Some improvements in the theory of faster-than-light particles

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The relationship between differentially conserved quantities (electric current density J^μ, stress tensor density $T^{\mu\nu}$, etc.) and the conserved integral quantities (charge Q, energy-momentum P^ν, etc.) is carefully studied for the case of faster-than-light particles. It is found that there is no problem of "negative-energy states" and no need for the "reinterpretation principle" used by previous authors. The central lesson learned is that the concept of "charge" or "momentum" or "energy" even for a free particle should not be taken as separable from the concept of the particle moving "in" or "out" relative to some interaction. Mathematically it is just a matter of certain minus signs in otherwise familiar formulas, but it is essential to include these factors in order to get a relativistically consistent theory for tachyons. Basic application is made to the classical theory of point particles with electromagnetic interactions, the classical theory of a free field, and the quantum theory of a free field. The earlier tachyon quantum theory of Feinberg is drastically revised.

The literature on the speculative theory of faster-than-light particles (tachyons) contains the general assumption that momentum (the four-vector p^μ) is the fundamental quantity to be carried over from the established theory of slower-than-light particles.[1,2] Since p^μ is taken to be a space-like vector, its time component can have different signs in different Lorentz frames and thus there appears a problem of negative-energy states. Additional rules, such as the "reinterpretation principle", have been used to alleviate some aspects of this problem; but I suggest that there is a deeper error in that approach.

In the present study the fundamental quantity will be taken to be the second-rank tensor $T^{\mu\nu}(x)$—called the stress-energy-momentum tensor, or stress tensor for short. This is a differentially conserved quantity,

$$\frac{\partial}{\partial x^\mu} T^{\mu\nu}(x) = \partial_\mu T^{\mu\nu}(x) = 0 , \qquad (1)$$

and what we call momentum (p^μ) is derived by integrating certain components of $T^{\mu\nu}$ over certain three-dimensional surfaces in four-dimensional space-time.[3] It is the choice of these components and surfaces which makes a crucial difference between the treatments of slow and fast particles ("slow" and "fast" relative to light).

In Sec. I below the general form of the intergrated conservation law is derived. In Sec. II the dynamical system of fast and slow point particles interacting with the electromagnetic field is studied and the correct conservation law for energy and momentum is derived. In Sec. III a noninteracting (classical) tachyon field is studied and a new expression for the field momentum is derived; and in Sec. IV these lessons are used to correct a fundamental error in the quantum field theory for tachyons. In the notation used here there is no need for imaginary numbers in any of the classical (nonquantum) theory. The relativity notation is

$$\mu, \nu, \ldots = 0, 1, 2, 3 ,$$

$$i, j, = 1, 2, 3 ,$$

the velocity of light = 1 ,

$$x^\mu = (x^0 = t, x^i) ,$$

$$g^{00} = +1, \quad g^{ii} = -1 .$$

The symbol x stands for the four-vector x^μ and the symbol $\vec{x}$ stands for the three-vector x^i, and we have the scalar products

$$x \cdot y = x^\mu g_{\mu\nu} y^\nu = x^\mu y_\mu = x^0 y^0 - \vec{x} \cdot \vec{y} ,$$

$$x^2 = x \cdot x ,$$

$$\vec{x} \cdot \vec{y} = x^i y^i .$$

$\mathfrak{v}$ represents a volume in four-space and $\mathfrak{s}$ represents its (three-dimensional) boundary surface; V represents a volume in three-space ($\vec{x}$) and S represents its (two-dimensional) boundary surface. The step functions are

$$\theta(u) = \begin{cases} +1, & u > 0 \\ 0, & u < 0 , \end{cases}$$

$$\epsilon(u) = \theta(u) - \theta(-u) = u/|u| ,$$

and the Dirac δ function is $\delta(u)$ in one dimension or $\delta^4(x)$ in four dimensions.

I. THE GENERAL CONSERVATION THEOREM

Consider some differentially conserved four-vector $J^\mu(x)$,

$$\partial_\mu J^\mu = 0 \,. \tag{2}$$

This might represent the electric charge, the energy-momentum, or any other conserved physical quantity; for the sake of concreteness and familiarity think of J^μ as electric charge density and current density.

Suppose we have a physical system consisting of various particles and fields (the latter perhaps partially localizable as wave packets) which interact with one another in some finite region of space-time. Figure 1 shows this region of interaction, outside of which are various trajectories of the (free) particles and wave packets which enter into or emerge from this interaction region. It is assumed that in this outer region the conserved current can be decomposed as

$$J^\mu(x) = \sum_n J_n^\mu(x) \,, \tag{3}$$

where the label n identifies the individual particles, and in this outer region each current $J_n^\mu(x)$ would be individually conserved.

What is indicated as the "region of interaction" in this figure may also contain trajectories of spectator particles which do not actually interact with others but are admitted by the aperture of the experimental device. The boundaries of the familiar timelike surfaces should not extend to infinity since any actual experiment requires the isolation of a particular region from the rest of the universe.

