{n]}, \tag{B.6c}$$

where the second term in C_i, which is of "spin–orbit" form, contains all the η derivatives in the supercharge. See App. G for further details.

Appendix C. Diagonalization of the Yukawa Term

In this appendix we discuss the exact diagonalization of the Yukawa term H_F in the Hamiltonian, keeping $\hbar = g = 1$.

We begin with the expression (4.5) for H_F in terms of the gauge-invariant fermions Λ',

$$H_F = -\frac{i}{2}\varepsilon_{abc}\Lambda_a\Gamma^m\phi_b^m\Lambda_c = \frac{1}{2}\Lambda_{i\alpha}'M_{i\alpha,j\beta}\Lambda_{j\beta}', \tag{C.1a}$$

$$i, j = 1, 2, 3, \qquad \alpha, \beta = 1\cdots 16, \tag{C.1b}$$

$$M = -i\begin{pmatrix} 0 & \lambda_3\Gamma_3 & -\lambda_2\Gamma_2 \\ -\lambda_3\Gamma_3 & 0 & \lambda_1\Gamma_1 \\ \lambda_2\Gamma_2 & -\lambda_1\Gamma_1 & 0 \end{pmatrix}, \tag{C.1c}$$

where we have noted that $\varepsilon_{abc}\psi_b^i\psi_c^j = \varepsilon_{ijk}\psi_a^k$ because the eigenvector ψ is a group element in the adjoint of SU(2). The gauge-invariant matrix M is Hermitian and imaginary, which means that its eigenvalues μ are real and occur in $\pm$ pairs: if U is one of the 48 eigenvectors of M with eigenvalue μ, then U^* is also an eigenvector, with eigenvalue $-\mu$.

The matrix M also satisfies

$$(M^2)_{ij} = (r^2 - 2\lambda_i^2)\delta_{ij} + \lambda_i \Gamma_i \lambda_j \Gamma_j \,, \tag{C.2a}$$

$$[M(M^2 - r^2)]_{ij} = 2\lambda_1 \lambda_2 \lambda_3 \Theta \delta_{ij} \,, \tag{C.2b}$$

where we have defined $r^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$ and

$$\Theta = -i\Gamma_1 \Gamma_2 \Gamma_3 \,. \tag{C.3}$$

The gauge-invariant matrix Θ (which occurs throughout this paper) is Hermitian and squares to 1. Then (C.2b) gives us a sixth order algebraic equation for the eigenvalues of M:

$$[\mu(\mu^2 - r^2)]^2 = 4W \,, \tag{C.4}$$

where $W = (\lambda_1 \lambda_2 \lambda_3)^2 = \det(\Phi)$ is the Claudson–Halpern variable. The solutions of this algebraic equation are six real numbers, in three $\pm$ pairs, so that each eigenvalue is eightfold degenerate.

Furthermore, Θ commutes with all of the Γ_i, and hence with the matrix M; so we can label the eigenvectors of M by their Θ eigenvalues ± 1. From Eq. (C.4), we find that the three eigenvalues $\mu_k, k = 1, 2, 3$, corresponding to the $+1$ eigenvalue of Θ satisfy

$$r \leq \mu_3 \leq \frac{2r}{\sqrt{3}} \,, \qquad -r \leq \mu_2 \leq -\frac{r}{\sqrt{3}} \,, \qquad -\frac{r}{\sqrt{3}} \leq \mu_1 \leq 0 \,, \tag{C.5a}$$

$$\mu_1 + \mu_2 + \mu_3 = 0 \,, \tag{C.5b}$$

and the roots $-\mu_k$ correspond to the -1 eigenvalue of Θ.

The origin of the linear relation (C.5b) is as follows: the algebraic equation (C.4) gives the eigenvalues as functions of the gauge-invariant λ's, but in fact only two combinations out of three occur, so that $\mu_k = \mu_k(r, W)$. We also note for use below that the positive eigenvalue μ_3 in (C.5a) behaves as

$$\mu_3 = R + \frac{(\lambda_1 + \lambda_2)^2}{2R} + \cdots = R + O(R^{-2}) \tag{C.6}$$

for large $R = \lambda_3$ and $\lambda_1, \lambda_2 = O(R^{-\frac{1}{2}})$.

