

Large- N classical solution for the one-matrix model

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Large- N quantum systems are known to be dominated by a single classical solution M_{ab} , which is the reduced transition matrix among the ground state and relevant "generalized-adjoint" eigenstates. For the one-matrix model, we obtain M_{ab} (and hence the spectrum) for transitions among the ground state and high-lying adjoint states. The method of solution is a new "quasiclassical" quantum mechanics, with no penetration of classically forbidden regions—yet another classical aspect of the large- N limit.

I. INTRODUCTION

Some time ago, Jevicki and Papanicolaou¹ conjectured that the large- N limit of quantum systems would be dominated by a single solution of the classical equations of motion. The original work was on $O(N)$ vector models, but subsequent work verified the conjecture for the one-matrix model,² and gauge theories.³ Finding that solution has proven more difficult; it has thus far been obtained only for the vector models.

Recently, an alternative verification of the conjecture, using large- N matrix mechanics,⁴ has provided the full physical interpretation of the large- N classical solution. It is the set of reduced transition matrix elements among the ground state and those "generalized-adjoint" states which dominate in the large- N limit. As a result, the time dependence of the large- N classical solution is now known to be

$$M_{ab}(t) = e^{i\omega_{ab}t} M_{ab}(0), \quad (1.1)$$

where $\hbar\omega_{ab} = E_a - E_b$ are the energy differences among these states. The program is interesting physically because the generalized-adjoint states correspond, in temporal-gauge quantum chromodynamics (QCD), to the heavy-quark meson states.

In the present work, we will extend the approach of Ref. 4, solving for the large- N classical solution (and spectrum, etc.) in the case of the one-matrix model.

The method of solution of the classical equation of motion (and constraint) is new. We relate the system to a new "quasiclassical" quantum mechanics, in which the equal-time operator algebra is of the form

$$[\hat{q}, \hat{p}] = i\hbar |0\rangle\langle 0|, \quad (1.2)$$

where $|0\rangle$ is the ground state of the system. This is an unfamiliar dynamics, but it (and probably variants thereof) is consistent. We call this dynamics quasiclassical because, as it turns out, with (1.2), classically forbidden regions are *truly*

forbidden. In the cases of interest, this dynamics yields a spectrum which is asymptotically linear, in agreement with previous work.⁵ The method also provides an appropriate pedagogical link with the "collective field" method of Sakita and Jevicki.⁶

II. MATRIX MECHANICS AND CONSTRAINED CLASSICAL SYSTEMS

In this section, we briefly repeat the derivation⁴ of the classical system that governs the large- N limit. One of the basic tools is matrix mechanics, and the first point to make is one which, despite initial appearances, is not solely pedantic: Matrix mechanics can *always* be viewed as a constrained classical system.

Consider, for example, ordinary ($N=1$) quantum mechanics of a single variable, with classical equation of motion

$$\ddot{q} + V'(q) = 0, \quad V'(q) = \frac{d}{dq}V(q). \quad (2.1)$$

Quantizing the system in the usual way ($q \rightarrow \hat{q}$) and taking matrix elements among energy eigenstates

$$q_{nm}(t) = \langle n | \hat{q}(t) | m \rangle \quad (2.2)$$

yields the system

$$\ddot{q} + V'(q) = 0, \quad [q, \dot{q}] = i\hbar, \quad (2.3)$$

where q is the matrix whose elements are q_{nm} . We may view this as a "constrained classical system." The object in question is a matrix; the classical equation is exactly that of the one-matrix model, and the commutator is the constraint.

The size of the matrix in question is obtainable from the constraint; the matrix must be infinite-dimensional. Because n and m label energy eigenstates, one is instructed to seek only that classical solution with time dependence

$$q_{nm}(t) = e^{i\omega_{nm}t} q_{nm}(0). \quad (2.4)$$

This example does not, of course, illustrate the important part of the large- N conjecture—that a solution of the original classical equations of mo-

tion is relevant. Indeed, here we must solve a $U(\infty)$ classical equation to describe a $U(1)$ quantum system. Can this be remedied?

Suppose we started with the $U(N)$ one-matrix model, whose quantized form is

$$\frac{d^2}{dt^2}\hat{M} + N^{1/2}V'(\hat{M}N^{-1/2}) = 0, \quad (2.5)$$

$$[\hat{M}_{ab}, \hat{P}_{cd}] = i\hbar\delta_{bc}\delta_{ad}.$$

Here $V(S\hat{M}S^\dagger) = V(\hat{M})$, $[V'(z)]_{ab} = (d/dz_{ba})V(z)$, and S is an element of $U(N)$. Imagine taking matrix elements among all energy eigenstates. The eigenstates are labeled with energy n , and many $SU(N)$ labels, so that, again, even at $N=\infty$, we arrive at a constrained classical system where the "classical matrix" has many more labels than the original classical matrix. This tail chasing can be terminated in the following manner.