The theorem—conservation of total charge—will be derived by integrating Eq. (2) over some volume $\mathfrak{v}$ bounded by a surface $\mathfrak{s}$. For convenience (both theoretical and experimental) the surface $\mathfrak{s}$ is chosen, as shown in Fig. 1, to consist of three portions,

$$\mathfrak{s} = \mathfrak{s}_1 + \mathfrak{s}_2 + \mathfrak{s}_3 \,, \tag{4}$$

sufficiently far away from the interaction region so that the slow particles (including those moving at the speed of light) enter through $\mathfrak{s}_1$ and leave through $\mathfrak{s}_2$ while the fast particles (tachyons) pierce the surface $\mathfrak{s}_3$.

From the divergence theorem and Eq. (2) we get

$$\int_{\mathfrak{v}} d^4x\, \partial_\mu J^\mu(x) = \int_{\mathfrak{s}} d^3x\, \eta_\mu J^\mu(x) = 0 \,, \tag{5}$$

where η_μ is a unit vector directed outward at each point of the surface. We now break up the current J^μ according to (3) and the surface $\mathfrak{s}$ according to (4). Simplifying further we take $\mathfrak{s}_1$ to the three-dimensional volume V_1 at the time t_1, $\mathfrak{s}_2$ to be the volume V_2 at time t_2, and $\mathfrak{s}_3$ is then the time interval $t_1 \leq t \leq t_2$ along with the two-dimensional surface $S(t)$ which bounds V_1 at t_1 and bounds V_2 at t_2.

The contribution to (5) of each slow particle crossing $\mathfrak{s}_1$ is thus, with $\eta_\mu = (-1, 0, 0, 0)$,

$$-\int_{V_1} d^3x\, J_n^0(\vec{x}, t_1) = -Q_n(t_1) \tag{6}$$

and each slow particle crossing $\mathfrak{s}_2$ is, with $\eta_\mu = (+1, 0, 0, 0)$,

$$+\int_{V_2} d^3x\, J_n^0(\vec{x}, t_2) = +Q_n(t_2) \,. \tag{7}$$

The plus and minus signs are determined by η_μ, and we have given conventional definitions of the total charges Q_n. For the fast particles crossing $\mathfrak{s}_3$ the contributions to (5) are, with $\eta_\mu = (0, \hat{n})$,

$$+\int_{t_1}^{t_2} dt \int_{S(t)} d^2x\, \hat{n}\cdot\vec{J}_n(\vec{x}, t) = Q_n(S) \,. \tag{8}$$

Here $\hat{n}$ is a unit three-vector pointing outward at the closed surface S.

The integrated conservation theorem is, putting all the parts together,

$$\sum_{n,\ \text{slow particles incoming}} Q_n(t_1) = \sum_{n,\ \text{slow particles outgoing}} Q_n(t_2) + \sum_{n,\ \text{fast particles}} Q_n(S) \,. \tag{9}$$

In words this reads as follows: (the sum of all the charges of the slow particles present inside V before the interaction) equals (the sum of all the charges of the slow particles inside V after the interaction) plus (the sum of all the charges carried out through the surface S, enclosing V, by the fast particles).

The essential and new feature of this calculation is that the "charge" of a fast particle is not defined by the usual three-space integral over J^0 that is used for slow particles—as in (6) and (7)—but is instead the "time integrated outflux" as given by (8). Note that the distinctions "before" or "after" the interaction (equivalently, "incoming" or "outgoing") are not applied to fast particles; we understand that this is not a Lorentz-invariant characterization for fast particles.

There are two further results, familiar from the usual theory of slow particles, which also apply here. Each Q_n is actually independent of the argument shown (t or S as the case may be) in the preceding formulas, and the defining integrals (6) and (8) can be taken over more complicated three-dimensional surfaces than the simple ones chosen above. Each Q_n is a Lorentz scalar.

Proofs will be left to the reader with only the following added comments: the time t_1 (t_2), or more generally the surface s_1 (s_2), must remain in the backward (forward) light cone, and the surface S must enclose the interaction region with the unit vector $\hat{n}$ pointing outward. (More generally, the normal η_μ to the surface s_3 should be a spacelike vector pointing outward from the interaction region.) These basic geometrical relationships shown in Fig. 1 are invariant under (proper) Lorentz transformations.