We are now ready to be more explicit about the eigenfunctions of M, which may be labeled as

$$M_{i\alpha, j\beta} U_{j\beta}^{k\nu} = \mu_k U_{i\alpha}^{k\nu} \,, \qquad \Theta_{\alpha\beta} U_{i\beta}^{k\nu} = +U_{i\alpha}^{k\nu} \,, \tag{C.7a}$$

$$M_{i\alpha, j\beta} U_{j\beta}^{k\nu*} = -\mu_k U_{i\alpha}^{k\nu*} \,, \qquad \Theta_{\alpha\beta} U_{i\beta}^{k\nu*} = -U_{i\alpha}^{k\nu*} \,, \tag{C.7b}$$

$$k = 1, 2, 3 \,; \qquad \nu = 1 \cdots 8 \,. \tag{C.7c}$$

These eigenvectors U and U^* form a complete orthonormal set,

$$U_{i\alpha}^{k\nu*} U_{i\alpha}^{k'\nu'} = \delta_{kk'} \delta_{\nu\nu'} \,, \qquad U_{i\alpha}^{k\nu} U_{i\alpha}^{k'\nu'} = 0 \,, \tag{C.8a}$$

$$U_{i\alpha}^{k\nu*} U_{j\beta}^{k\nu} = \delta_{ij} \left(\frac{1 - \Theta}{2} \right)_{\alpha\beta} \,, \qquad U_{i\alpha}^{k\nu} U_{j\beta}^{k\nu*} = \delta_{ij} \left(\frac{1 + \Theta}{2} \right)_{\alpha\beta} \,, \tag{C.8b}$$

so we can use them to define gauge-invariant creation and annihilation operators

$$\Lambda'_{i\alpha} = \sum_{k,\nu} (U_{i\alpha}^{k\nu} a_{k\nu}^+ + U_{i\alpha}^{k\nu*} a_{k\nu}) \,, \tag{C.9a}$$

$$\{a_{k\nu}, a_{k'\nu'}^+\} = \delta_{kk'} \delta_{\nu\nu'} \,, \tag{C.9b}$$

and we emphasize the pivotal role of the matrix Θ in the separation into creation and annihilation terms.

With this expansion, the original Yukawa term is completely diagonalized,

$$H_F = -\sum_{k,\nu} \mu_k a_{k\nu}^+ a_{k\nu} \,, \tag{C.10}$$

and this is the main result of this appendix. Defining $|\tilde{0}\rangle$ by $a_{k\nu}|\tilde{0}\rangle = 0$ as usual, we find that the state with the lowest fermionic energy, $H_F \Rightarrow E_0^F$, is

$$\left(\prod_{\nu=1}^{8} a_{3\nu}^+ \right) |\tilde{0}\rangle \;:\; E_0^F = -8\mu_3 \,, \tag{C.11a}$$

$$E_0^F = -8R + O(R^{-2}) \,, \tag{C.11b}$$

and we note that the asymptotic form of this energy is the negative of the bosonic energy $E_0(R)$ in (2.18).

In this case, one can also make a canonical transformation to independent canonical momenta $\tilde{\pi}_a^m$ which commute with the fermion creation and annihilation operators,

$$\tilde{\pi}_a^m = \pi_a^m + \frac{1}{2} \Lambda'_{i\alpha} (R_a^m)_{i\alpha,j\beta} \Lambda'_{j\beta} \,, \tag{C.12a}$$

$$(R_a^m)_{i\alpha,j\beta} = i \sum_{k,\nu} (\partial_a^m U_{i\alpha}^{k\nu*} U_{j\beta}^{k\nu} + \partial_a^m U_{i\alpha}^{k\nu} U_{j\beta}^{k\nu*}) \,, \tag{C.12b}$$

$$[\tilde{\pi}_a^m, a_{k\nu}] = [\tilde{\pi}_a^m, a_{k\nu}^+] = 0 \,, \tag{C.12c}$$

where R_a^m is again a flat connection.

We have used this transformation and the decomposition

$$U_{ia}^{k\nu} = u_i^k(\lambda)(\Gamma_i)_{\alpha\beta} \chi_\beta^\nu \,, \qquad U_{i\alpha}^{k\nu*} = u_i^k(\lambda)(\Gamma_i)_{\alpha\beta} \chi_\beta^{\nu*} \,, \tag{C.13a}$$

$$\sum_i u_i^k(\lambda) u_i^{k'}(\lambda) = \delta^{kk'} \,, \tag{C.13b}$$

$$\Theta \chi^\nu = +\chi^\nu \,, \qquad \Theta \chi^{\nu*} = -\chi^{\nu*} \tag{C.13c}$$

to study the rotational properties of the states in this fermionic Hilbert space. The explicit form of the functions $u_i^k(\lambda)$ is easily obtained, but is not needed here. The rotation generators take the form

