A class of discontinuous integrals involving Bessel functions

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A general theorem, which appears to be newly discovered although it is of a very classical sort, gives simple evaluations for a large class of infinite integrals containing Bessel functions in product with other suitably constrained analytic functions.

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Consider the following two classes of infinite integrals involving Bessel functions, where b is a real positive number:

$$I_1 = \int_0^\infty dt\, t^{\mu+1} J_\mu(bt) f(t),$$
$$I_2 = \int_0^\infty dt\, t^{\mu} N_\mu(bt) f(t). \tag{1}$$

There may be constraints upon the values of μ and upon $f(t)$ in order to give convergence of the integrals at the end points; although in many instances one may add the convergence factor $e^{-\epsilon t}$ and take the limit $\epsilon \to 0+$. The following three conditions are imposed upon the otherwise arbitrary functions $f(t)$.

(i) $f(t)$ is an analytic function in the right half-plane, Re $t \geqslant 0$;
(ii) $f(t)$ is an even function along the imaginary axis, $f(it) = f(-it)$; (2)
(iii) $f(t)$ is bounded for large $|t|$ by $e^{a|\mathrm{Im}\, t|}$ and $b > a \geqslant 0$.

In the nomenclature of Boas,[1] $f(t)$ is an even entire function of exponential type a.

Theorem: The integrals I_1 and I_2 vanish.

The proof involves writing the Bessel functions J_μ and N_μ as linear combinations of the Hankel functions $H_\mu^{(1)}$ and $H_\mu^{(2)}$ and then, by virtue of the assumed analyticity, moving the contour of the $H_\mu^{(1)}$ integral up to the positive imaginary axis and the contour of the $H_\mu^{(2)}$ integral down to the negative imaginary axis. The portions of the contour integrals along the arc at infinity vanish by condition (iii); and the portions along the two halves of the imaginary axis cancel by virtue of condition (ii) plus the identity

$$(+i)^\mu H_\mu^{(1)}(+it) = -(-i)^\mu H_\mu^{(2)}(-it). \tag{3}$$

COMMENTS

As regards the integral I_1, the theorem can be extended to include negative real values of b and the condition (which gives rise to the identification of these as "discontinuous" integrals) reads

$$|b| > a \geqslant 0. \tag{4}$$

A familiar theorem in Fourier transforms gives

$$\int_{-\infty}^\infty dt\, e^{ibt} F(t) = 0, \tag{5}$$

where $F(t)$ is any entire function of exponential type a and $|b| > a$. This may be seen as a special case of our theorem, by combining the integrals I_1 for the two cases $\mu = +\frac{1}{2}$ and $\mu = -\frac{1}{2}$. It is also possible to restate our theorem in the language of Hankel transforms but this does not appear to add anything new.

The most powerful application of the theorem comes from the following.

Corollary: If condition (i) is relaxed to allow some poles in the integrals I_1 and I_2, then we get the residues picked up in moving the contours as described in the above proof.

For example, assuming the complex number z lies in the first quadrant [and f subject to the conditions (2)]

$$\int_0^\infty dt\, t^{\mu+1} J_\mu(bt) f(t)/(t^2 - z^2) = \tfrac{1}{2}\pi i z^\mu H_\mu^{(1)}(bz) f(z), \tag{6a}$$

$$\int_0^\infty dt\, t^{\mu} N_\mu(bt) f(t)/(t^2 - z^2) = \tfrac{1}{2}\pi z^{\mu-1} H_\mu^{(1)}(bz) f(z). \tag{6b}$$

Alternatively, taking x on the real positive axis and using the Principal Value prescription, we find the pair of integral transforms:

$$\mathrm{P.V.}\int_0^\infty dt\, t^{\mu+1} J_\mu(bt) f(t)/(t^2 - x^2) = -\tfrac{1}{2}\pi x^\mu N_\mu(bx) f(x), \tag{7a}$$

$$\mathrm{P.V.}\int_0^\infty dt\, t^{\mu} N_\mu(bt) f(t)/(t^2 - x^2) = \tfrac{1}{2}\pi x^{\mu-1} J_\mu(bx) f(x). \tag{7b}$$

FURTHER COMMENTS

Other interesting relations can be gotten from (6) by differentiating with respect to z, or by taking the Fourier transform with respect to z, or by taking the limit $z \to 0$:

$$\int_0^\infty dt\, t^{\mu-1} J_\mu(bt) f(t) = \frac{2^{\mu-1}}{b^\mu} \Gamma(\mu) f(0), \tag{8}$$

Re $\mu > 0$.

Looking at the standard reference books[2,3] and tables[4–8] of known integrals with Bessel functions one will find a great many particular results that fall within the theorem and corollary given here. Examples of the sorts of functions $f(t)$ encountered are

$$t^{2N};\ J_\nu(at) t^{-\nu};\ J_\nu(a(t^2+p^2)^{(1/2)})(t^2+p^2)^{-(1/2)\nu} \tag{9}$$

and products of these. Generalized hypergeometric functions of the type

$${}_nF_{n+1}(r_1, r_2, \ldots, r_n; s_1, s_2, \ldots, s_{n+1};\ -\tfrac{1}{4}a^2(t^2+p^2)) \tag{10}$$

also satisfy the conditions given. The possibilities are endless.

It is most surprising that I have not been able to find any prior mention of the general results reported in this paper.

I would also guess that the theorem could be extended to consider a class of functions larger than the Bessel functions as the kernel of the integrals.

[1] R. P. Boas, *Entire Functions* (Academic, New York, 1954).
[2] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed. (Cambridge University, Cambridge, England, 1962), Chap. XIII.
[3] W. Magnus and F. Oberhettinger, *Formulas and Theorems for the Functions of Mathematical Physics* (Chelsea, New York, English translation, 1949). This is the reference for the Bessel function notation used in this paper.
[4] A. Erdélyi *et al.*, *Higher Transcendental Functions* (3 vols.) and *Tables of Integral Transforms* (2 vols.) (McGraw-Hill, New York, 1953–55).
[5] Y. L. Luke, *Integrals of Bessel Functions* (McGraw-Hill, 1962).
[6] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products* (English translation, Academic, New York, 1965 and 1980).
[7] A. D. Wheelon, *Tables of Summable Series and Integrals Involving Bessel Functions* (Holden-Day, New York, 1968).
[8] F. Oberhettinger, *Tables of Bessel Transforms* (Springer, Berlin, 1972).