As an example, consider a point particle which moves along a trajectory (world line) with coordinates $\xi^\mu(\tau)$ where τ is some monotonic scalar parameter. The current is written as

$$J^\mu(x) = e\int_{\tau_1}^{\tau_2} d\tau \frac{d\xi^\mu}{d\tau}\,\delta^4(x-\xi(\tau)) \tag{10}$$

and one readily proves that $\partial_\mu J^\mu = 0$ provided that the end points of the trajectory are away from the region of space of interest. Suppose this is a slow particle ($d\xi^\mu$ is a timelike vector); we calculate the charge:

$$\begin{aligned} Q &= \int_V d^3x\, J^0(\vec{x},t) = e\int d\tau \frac{d\xi^0}{d\tau}\,\delta(t-\xi^0(\tau)) \\ &= e\frac{d\xi^0}{d\tau}\left|\frac{d\xi^0}{d\tau}\right|^{-1} = \pm e\,. \end{aligned} \tag{11}$$

This assumes that the trajectory locates the particle inside the volume V at the time t, or else the δ function gives zero. It is common to assume that the parameter τ is the "proper time" which increases with time along the trajectory, thus giving the plus sign in the last part of (11); but this is a quite unnecessary, and perhaps restrictive, assumption.

Now return to (10) for the case of a fast particle ($d\xi^\mu$ spacelike).

$$\begin{aligned} Q &= \int dt \int_S d^2x\, \hat{n}\cdot\vec{J}(\vec{x},t) \\ &= e\int d\tau\, \hat{n}\cdot\frac{d\vec{\xi}}{d\tau}\,\delta(\hat{n}\cdot\vec{x}-\hat{n}\cdot\vec{\xi}) \\ &= e\hat{n}\cdot\frac{d\vec{\xi}}{d\tau}\left|\hat{n}\cdot\frac{d\vec{\xi}}{d\tau}\right|^{-1} = \pm e\,. \end{aligned} \tag{12}$$

This assumes that the trajectory passes through the surface S, and the calculation is easiest if one uses a local coordinate system with one axis along $\hat{n}$ at the point where the trajectory pierces the surface. The plus or minus sign here is the result of not only the choice of sign for $d\tau$ but also the relative orientation of the vectors $\hat{n}$ and $d\vec{\xi}$.

The above results, (11) and (12), depended on particular choices of surfaces and are not obviously Lorentz invariant. The general calculation is (see the Appendix)

$$\begin{aligned} \int d^3x\, \eta_\mu J^\mu(x) &= \int d^4x\, \delta[f(x)]\frac{\partial f}{\partial x^\mu}\, e\int d\tau \frac{d\xi^\mu}{d\tau}\,\delta^4[x-\xi(\tau)] \\ &= e\int d\tau\, \delta[f(\xi)]\frac{df}{d\tau} = e\epsilon\left(\frac{df}{d\tau}\right) \\ &= e\epsilon\left(\eta_\mu\frac{d\xi^\mu}{d\tau}\right), \end{aligned} \tag{13}$$

where the resulting quantities are evaluated at the point where the trajectory intersects the surface.

Some further discussion may help in the understanding of what has been learned here. The standard integral (6) used for slow particles would be inappropriate for fast particles: the volume V used for the three-space integral should be finite but a tachyon may move arbitrarily fast, thus taking itself outside of this volume in any short time interval. Similarly, the integral (8) would be inappropriate for slow particles since they could be standing still and never cross through the surface S.

The definition of charge given by (8) may seem wrong by the following argument: go to another Lorentz frame where the tachyon moves in the opposite direction, then the charge has changed sign. The difficulty here is that ideas familiar from the study of slow particles are being improperly used for fast particles. For slow particles we define separately the two concepts of "the charge of a particle" and "the direction in time of its motion." Note that the conservation theorem (9) involves only the product of these two concepts, and for fast particles it appears that only the combined concept, call it "charge flow", is meaningful. This discussion will be continued in the next section when we discuss energy and momentum.

II. DYNAMICS OF POINT CHARGES

Consider a system of point particles, moving along trajectories

$$\xi_n^\mu = \xi_n^\mu(\tau_n)\,,\quad n = 1, 2, \ldots$$

which interact with Maxwell's field

$$F^{\mu\nu}(x) = -F^{\nu\mu}(x) \tag{14}$$

according to the familiar dynamical equations

$$M_n\frac{d^2\xi_n^\mu}{d\tau_n^{\,2}} = e_n\frac{d\xi_{n\nu}}{d\tau_n}F^{\mu\nu}(\xi_n)\,, \tag{15}$$

$$\partial_\mu F^{\mu\nu}(x) = 4\pi J^\nu(x)\,, \tag{16}$$

$$\partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} + \partial^\lambda F^{\mu\nu} = 0\,, \tag{17}$$

and

$$J^\nu(x)=\sum_n e_n\int d\tau_n \frac{d\xi_n^\nu}{d\tau_n}\,\delta^4(x-\xi_n)\,. \tag{18}$$

The current $J^\nu(x)$ is conserved (actually each term is separately conserved) and we have the discussion of conserved charge exactly as in the preceding section. An additional conserved quantity is the stress tensor:

$$T^{\mu\nu}(x)=T_F^{\mu\nu}(x)+\sum_n T_n^{\mu\nu}(x)\,, \tag{19}$$

where the field portion $T_F^{\mu\nu}$ is familiar,

$$T_F^{\mu\nu}(x)=\frac{1}{4\pi}F^{\mu\lambda}F_\lambda{}^\nu+\frac{1}{16\pi}g^{\mu\nu}F^{\lambda\kappa}F_{\lambda\kappa}\,, \tag{20}$$

and the particle portion is

$$T_n^{\mu\nu}(x)=M_n\int d\tau_n \frac{d\xi_n^\mu}{d\tau_n}\frac{d\xi_n^\nu}{d\tau_n}\,\delta^4[x-\xi_n]\,. \tag{21}$$