$$J^{mn} = \tilde{\pi}_a^{[m}\phi_a^{n]} + \frac{i}{2}\sum_{k,\nu,\nu'}[a_{k\nu}, a_{k\nu'}^+]\chi^{\nu*}(D^{mn} + i\Sigma^{mn})\chi^{\nu'}, \qquad (\text{C.14a})$$

$$D^{mn} = \phi_a^{[m}\partial_a^{n]}, \qquad (\text{C.14b})$$

in this case, and the following list collects the states which are singlets ($J^{mn} = 0$) under spin (9):

$$|\tilde{0}\rangle e^{-\frac{3\omega}{2}}, \qquad A_k|\tilde{0}\rangle e^{-\frac{\omega}{2}}, \qquad (\text{C.15a})$$

$$A_k A_{k'}|\tilde{0}\rangle e^{+\frac{\omega}{2}}, \qquad A_1 A_2 A_3|\tilde{0}\rangle e^{+\frac{3\omega}{2}}. \qquad (\text{C.15b})$$

Here we have defined

$$A_k \equiv \prod_{\nu=1}^{8} a_{k\nu}^+, \qquad \omega_a^m \equiv \chi_\alpha^{\nu*}\partial_a^m\chi_\alpha^\nu = \partial_a^m\omega \qquad (\text{C.16})$$

and the last relation follows because ω_a^m is a flat connection. The "lowest" state (C.11a) appears in this list, and, owing to (C.11b) this set of states may provide an alternative description of the spin (9) singlet ground state candidate obtained in Sec. 7.

Appendix D. Assessment of Terms in the Hamiltonian

Here we examine individual terms, or groups of terms, in the transformed Hamiltonian (4.19) and note for each:

(1) Its selection rule with respect to the fermion number operator

$$N_F = \Sigma a_\alpha^+ a_\alpha; \qquad (\text{D.1})$$

(2) Its order of magnitude in powers of R, using $\lambda_3 = R$ and the fact that λ_1 and λ_2 are of order $R^{-\frac{1}{2}}$ at large R;
(3) Its contribution to the asymptotic computation, keeping only terms through $O(R^{-2})$ in the effective Hamiltonian.

The details below are given for the "first computation" of the text, and comments are added at the end which discuss the changes needed for the second computation (which allows η_3 dependence in the reduced state vector).

In this discussion, we will use the shorthand PHP, QHQ, PHQ and QHP for the various terms in the basic equations, where PHP refers to $\langle \cdot | H | \cdot \rangle$ in (5.8), QHQ refers to $QH|\Psi_Q\rangle$ with the ansatz (5.16) for $|\Psi_Q\rangle$, etc. In this language it will be helpful to state in advance the large R systematics

$$H = O(R), \tag{D.2a}$$

$$PHP = O(R^{-2}), \tag{D.2b}$$

$$QHP, PHQ = O(R^{-\frac{1}{2}}), \tag{D.2c}$$

$$QHQ = O(R), \tag{D.2d}$$

which we will verify below. These orders of magnitude (and the fact that $P|\Psi_Q\rangle = 0$) tell us that

$$Q(H - E)|\Psi_Q\rangle = (H - E)|\Psi_Q\rangle + O(R^{-\frac{1}{2}}) \tag{D.3}$$

and since (as explained in the text) we are interested only in the order R contributions to these terms, the asymptotic results given here for QHQ come entirely from the first term of (D.3).

Finally, it will be useful to note that the shift terms F_a^m in (4.20c) and their squares can be written as

$$F_a^m = (a^+\Gamma_3 a)(T_a^m)_{12} + i(\Lambda_1\Gamma_3\Lambda_3)(T_a^m)_{13} + i(\Lambda_2\Lambda_3)(T_a^m)_{23}, \tag{D.4a}$$

$$F_a^m F_a^m = (a^+\Gamma_3 a)^2 U_{12} - (\Lambda_1\Gamma_3\Lambda_3)^2 U_{13} - (\Lambda_2\Lambda_3)^2 U_{23}, \tag{D.4b}$$

where U_{ij} is as defined in (E.6) and we have used (E.10) to verify that there are no cross terms in (D.4b).