To achieve correspondence between the original and final classical equations, one must work on a *subset* of the eigenstates, the natural set being singlet and adjoint eigenstates. This idea parallels Bardakci,³ who used fixed time eigenstates of invariant position, instead of energy eigenstates. His group theory (and reduced matrix elements) carries over immediately, however, to the energy eigenstates. These energy eigenstates suffice to saturate any "ordered" Wightman function, e.g.,

$$\langle 0 | \text{Tr}[\hat{M}(t_1)\hat{M}(t_2)\hat{M}(t_3)] | 0 \rangle, \quad (2.6)$$

where $|0\rangle$ is the ground state of the system. At this stage of the argument, any other singlet state may be used on the outside.

We will not work out the equations for the full singlet and adjoint sectors here, but specialize to the large- N limit, where there is an important simplification: factorization.

Consider, in the large- N limit, a singlet channel in an ordered *ground-state expectation value*, such as (2.6). It is known then (*only* for ground-state expectation value now) that, of all possible singlet states in a singlet channel, the ground state dominates (factorization). Thus, in the large- N limit, the completeness relation for these functions may be written

$$1 = |0\rangle\langle 0| + \sum_{ab,n} |ab,n\rangle\langle ab,n|. \quad (2.7)$$

Here n labels energy and a, b are the adjoint-state labels. The large- N simplification is simply the omission of the other singlets.

These are the states we sandwich about the equation of motion. Bardakci's³ reduced matrix elements are useful, e.g.,

$$\begin{aligned} \langle 0 | \hat{M}_{ab}(t) | a'b', n \rangle &\equiv (\delta_{ba'}\delta_{ab'} - N^{-1}\delta_{ab}\delta_{a'b'}) \\ &\times N^{1/2}(N^2 - 1)^{-1/2}M_{0n}(t), \end{aligned} \quad (2.8)$$

$$\langle 0 | \hat{M}_{ab} | 0 \rangle \equiv \delta_{ab}M_{00}(t),$$

etc. The resulting constrained classical system for the reduced matrix elements is⁷

$$\begin{aligned} \ddot{M} + N^{1/2}V'(MN^{-1/2}) &= 0, \\ [M, \dot{M}]_{\alpha\beta} &= iN\hbar\delta_{\alpha 0}\delta_{\beta 0}. \end{aligned} \quad (2.9)$$

Once again the constraint determines the size of the matrix M to be infinite. Therefore, at $N=\infty$, we have an isomorphism between the original and final classical equations of motion.

The particular classical solution we seek must have the time dependence

$$M_{ab}(t) = e^{i\omega_{ab}t} M_{ab}(0), \quad (2.10)$$

and we will be able to interpret $\hbar\omega_{ab} = E_a - E_b$ as the energy differences among the ground state and those adjoint eigenstates which dominate at large N .

We finally note that the true ground-state energy is computable from

$$E_0 = \frac{N}{2} \sum_n |M_{0n}|^2 \omega_{0n}^2 + N^2 V_{00}(MN^{-1/2}) \quad (2.11)$$

while the collective field of Sakita and Jevicki⁶ is

$$\begin{aligned} \phi_0(x) &= \left\langle 0 \left| \frac{1}{2\pi} \int dk e^{ikx} \text{Tr}(e^{-ik\hat{M}}) \right| 0 \right\rangle \\ &= \frac{N}{2\pi} \int dk e^{ikx} (e^{-ikM})_{00}. \end{aligned} \quad (2.12)$$

III. OPERATOR FORMULATION OF CLASSICAL SYSTEM

Our task is to solve the constrained classical system

$$\ddot{q} + V'(q) = 0, \quad [q, \dot{q}]_{ab} = i\delta_{a0}\delta_{b0}, \quad (3.1)$$

where $q \equiv MN^{-1/2}$, and we have set $\hbar = 1$. We are only interested, however, in that solution with the correct time dependence $q_{ab}(t) = e^{i\omega_{ab}t} q_{ab}(0)$. The system is evidently a matrix mechanics of the following operator system:

$$i[\hat{H}, \hat{q}] = \dot{\hat{q}} = \hat{p}, \quad (3.2a)$$

$$i[\hat{H}, \hat{p}] = \dot{\hat{p}} = -V'(\hat{q}), \quad (3.2b)$$

$$[\hat{q}, \hat{p}] = i|0\rangle\langle 0|, \quad (3.2c)$$

and

$$\hat{H}|n\rangle = \omega_{n0}|n\rangle, \quad \omega_{n0} = E_n - E_0, \quad \langle m|n\rangle = \delta_{mn}. \quad (3.3)$$

We do not yet, however, know the form of the (reduced) Hamiltonian \hat{H} . It should be emphasized

that these states and operators are *not* the states of the original one-matrix model. These operators are *reduced* operators, from which the reduced matrix elements are computed by sandwiching between the *reduced* states $|n\rangle$, i.e., $q_{ab} = \langle a|\hat{q}|b\rangle$.