It is left as an exercise to prove that

$$\partial_\mu T^{\mu\nu}(x)=0\,. \tag{22}$$

Before proceeding to derive the energy-momentum conservation theorem from this, it is worth remarking on the various parameters appearing in the above equations. The masses M_n and the charges e_n are real numbers that will presumably be determined by experiments. The variables τ_n are arbitrary except for the requirements that each one be a continuous, monotonic real variable along the trajectory of the particle in question and they should be invariant under proper Lorentz transformations. Equations (15) and (14) do give further constraints, however, when we contract (15) with $d\xi_n^\nu/d\tau$ to get

$$\tfrac{1}{2}M_n\frac{d}{d\tau_n}\left(\frac{d\xi_n^\nu}{d\tau_n}\frac{d\xi_{n\nu}}{d\tau_n}\right)=0 \tag{23}$$

which implies, for some constants C_n,

$$d\xi_n^\nu d\xi_{n\nu}=C_n(d\tau_n)^2\,. \tag{24}$$

By a scale transformation on the parameters τ_n (and the M_n as well) the constants C_n can be reduced to ±1:

$$d\xi_n^\nu d\xi_{n\nu}=\pm(d\tau_n)^2\,. \tag{25}$$

Thus we have the two distinct possibilities of slow and fast particles.

This also shows that the variables τ_n are dependent upon the coordinates ξ_n^ν of the particle trajectory, except that there remains an ambiguity of sign in taking the square root:

$$d\tau_n=\pm\left|d\xi_n^\nu d\xi_{n\nu}\right|^{1/2}\,. \tag{26}$$

Whatever choice is made in this sign must be invariant under proper Lorentz transformations but may be changed under discrete symmetries. From the equations of motion, (15), (16), (17), and (18), follow the reflection symmetries[4]:

space inversion, or parity (P)—

change the signs of x_n^i, ξ_n^i, J^i, F^{i0}, (27a)

time reversal (T)—

change the signs of x_n^0, ξ_n^0, J^i, F^{ij}, τ_n, (27b)

charge conjugation (C)—

change the signs of J^μ, $F^{\mu\nu}$, τ_n. (27c)

The customary fixing of $d\tau_n$ as proportional to the time $d\xi_n^0$ leaves only two reflection symmetries, P and T. Thus it appears that the study of fast particles has lead to something new for slow particles as well: the possibility of antiparticles suggested by classical theory.

Now let us return to the differential conservation law (22) for the stress tensor and subject this to the analysis of the preceding section. There will be four integrated conservation laws (one for each value of ν) and four conserved charge flows for each particle or wave packet connected to the interaction region (see Fig. 1). Comparing (21) to (10) one can read off the results from (11) and (12). For slow particles crossing s_1 (incoming) the contributions are

$$-M_n\frac{d\xi_n^\nu}{d\tau_n}\frac{d\xi_n^0}{d\tau_n}\left|\frac{d\xi_n^0}{d\tau_n}\right|^{-1}\Bigg|_{t_1}\equiv -P_n^\nu(t_1) \tag{28}$$

and for those crossing s_2 only the overall sign is changed and the evaluation is at time t_2. For fast particles crossing s_3 the result is

$$M_n\frac{d\xi_n^\nu}{d\tau_n}\frac{\hat{n}\cdot d\vec{\xi}_n}{d\tau_n}\left|\frac{\hat{n}\cdot d\vec{\xi}_n}{d\tau_n}\right|^{-1}\Bigg|_S\equiv P_n^\nu(S)\,. \tag{29}$$

The integrated conservation theorem reads

$$\sum_{n,\text{ slow particles (including light) incoming}} P_n^\nu(t_1) = \sum_{n,\text{ slow particles (including light) outgoing}} P_n^\nu(t_2) + \sum_{n,\text{ fast particles}} P_n^\nu(S)\,. \tag{30}$$

The contributions of the electromagnetic field have been taken in the standard way

$$P_F^\nu(t)=\int d^3x\, T_F^{0\nu}(\vec{x},t) \tag{31}$$

and added into (30) as "light".

Now the interesting study is to see how the quantities defined in (28) and (29) compare to the usual definitions of momentum; clearly these are the usual definitions except, perhaps, for the signs. First, note that the formulas (28) and (29) do not depend on the choice of ± sign given to the τ_n. Next, look at the $\nu=0$ components (the "energy"). For each slow particle,

$$\operatorname{sign}(P_n^0)=\operatorname{sign}(M_n) \tag{32}$$

and for each fast particle,

$$\operatorname{sign}(P_n^0)=\operatorname{sign}\left(\frac{M_n\hat{n}\cdot d\vec{\xi}_n}{d\xi_0}\right) = \operatorname{sign}(M_n\hat{n}\cdot\vec{v}_n), \tag{33}$$

where $\vec{v}=d\vec{\xi}/d\xi_0$ is the velocity of the particle.