(1) H_B and the first term of H_F:

$$H_0 + H_1 + \lambda_3(N_F - 8) \tag{D.5}$$

where $H_B = H_0 + H_1$ is the bosonic Hamiltonian in (4.19). The decomposition of H_B is given in (2.17), now written in terms of independent bosonic derivatives. This group of terms is $O(R)$ and diagonal in N_F, but the terms of order R cancel in HP because

$$(H_0 - 8R)u_R = 0. \tag{D.6}$$

The $O(R^{-2})$ contributions of these terms to PHP are the derivative terms $(\frac{d^2}{dR^2}, \frac{d}{dR})$ in (5.10), as in the bosonic computation of Subsec. 2.3. The contributions of these terms to QHP and PHQ are negligible in this computation. For QHQ, it is important to note first that $H_1 = O(R^{-2})$, and so these terms can be ignored in the present computation. We find that the remaining terms contribute the $O(R)$ terms which are the first four terms on the left of each of (5.17a,b), plus the R terms and half of the U terms: the term $\frac{1}{2}U$ comes from the operation of Δ on each Γ_i in $|\Psi_Q\rangle$ [using (E.12)], while the R term follows from the $\lambda_3 N_F$ term of (D.5) and the fact that $|\Psi_Q\rangle$ has $N_F = 1$. The RD terms come from Δ acting as one derivative on the f's and one derivative on u_R. Other "cross derivatives" vanish by virtue of (E.14).

(2) The second and third terms of H_F:

$$i(\Lambda_2\Gamma_1\Lambda_3)\lambda_1 + i(\Lambda_3\Gamma_2\Gamma_3\Lambda_1)\lambda_2. \tag{D.7}$$

This operator changes N_F by $+1$ or -1 and is of order $R^{-\frac{1}{2}}$. It gives the entire asymptotic contribution to QHP (and to PHQ) for our calculation and is written out in Eq. (5.14).

(3) The first term in $-F_a^m \pi_a^m$:

$$i(a^+ \Gamma_3 a)(T_a^m)_{12} \partial_a^m .$$ (D.8)

This is diagonal in N_F and of order R. However, it is zero when acting on P, owing to (E.9). Its only significant contribution is in QHQ, where it acts upon the matrices Γ_1 and Γ_2, according to (E.15). This term exchanges the fermion bilinears

$$(T_a^m)_{12} \partial_a^m [(a^+ \Gamma_3 a) \left\{ \begin{array}{c} (a^+ \Gamma_1 \Lambda_3) \\ (\Lambda_3 \Gamma_2 \Gamma_3 a^+) \end{array} \right\} |0\rangle]$$

$$= Z \left\{ \begin{array}{c} (\Lambda_3 \Gamma_2 \Gamma_3 a^+) \\ -(a^+ \Gamma_1 \Lambda_3) \end{array} \right\} |0\rangle$$ (D.9)

to leading order in R and produces the "mixing" terms in Eqs. (5.17) proportional to Z.

(4) The second and third terms in $-F_a^m \pi_a^m$: These terms [see (D.4a)] raise or lower N_F by 1 and are 0 acting on P [see (E.9)]; they are too small to make any contribution to the present calculation.

(5) The term $-G_a^m \pi_a^m$: This gives zero in PHP by (E.14) and is too small to contribute elsewhere.

(6) The first term in $\frac{1}{2} F_a^m F_a^m$: This term [see (D.4b)] is diagonal in N_F but zero when acting on P. A useful fact here is

$$(a^+ \Gamma_3 a)^2 a_\alpha^+ |0\rangle = a_\alpha^+ |0\rangle$$ (D.10)

and the asymptotic contribution $\frac{1}{2} U_{12} |\Psi_Q\rangle$ is obtained for this term in $QH|\Psi_Q\rangle$. This gives the remaining half of the U terms in (5.17).

(7) The second and third terms in $\frac{1}{2} F_a^m F_a^m$: These contribute to PHP as

$$\langle 0| \frac{1}{2} F_a^m F_a^m |0\rangle = 2(U_{13} + U_{23}) = \frac{4}{R^2} + \cdots$$ (D.11)

and hence make a contribution of $\frac{4}{R^2}$ to (5.9).

(8) The term $\frac{1}{2} G_a^m G_a^m$: This contributes to PHP as follows. Using (E.16) and (E.12) we compute

$$\langle 0| \frac{1}{2} G_a^m G_a^m |0\rangle = \frac{1}{16} \mathrm{Tr}\, [(\partial_a^m \Gamma_3)(\partial_a^m \Gamma_3)] = -\frac{1}{16} \mathrm{Tr}\, [(\Gamma_3 \Delta \Gamma_3)]$$

$$= \frac{6}{\lambda_3^2} + U_{13} + U_{23} = \frac{8}{R^2} + \cdots$$ (D.12)

and hence this group of terms contributes $+\frac{8}{R^2}$ to Eq. (5.9).

(9) The term $F_a^m G_a^m$. Using (E.15), this term is negligible in this calculation.

For the second computation, we must allow for the fact that the reduced state vector $|\psi(R, \Lambda_3, \eta_3)\rangle$ is a function also of the gauge-invariant angular variable η_3. This means that we must re-examine those terms above which involve derivatives with respect to η_3, namely Δ and the shift terms $F\pi$ and $G\pi$. The result for Δ is discussed in Sec. 7, and, because derivatives of η_3 are at least one power of R^{-1} smaller than the terms we have kept, we find no new contributions from the shift terms.

Appendix E. Derivatives

We list here a number of useful formulas for the differentiation of the bosonic variables introduced in the text. The notation is

$$\partial_a^m = \frac{\partial}{\partial \phi_a^m}, \qquad \Delta = \partial_a^m \partial_a^m$$

and we adopt here the generalization

$$m, n = 1 \cdots d; \qquad a, b, c = 1 \cdots g; \qquad i, j, k = 1 \cdots g \qquad (g \leq d),$$

although only $d = 9$ and $g = 3$ apply for SU(2) matrix theory.

Using the familiar method of matrix-perturbation theory, one derives the following two basic formulas for differentiation of λ_i and ψ_a^i, defined in (2.2):

$$\partial_a^m \lambda_i = \psi_a^i \psi_b^i \frac{\phi_b^m}{\lambda_i}, \tag{E.1}$$

$$\partial_a^m \psi_b^i = \sum_{j \neq i} \psi_b^j \phi_c^m \frac{\psi_a^i \psi_c^j + \psi_c^i \psi_a^j}{\lambda_i^2 - \lambda_j^2}. \tag{E.2}$$

All that follows is derived by repeated application of these relations and the prior definitions.

When $f(\lambda)$ is any function of the λ_i, we have

$$\Delta f(\lambda) = \sum_i \left[\frac{\partial^2}{\partial \lambda_i^2} + \left(\frac{d - g}{\lambda_i} + \sum_{j \neq i} \frac{2\lambda_i}{\lambda_i^2 - \lambda_j^2} \right) \frac{\partial}{\partial \lambda_i} \right] f(\lambda), \quad \text{(E.3)}$$

$$(\partial_a^m \psi_b^i)(\partial_a^m f(\lambda)) = 0, \tag{E.4}$$

$$\Delta \psi_a^i = \psi_a^i \left\{ -\sum_{j \neq i} U_{ij} \right\}, \tag{E.5}$$

$$U_{ij} \equiv \frac{\lambda_i^2 + \lambda_j^2}{(\lambda_i^2 - \lambda_j^2)^2}. \tag{E.6}$$

The flat matrix connections T were introduced in (4.7) and App. A:

$$(T_a^m)_{ij} = (\psi_b^i \partial_a^m \psi_b^j) = (1 - \delta_{ij}) \phi_b^m \frac{\psi_b^i \psi_a^j + \psi_a^i \psi_b^j}{\lambda_j^2 - \lambda_i^2}, \tag{E.7}$$

$$\partial_a^m (T_a^m)_{ij} = 0, \tag{E.8}$$

$$(T_a^m)_{ij}(\partial_a^m f(\lambda)) = 0, \tag{E.9}$$

$$(T_a^m)_{ij}(T_a^m)_{kl} = (1 - \delta_{ij})(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})U_{ij}. \tag{E.10}$$

The gauge-invariant matrices Γ_i are defined in (4.6). They are real, symmetric, traceless, anticommuting and satisfy $(\Gamma_i)^2 = 1$:

$$\partial_a^m \Gamma_i = \frac{\Gamma^n}{\lambda_i}\left(\delta_{nm}\psi_a^i - \phi_b^n \phi_c^m \psi_b^i \psi_c^i \frac{\psi_a^i}{\lambda_i^2} + \phi_b^n \phi_c^m \sum_{j\neq i} \psi_b^j \frac{\psi_c^i \psi_a^j + \psi_a^i \psi_c^j}{\lambda_i^2 - \lambda_j^2} \right),$$

(E.11)

$$\Delta\Gamma_i = \Gamma_i\left(-\frac{d-g}{\lambda_i^2} - \sum_{j\neq i} U_{ij} \right),$$

(E.12)

$$\partial_a^m(\Gamma_i \partial_a^m \Gamma_i) = 0 \qquad (\text{no sum on } i)\,,$$

(E.13)

$$(\partial_a^m \Gamma_i)(\partial_a^m f(\lambda)) = 0\,,$$

(E.14)

$$(T_a^m)_{ij}(\partial_a^m \Gamma_k) = (1 - \delta_{ij})\frac{2\lambda_k}{(\lambda_i^2 - \lambda_j^2)^2}(\delta_{kj}\lambda_i \Gamma_i - \delta_{ki}\lambda_j \Gamma_j)\,,$$