Now we make the assertion that all the mass parameters M_n should be positive, and read the $\nu=0$ component of the conservation law (30) as follows:

(The sum of the energies of all the incoming slow particles)
equals (the sum of the energies of all the outgoing slow particles
plus (the sum of the energies of all the fast particles moving outward minus the sum of the energies of all the fast particles moving inward). (34)

The word energy in every case means a *positive* (at least non-negative) quantity. This means that the system is stable, and there is no such thing as particles carrying "negative energy" in or out of a reaction. This result conforms to all our expectations of what a sensible theory ought to yield. What is significant here is that we had no need for the "reinterpretation principle" used in previous studies of fast particles.

The crux of the matter lies in the discussion given at the end of Sec. I. Customarily it is assumed that momentum (p^ν) of a free particle is one attribute of that particle and its relation to an interaction (going in or coming out) is another separate attribute; but for tachyons we now see that only the product of these two ideas—the "momentum flow"—is meaningful. The reinterpretation principle was needed only when one tried to take these two concepts apart and I suggest its invention has caused more harm than good in the development of tachyon theory. The following sections will give an example of this.

Again, the results above, derived for the special surfaces, can be generalized to read, like (13) for the charge flow,

$$P^\nu(s)=M\frac{d\xi^\nu}{d\tau}\epsilon\left(\eta_\mu\frac{d\xi^\mu}{d\tau}\right) \tag{35}$$

and this has a manifestly covariant form.

For an additional exercise, look at (29) for the component of $\vec{P}_n(S)$ in the direction of $\hat{n}$—it is always a positive quantity. Does this mean that a tachyon's momentum is always positive, in the direction normal to the surface S, regardless of whether it is traveling in or out?

III. CLASSICAL FIELD THEORY

Consider a real scalar field $\phi(x)$, without interactions, which satisfies the wave equation

$$(\partial^\mu\partial_\mu \pm M^2)\phi=0 \tag{36}$$

in which the choice of sign $+M^2$ (the usual Klein-Gordon equation) refers to a slow particle and the choice $-M^2$ refers to tachyon waves. The conserved stress tensor is

$$T^{\mu\nu}(x)=\tfrac{1}{2}(\partial^\mu\phi\partial^\nu\phi+\partial^\nu\phi\partial^\mu\phi) - \tfrac{1}{2}g^{\mu\nu}(\partial^\lambda\phi\partial_\lambda\phi\mp M^2\phi\phi), \tag{37}$$

and it is customary to introduce a Fourier transform of the wave,

$$\phi(x)=\int d^4k\,\delta(k^2\mp M^2)\,a(k)\,e^{-ik\cdot x}. \tag{38}$$

A proper homogeneous Lorentz transformation L is expressed by the mapping

$$a(k)\to a(Lk) \tag{39a}$$

which brings about the transformation

$$\phi(x)\to\phi(Lx) \tag{39b}$$

It is important to understand the meaning of the Fourier transform in relation to the picture, Fig. 1, of the general reaction process. The wave equation, without interaction, describes what is happening only *outside* the interaction region. Thus the domain of space-time being described by the Fourier integral representation is not all of space-time, but rather only some restricted region in some neighborhood of one of the surfaces s_1, s_2, or s_3. Thus, the derivation of the conserved four-vector of momentum flow

$$P^\nu=\int_s d^3x\,\eta_\mu T^{\mu\nu}(x) \tag{40}$$

will need to specify a particular surface s and normal vector η_μ, and in order to maintain translation invariance this will be chosen to be a plane surface with a constant normal vector. The

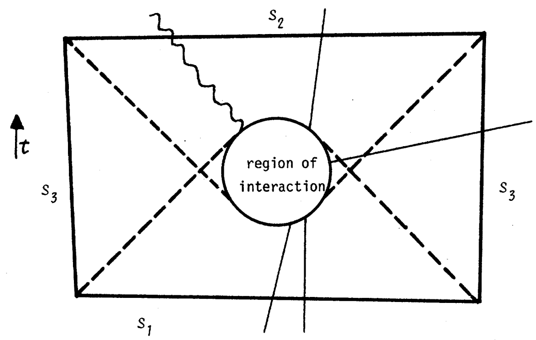

FIG. 1. Schematic diagram of a general interaction. The dashed lines represent the envelopes of the future and past light cones from all points within the region of interaction. The diagram shows two slow particles entering through the (three-dimensional) surface s_1, one slow particle and one wave packet of light leaving through the surface s_2 and one tachyon leaving through the surface s_3.