(E.15)

$$(\partial_a^m \Gamma_i)_{\alpha\beta}(\partial_a^m \Gamma_j)_{\gamma\delta} = \delta_{ij}\left[\frac{(\Gamma^m)_{\alpha\beta}(\Gamma^m)_{\gamma\delta} - \sum_k (\Gamma_k)_{\alpha\beta}(\Gamma_k)_{\gamma\delta}}{\lambda_i^2} \right.$$

$$\left. + \sum_{k\neq i} U_{ik}(\Gamma_k)_{\alpha\beta}(\Gamma_k)_{\gamma\delta} \right] - (1 - \delta_{ij})U_{ij}(\Gamma_j)_{\alpha\beta}(\Gamma_i)_{\gamma\delta}\,, \quad \text{(E.16)}$$

$$\Delta(\Gamma_i\Gamma_j) = (1 - \delta_{ij})\Gamma_i\Gamma_j\left(-\frac{d-g}{\lambda_i^2} - \sum_{k\neq i}U_{ik} - \frac{d-g}{\lambda_j^2} - \sum_{k\neq j}U_{jk} + 2U_{ij} \right).$$

(E.17)

In the case of SU(2), the special gauge-invariant matrix

$$\Theta = -i\Gamma_1\Gamma_2\Gamma_3 = -\frac{i}{6}\varepsilon_{abc}\Gamma^m\Gamma^n\Gamma^p\frac{\phi_a^m \phi_b^n \phi_c^p}{\lambda_1\lambda_2\lambda_3}$$

(E.18)

is imaginary, antisymmetric, traceless, has square equal to the unit matrix, and commutes with the matrices Γ_i:

$$\phi_a^m \partial_b^m \Theta = 0\,,$$

(E.19)

$$\Delta\Theta = \Theta\left[-(d-3)\sum_i \frac{1}{\lambda_i^2} \right],$$

(E.20)

$$\partial_a^m(\Theta \partial_a^m \Theta) = 0\,.$$

(E.21)

Appendix F. Integrals

When we average over the fast variables λ_1 and λ_2 with the Gaussian function (2.18a), the following class of two-dimensional integrals occurs:

$$\int_0^\infty ds \int_0^s dt(s^2 - t^2)(st)^M (s^2 + t^2)^N \exp(-s^2 - t^2) = \frac{(N+M+1)!}{(M+1)2^{M+2}}\,. \quad \text{(F.1)}$$

This formula gives us useful averages for our asymptotic calculation. Using the notation

$$\langle f(\lambda_1, \lambda_2) \rangle = \int d^2 \lambda \sigma_\infty |u_R|^2 \frac{f(\lambda_1, \lambda_2)}{\int d^2 \lambda \sigma_\infty |u_R|^2} \tag{F.2}$$

[see Eq. (2.21) and set $\hbar = g = 1$] we find, for general values of d, that

$$\langle \lambda_1^2 + \lambda_2^2 \rangle = \frac{d-1}{R}, \tag{F.3}$$

$$\langle (\lambda_1^2 + \lambda_2^2)^2 \rangle = \frac{d(d-1)}{R^2}, \tag{F.4}$$

$$\langle \lambda_1 \lambda_2 \rangle = \frac{d-2}{2R}, \tag{F.5}$$

$$\langle \lambda_1^2 \lambda_2^2 \rangle = \frac{(d-1)(d-2)}{4R^2}, \tag{F.6}$$

where $d = 9$ for matrix theory.

Appendix G. Gauge-Invariant Angular Variables

Here we will express the bosonic Laplacian in terms of the complete set (λ_i, η_i^m) of gauge-invariant variables, regarding the λ's and η's respectively as radial and angular variables. The result can be arranged in several ways and the one shown below has particular advantages for our work.

We start, as in App. B, with the gauge-invariant bosonic variables

$$\phi_i^m \equiv \psi_a^i \phi_a^m \tag{G.1}$$

and the Lie derivatives (which do not act on the gauge-invariant fermions)

$$i\pi_i'^m = D_i^m \equiv \psi_a^i \partial_a^m = \psi_a^i \frac{\partial}{\partial \phi_a^m}, \tag{G.2}$$

so that the Laplacian can be written as

$$\Delta = D_i^m D_i^m + (\partial_a^m \psi_a^j) D_j^m = D_i^m D_i^m - \sum_{i \neq j} y_{ij} \phi_j^m D_j^m. \tag{G.3}$$

Here we have made use of (E.2) and

$$y_{ij} \equiv \frac{1}{\lambda_i^2 - \lambda_j^2}. \tag{G.4}$$

The formula

$$D_i^m \phi_j^n = \delta_{ij} \left[\delta_{mn} + \sum_{k \neq i} y_{ik} \phi_k^m \phi_k^n \right] - (1 - \delta_{ij}) y_{ij} \phi_j^m \phi_i^n \tag{G.5}$$

also follows from results in App. E and will be used below.

Next, we write the orbital angular momentum operator M^{mn} in terms of these new derivatives,

$$M^{mn} = -i[\phi_a^m \partial_a^n - \phi_a^n \partial_a^m] = \sum_i M_i^{mn} \,, \tag{G.6a}$$

$$M_i^{mn} \equiv -i[\phi_i^m D_i^n - \phi_i^n D_i^m] \,, \tag{G.6b}$$

and note that the operators M_i^{mn} are Hermitian in the measure $(d\phi)$, although their algebra is not simple. Now calculate the trace of the square of each M_i:

$$M_i^2 = - \sum_{m<n} [\phi_i^m D_i^n - \phi_i^n D_i^m]^2 \tag{G.7a}$$

$$= -\lambda_i^2 (D_i^n)^2 + (\phi_i^m D_i^m)^2 + \left[d - 2 + \sum_{k \neq i} y_{ik} \lambda_k^2 \right] \phi_i^m D_i^m \,. \tag{G.7b}$$

Combining this result with Eq. (G.3) for the Laplacian, we find that

$$\Delta = \frac{1}{\lambda_i^2} \left\{ (\phi_i^m D_i^m)^2 - M_i^2 + \left[d - 2 + \sum_{j \neq i} y_{ij}(\lambda_i^2 + \lambda_j^2) \right] \phi_i^m D_i^m \right\} \,, \tag{G.8}$$

and then noting that

$$\phi_i^m D_i^m = \lambda_i \frac{\partial}{\partial \lambda_i} \qquad \text{(no sum on } i) \tag{G.9}$$

we can simplify this formula to the nice form

$$\Delta = \Delta_\lambda + \Delta_\eta \,, \tag{G.10a}$$

$$\Delta_\eta = - \sum_i \frac{M_i^2}{\lambda_i^2} \,. \tag{G.10b}$$

Here Δ_λ, which contains the λ derivatives, is as given earlier in Eq. (E.3) and we will see that Δ_η, which is negative semidefinite, contains only derivatives with respect to the angular variables η,

$$\eta_i^m \equiv \frac{\phi_i^m}{\lambda_i} \,, \tag{G.11}$$

which complement the radial variables λ.

For any function $f(\lambda, \eta)$, the chain rule gives

$$D_i^m f = \eta_i^m \frac{\partial}{\partial \lambda_i} f + (D_i^m \eta_j^n) \partial_j^n f \,, \tag{G.12a}$$

$$D_i^m \eta_j^n = \frac{\delta_{ij}}{\lambda_i} \left[\delta_{mn} - \sum_k \eta_k^m \eta_k^n + \sum_{k \neq i} y_{ik} \lambda_i^2 \eta_k^m \eta_k^n \right]$$
$$- (1 - \delta_{ij}) y_{ij} \lambda_i \eta_j^m \eta_i^n \,, \tag{G.12b}$$

where we have defined the η derivative,

$$\partial_i^m \equiv \frac{\partial}{\partial \eta_i^m}, \qquad \partial_i^m \eta_j^n = \delta_{mn} \delta_{ij}, \tag{G.13}$$

and (G.12b) is closely related to (E.11).

Using (G.12), we re-express the operators M_i^{mn} in terms of the variables λ and η. As expected, all $\frac{\partial}{\partial \lambda_i}$ terms cancel out and we find that

$$M_i^{mn} = L_i^{mn} + i \sum_{j \neq i} \eta_i^{[m} \eta_j^{n]} \{ x_{ij} (\eta_i \partial_j) + x_{ji} (\eta_j \partial_i) \}, \tag{G.14a}$$

$$L_i^{mn} \equiv -i \eta_i^{[m} \partial_i^{n]}, \tag{G.14b}$$

where L_i^{mn} is the naive angular momentum operator for the η variables and

$$x_{ij} \equiv \frac{\lambda_i^2}{\lambda_i^2 - \lambda_j^2}, \qquad (\eta_i \partial_j) \equiv \eta_i^m \partial_j^m. \tag{G.15}$$

Using the naive η derivative in (G.13), it is not difficult to check that the operators M_i in (G.14) respect the η constraints,

$$M_i^{mn} (\eta_j^p \eta_k^p) = 0, \tag{G.16}$$

and it follows directly that the operators M_i^{mn} are Hermitian in the gauge-invariant measure $d^3 \lambda (d^3 \eta)$ [see (7.4)]. Taken with (G.14), the form of the Laplacian in (G.10) is the central result of this appendix.