result of the calculation (see the Appendix) is

$$P^\nu = \int d^4k\, \delta(k^2 \mp M^2)\tfrac{1}{2}\, a(k)\, a(-k)\, k^\nu \epsilon(\eta \cdot k)(2\pi)^3 \quad (41)$$

and with

$$\phi^*(x) = \phi(x) \quad (42a)$$

we have the constraint

$$a^*(k) = a(-k)\,. \quad (42b)$$

This result looks quite familar—except for the factor $\epsilon(\eta \cdot k)$. In the case of slower-than-light fields both the vectors η and k are timelike and so their dot product cannot change sign as the k integral moves over either portion of the mass-shell. This means that the ϵ factor is irrelevant. In the case of tachyon waves, however, this is certainly not the case and the ϵ factor is all important. For example, with the choice

$$\eta_\mu = (0, \hat{n}) \quad (43)$$

we can write the momentum flow as

$$P^\nu = \int d^4k\, \delta(k^2 + M^2)\theta(k^0)\frac{(2\pi)^3}{2}[a^*(k)a(k) + a(k)a^*(k)]k^\nu \epsilon(\hat{n}\cdot\vec{k}) \quad (44)$$

and this is interpreted just as was done for point particle motion in the preceding section. The energy flow ($\nu = 0$) is positive for those Fourier components of the wave flowing with $\vec{k}$ vector (and group velocity) going one way through the surface, and negative for those flowing the other way; the component $\hat{n}\cdot\vec{P}$ is always positive (analogous to $P^0 \geq 0$ in the case of slower-than-light waves).

Angular momentum and boost components derive from the conserved tensor

$$M^{\mu\nu\lambda}(x) = T^{\mu\nu}(x)\, x^\lambda - T^{\mu\lambda}(x) x^\nu\,, \quad (45a)$$

$$\partial_\mu M^{\mu\nu\lambda} = 0 \quad (45b)$$

and the calculation of the angular momentum flow tensor

$$J^{\nu\lambda} = \int_s d^3x\, \eta_\mu M^{\mu\nu\lambda}(x) \quad (46)$$

yields the result

$$J^{\nu\lambda} = \int d^4k\, \delta(k^2 \mp M^2)\frac{(2\pi)^3}{2i}\left(k^\nu \frac{\partial a(k)}{\partial k_\lambda} - k^\lambda \frac{\partial a(k)}{\partial k_\nu}\right) a(-k)\epsilon(\eta\cdot k) \quad (47)$$

IV. QUANTUM FIELD THEORY

Quantization of the classical field theory of the preceding section follows from identification of the quantities $\phi(x)$ and $a(k)$ as noncommuting operators (the algebra of the preceding section was arranged with this in mind) and the replacement of the complex conjugation symbol * by the Hermitian adjoint symbol +

The original work by Feinberg[5] on quantum field theory of tachyons was built upon the mistake of borrowing the formula for P^ν from the slow-particle theory, lacking the ϵ factor. In order to make the formula for P^ν transform as a Lorentz four-vector he had to postulate anticommutator relations (Fermi statistics) for the $a(k)$'s and deal with a number of ensuing problems: a vacuum state which changed particle occupation number in different Lorentz frames; an intricate set of rules to recover physically sensible results for transition probabilities; and a continuing argument with various critics over whether the resulting theory really was Lorentz invariant.[6]

Using what has been learned about tachyons from the earlier parts of this paper one can proceed with quantization of the field theory along quite different lines. The basic algebra of the operators $a(k)$ is postulated to be the commutation relation

$$\delta(k^2 \mp M^2)\delta(k'^2 \mp M^2)[a(k), a(k')] = \frac{\delta^4(k+k')}{(2\pi)^3}\,\delta(k^2 \mp M^2)\,\epsilon(\eta\cdot k) \quad (48)$$

which indicates that $a(k)$ is a destruction (creation) operator for $\eta\cdot k$ positive (negative) and there will be a vacuum state $|0\rangle$ such that

$$a(k)\,\theta(\eta\cdot k)\,|0\rangle=0. \tag{49}$$

The order of factors in (41) is rearranged in defining the operator P^ν so that there will be no "zero-point energies,"

$$P^\nu=(2\pi)^3\int d^4k\;\delta(k^2\mp M^2)\theta(\eta\cdot k)k^\nu a(-k)a(k), \tag{50}$$

and the commutator turns out to have the familiar form

$$[P^\nu, a(k)]=-k^\nu a(k). \tag{51}$$

A number operator, which is non-negative and annihilates the vacuum state, is

$$N(k)=(2\pi)^3[\theta(\eta\cdot k)a(-k)a(k)+\theta(-\eta\cdot k)a(k)a(-k)]=N(-k) \tag{52}$$

and its commutator is

$$\delta(k^2\mp M^2)[N(k),a(k')]=-\epsilon(\eta\cdot k')a(k')[\delta^4(k+k')+\delta^4(k-k')]. \tag{53}$$

The total particle number operator N is defined as

$$N=\tfrac{1}{2}\int d^4k\;\delta(k^2\mp M^2)N(k) \tag{54}$$

and its commutator

$$[N,a(k)]=-\epsilon(\eta\cdot k)a(k) \tag{55}$$

leads to the conclusion that the spectrum of N is $0,1,2,\ldots$.