For the discussion below, we will also need the form of the operators M_i,

$$M_3^{mn} = L_3^{mn} + i \sum_{j=1,2} \eta_3^{[m} \eta_j^{n]} (\eta_3 \partial_j) + \cdots, \tag{G.17a}$$

$$M_1^{mn} = -i \{ \eta_1^{[m} \partial_1^{n]} - \eta_1^{[m} \eta_2^{n]} [x_{21} (\eta_2 \partial_1) + x_{12} (\eta_1 \partial_2)]$$
$$+ \eta_1^{[m} \eta_3^{n]} (\eta_3 \partial_1) \} + i \left(\frac{\lambda_1^2}{R^2} \right) \eta_1^{[m} \eta_3^{n]} (\eta_{[3} \partial_{1]}) + \cdots, \tag{G.17b}$$

$$M_2^{mn} = -i \{ \eta_2^{[m} \partial_2^{n]} - \eta_2^{[m} \eta_1^{n]} [x_{12} (\eta_1 \partial_2) + x_{21} (\eta_2 \partial_1)]$$
$$+ \eta_2^{[m} \eta_3^{n]} (\eta_3 \partial_2) \} + i \left(\frac{\lambda_2^2}{R^2} \right) \eta_2^{[m} \eta_3^{n]} (\eta_{[3} \partial_{2]}) + \cdots, \tag{G.17c}$$

in the asymptotic region, $R = \lambda_3 \gg \lambda_1, \lambda_2 = O(R^{-\frac{1}{2}})$. The extra term (7.14) of the second computation in the text,

$$M_3^{mn} f(\eta_3) = L_3^{mn} f(\eta_3) + O(R^{-1}), \tag{G.18a}$$

$$-\frac{1}{2} \Delta f(\eta_3) = \left[\frac{L_3^2}{2R^2} + O(R^{-3}) \right] f(\eta_3), \tag{G.18b}$$

follows immediately from the asymptotic form of M_3 in (G.17), the $M_{1,2}$ terms failing to contribute at this order.

In what follows, we will use the results above to outline a strategy for proving the following conjecture:

(a) The eigenvalues ε of the bosonic operator H_0 in (2.17) satisfy

$$\varepsilon \geq (d-1)R = 8R \tag{G.19}$$

and u_R in (2.18) is the only state which realizes the minimum.

(b) The eigenvalues of the bosonic operator

$$H_0' = H_0 + \frac{M_1^2}{2\lambda_1^2} + \frac{M_2^2}{2\lambda_2^2} \tag{G.20}$$

also satisfy $\varepsilon \geq (d-1)R = 8R$ and u_R is the only state which realizes the minimum. Here $M_{1,2}$ are given by their leading (first four) terms in (G.17b) and (G.17c).

The operator H_0 is the dominant part (i.e. it contains all terms of order R) of the bosonic Hamiltonian in the gauge- and rotation-invariant sector; it contains only the fast derivatives $\frac{\partial}{\partial \lambda_{1,2}}$ with the slow variable $R = \lambda_3$ as a parameter. The operator H_0' is the dominant part (in the same sense) of the full bosonic Hamiltonian H_B at large R, including the gauge-invariant angular excitations; it involves only the fast derivatives $\frac{\partial}{\partial \lambda_{1,2}}$ and $\frac{\partial}{\partial \eta_{1,2}}$, with the slow variables $R = \lambda_3$ and η_3 as parameters.

If true, this conjecture implies that our η_1, η_2-independent projector state $|\cdot\rangle$ in (7.3) is the only state whose associated effective Hamiltonian (including $-8R$ from the fermions) has no linear term in R.

There is strong evidence for (a), though we have not tried to prove it: it is straightforward to find a large class of radial eigenfunctions $u_{m,n}(\lambda_1, \lambda_2)$ of H_0 (or H_0') with

$$\varepsilon = R[d-1+2(m+n)], \qquad m,n = 0,1,2,\ldots, \tag{G.21}$$

where $u_{0,0} = u_R$. Assuming (a), we can prove (b) as follows. The positive semidefinite operators M_i^2/λ_i^2, $i = 1,2$, can only give additional positive semidefinite contributions to ε, beyond $(d-1)R$. So, to prove (b), we only need to show that there are no nonconstant solutions to the differential equations

$$M_i^{mn} \mathrm{v}(\eta_1, \eta_2) = 0, \qquad i = 1,2, \qquad \forall mn, \tag{G.22}$$

where the M's are given by their leading terms at large R. We have explicitly checked that this is true.

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