The operator $J^{\nu\lambda}$ may be taken exactly as given by (47) and the careful calculation of the commutator with $a(k)$ gives the result

$$[J^{\nu\lambda},a(k)]=-\epsilon^2(\eta\cdot k)\,L^{\nu\lambda}(k)\,a(k)+i\epsilon(\eta\cdot k)\,\delta(\eta\cdot k)(k^\nu\eta^\lambda-k^\lambda\eta^\nu)\,a(k), \tag{56}$$

where

$$L^{\nu\lambda}(k)=\frac{1}{i}\left(k^\nu\frac{\partial}{\partial k_\lambda}-k^\lambda\frac{\partial}{\partial k_\nu}\right). \tag{57}$$

The second term on the right-hand side of (56) is clearly of a delicate nature, indicating that this analysis needs to be done over using wave packets and appropriately smooth distributions in place of the plane waves and discontinuous functions used here. Nevertheless, I proceed to work as follows. Using the relations

$$\theta(\pm u)\epsilon(u)=\pm\,\theta(\pm u), \tag{58a}$$

$$\theta(\pm u)\delta(u)=\tfrac{1}{2}\,\delta(u)=\pm\tfrac{1}{2}\frac{d}{du}\,\theta(\pm u), \tag{58b}$$

we can write

$$\begin{aligned}[J^{\nu\lambda},a(k)]\,\theta(\pm\eta\cdot k)&=-\theta(\pm\eta\cdot k)L^{\nu\lambda}(k)\,a(k)-\tfrac{1}{2}a(k)\,L^{\nu\lambda}(k)\theta(\pm\eta\cdot k)\\&=-\theta^{1/2}(\pm\eta\cdot k)\,L^{\nu\lambda}(k)\,a(k)\theta^{1/2}(\pm\eta\cdot k)\,.\end{aligned} \tag{59}$$

From this result follow

$$[J^{\nu\lambda},\,N(k)]=-L^{\nu\lambda}(k)N(k)\,, \tag{60}$$

$$[J^{\nu\lambda},\,N]=0\,, \tag{61}$$

$$[J^{\nu\lambda},\;P^\mu]=-i\,(P^\nu g^{\lambda\mu}-P^\lambda g^{\nu\mu}), \tag{62}$$

and the one-particle state (unnormalized) will be constructed as

$$|k\rangle_\eta=[a(k)\theta^{1/2}(-k\cdot\eta)+a(-k)\theta^{1/2}(k\cdot\eta)]\,|0\rangle=|-k\rangle_\eta\,. \tag{63}$$

The subscript η is placed on this state vector to remind us of the surface s with respect to which measurements are to be made; this is the generalization of the familiar "in" and "out" labels used for states of a free slow particle.

The various operators act on this one-particle state as follows:

$$N\,|k\rangle_\eta=|k\rangle_\eta\,, \tag{64}$$

$$P^\nu|k\rangle_\eta=k^\nu\epsilon(\eta\cdot k)|k\rangle_\eta\,, \tag{65}$$

$$J^{\nu\lambda}|k\rangle_\eta=-L^{\nu\lambda}(k)|k\rangle_\eta\,. \tag{66}$$

Exponentiation of the operator P^ν gives the unitary operator

$$U(d)=e^{iP\cdot d} \tag{67}$$

which is identified as the translation operator, having eigenvalues

$$e^{ik\cdot d\epsilon(\eta\cdot k)} \tag{68}$$

on the one-particle states. Exponentiation of $J^{\nu\lambda}$ gives the unitary operator $U(L)$ which acts on the one-particle states as

$$U(L)|k\rangle_\eta=|Lk\rangle_\eta \tag{69}$$

which is interpreted as a boost—changing the particle state (via a "Lorentz transformation" of the k vector) but leaving the reference surface (represented by η) unchanged. To describe a true Lorentz transformation—which relates the descriptions of two observers in different inertial reference frames—we must supplement the above operator transformation $U(L)$ by an explicit transformation on the coordinates of the vector η:

$$|k\rangle_\eta\to U(L)|k\rangle_\eta=|Lk\rangle_\eta\to|Lk\rangle_{L\eta}\,. \tag{70}$$

Continuing from (62) one can calculate the complete commutator algebra of the ten operators P^{ν} and $J^{\nu\lambda}$; it turns out to have the familiar structure of the Lie algebra of the Poincaré group, without any reference to η appearing. Yet, when one adds explicit transformations of η then new structure appears. Obviously there is more to be learned here.

This leads into consideration of tachyons with spin other than zero, which are usually excluded because of the lack of unitary (and finite dimensional) representations of the little group SO(2, 1) appropriate to a spacelike momentum vector. Perhaps the ϵ factors can play a role in the discovery of new representations. Or perhaps the requirement of unitarity—derived as it is from the standard measurement theory which appears to have some bias as regards the time evolution of systems–needs to be revised for tachyons. These questions will be left for future study.

Many questions about the possibilities of faster-than-light particles remain to be explored and I hope that the clarifications achieved in the present work will be useful.

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APPENDIX: SURFACES AND NORMAL VECTORS

A surface s is defined by the equation $f(x)=0$ in a given reference frame K, and the normal to this surface is

$$\eta_\mu = C\frac{\partial f}{\partial x^\mu} \tag{A1}$$

evaluated at the point on the surface in question. The normal vector η_μ is normalized in the Euclidean sense (for use in the divergence theorem),

$$1=\sum_\mu \eta_\mu\eta_\mu = C^2\sum_\mu\left(\frac{\partial f}{\partial x^\mu}\right)^2 \tag{A2}$$

and the sign of C is chosen in accord with the specification that η_μ point "outward" from some designated region. Looked at from another frame K', with coordinates differing by a Lorentz transformation L, the same point on the same surface will have the normal vector

$$\eta'_\mu = C'\frac{\partial f}{\partial x'^\mu} = C'\frac{\partial f}{\partial x^\nu}\frac{\partial x^\nu}{\partial x'^\mu}; \tag{A3}$$

thus we conclude that the normal vector η_μ transforms as C'/C times a covariant four-vector. If we consider only the set of proper Lorentz transformations, these are continuously connected to the identity transformation and it is clear that the sign of the normalizing constant C cannot change. Therefore, for any contravariant four-vector R^μ the *sign* of $\eta_\mu R^\mu = \eta\cdot R$ is invariant under proper Lorentz transformations, although the magnitude of this quantity is not invariant.

In addition, one can raise and lower indices on η_μ in the usual way and consider the Minkowski square of this normal vector

$$\eta^2 \equiv \eta_\mu \eta^\mu \tag{A4}$$

which, although not a Lorentz scalar, has important characteristics. If η^2 is positive, this is a Lorentz-invariant statement which characterizes the surface (like s_1 and s_2 in Fig. 1) as having a timelike normal, or more commonly said to be a "spacelike surface". If η^2 is negative, this is again a Lorentz-invariant statement which characterizes the surface (like s_3 in Fig. 1) as having a spacelike normal, although the naming of such a surface as being "timelike" would not be sensible. (Surfaces with $\eta^2=0$ exist but are not used in this study.) Furthermore, an expression as

$$\eta^\mu \eta\cdot R/\eta^2 \tag{A5}$$

does transform as a (contravariant) Lorentz four-vector.

The following manipulation which differs from the usual textbook presentation of the divergence theorem makes some calculations simpler. Consider the volume v enclosed by the surface s such that $f(x)$ is negative inside v, zero on s, and positive outside v,

$$\begin{aligned}\int_v d^4x\,\partial_\mu J^\mu(x) &= \int d^4x\,\theta(-f)\partial_\mu J^\mu(x)\\ &= -\int d^4x\,J^\mu(x)\partial_\mu\theta(-f)\\ &= \int d^4x\,J^\mu(x)\,\frac{\partial f}{\partial x^\mu}\,\delta(f).\end{aligned} \tag{A6}$$

For example, consider a plane surface s, with $f=\eta_\mu x^\mu + c$

$$\begin{aligned}\int_s d^3x\,e^{ik\cdot x} &= \int d^4x e^{ik\cdot x}\delta(f)\\ &= \int d^4x\,e^{ik\cdot x}\int\frac{du}{2\pi}\,e^{iuf}\\ &= \int du(2\pi)^3\delta^4(k+u\eta)e^{iuc}.\end{aligned} \tag{A7}$$

[1]O. M. P. Bilaniuk, V. K. Deshpande, and E. C. G. Sudarshan, Am. J. Phys. 30, 718 (1962).
[2]*Tachyons, Monopoles, and Related Topics,* Proceedings of the First Session of the Interdisciplinary Seminars, Erice, Sicily, 1976, edited by E. Recami (North-Holland, Amsterdam, 1978). Many references to other works are contained here.
[3]For a review of the standard treatments of light and slower particles, see, for example, the following texts. E. J. Saletan and A. H. Cromer, *Theoretical Mechanics* (Wiley, New York, 1971), especially Chap. VIII; J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), expecially Chap. 12.
[4]There is another symmetry of the equations. Change the sign of e_n and τ_n independently for each particle, but this is symmetry of a different sort since it involves numbers e_n rather than just dynamical variables.
[5]G. Feinberg, Phys. Rev. 159, 1089 (1967).
[6]M. E. Arons and E. C. G. Sudarshan, Phys. Rev. 173, 1622 (1968); G. Feinberg, Phys. Rev. D 17, 1651 (1978); W. B. Rolnick, *ibid.* 19, 3811 (1979); G. Feinberg, *ibid.* 19, 3812 (1979).