The system (3.2) and (3.3) defines an unusual quantum mechanics, and it defines it in an unfamiliar manner. In fact, we shall *construct* \hat{H} directly from (3.2) and (3.3) in what follows. It is not easy to guess \hat{H} because of the unusual algebra (3.2c).⁸ Indeed, this algebra is quasidynamical, in the sense that we do not yet know $|0\rangle$.

It is amusing to hunt for algebraic inconsistencies in the system (3.2) and (3.3). Far from discovering any, we were able to see that there is every reason to expect consistency if the algebra were generalized to

$$[\hat{q}, \hat{p}] = i \sum_n \epsilon_n |n\rangle \langle n|, \quad (3.4)$$

where ϵ_n are a set of real numbers. This class of algebras (3.4) includes both our case, and the usual case ($\epsilon_n = 1$), but we will not pursue this generalized dynamics here.

Before solving for \hat{H} , we record a number of useful results. The ground-state energy Eq. (2.11) now appears as

$$E_0 = N^2 \langle 0 | \frac{1}{2} \hat{p}^2 + V(\hat{q}) | 0 \rangle. \quad (3.5)$$

It is not hard to check that the quantum virial theorem is independent of the algebra of \hat{p} and \hat{q} , so we have the alternative form

$$E_0 = N^2 \langle 0 | \hat{V} + \frac{1}{2} \hat{V}' \hat{q} | 0 \rangle. \quad (3.6)$$

The collective field, Eq. (2.12), is now

$$\phi_0(x) = \frac{N}{2\pi} \int dk e^{ikx} \langle 0 | e^{-i\hat{q}\sqrt{N}} | 0 \rangle. \quad (3.7)$$

Finally,

$$\langle n | \hat{p} | n \rangle = 0 \quad (3.8)$$

follows directly from Eq. (3.2a).

IV. COORDINATE REPRESENTATION AND \hat{H}

We choose to study our operator system in the coordinate representation

$$\hat{q}|q\rangle = q|q\rangle, \langle q|q'\rangle = \delta(q-q'), \quad (4.1)$$

$$\langle q|n\rangle \equiv \psi_n(q), H_{qq'} = \langle q|\hat{H}|q'\rangle,$$

etc. In this representation, the ground-state energy (3.6) and the collective field (3.7) become

$$E_0 = N^2 \int dq \rho(q) (V + \frac{1}{2} V'q), \quad (4.2a)$$

$$\phi_0(x) = N^{1/2} \rho(xN^{-1/2}), \quad (4.2b)$$

where $\rho(q) = |\psi_0(q)|^2$. Notice that the normalization of the ground-state wave function

$$\int dq \rho(q) = \langle 0|0\rangle = 1 \quad (4.3)$$

guarantees $\int dx \phi_0(x) = N$.⁶

We proceed now to construct $H_{qq'}$. Equation (3.2c) may be solved immediately

$$p_{qq'} = i \frac{P}{q-q'} \psi_0(q) \psi_0^*(q') + f(q) \delta(q-q'). \quad (4.4)$$

We have chosen to use the principal-value prescription

$$\frac{P}{z} = \frac{1}{2} \left(\frac{1}{z+i\epsilon} + \frac{1}{z-i\epsilon} \right) \quad (4.5)$$

to define the singularity. Any other choice would simply lead to a redefinition of the (real) function $f(q)$. Equation (3.2a) then yields

$$H_{qq'} = \frac{-P}{(q-q')^2} \psi_0(q) \psi_0^*(q') + g(q) \delta(q-q') - \frac{i}{2} [f(q) + f(q')] \frac{\partial}{\partial q} \delta(q-q'), \quad (4.6)$$

where

$$\frac{P}{z^2} = \frac{1}{2} \left(\frac{1}{(z+i\epsilon)^2} + \frac{1}{(z-i\epsilon)^2} \right). \quad (4.7)$$

The Schrödinger equation for the ground state follows from (3.3) and (4.6),

$$\left(-\int \frac{\rho(q') dq'}{(q-q')^2} + g(q) + \frac{1}{i} f(q) \frac{\partial}{\partial q} + \frac{1}{2i} f'(q) \right) \psi_0(q) = 0, \quad (4.8)$$

where the slash on the integral sign means principal value. Multiplying this equation by $\psi_0^*(q)$ and taking the imaginary part leads to

$$\frac{d}{dq} [f(q) \rho(q)] = 0, \quad (4.9)$$

while the identity (3.8) gives the condition

$$\int dq f(q) \rho(q) = 0. \quad (4.10)$$

Together, (4.9) and (4.10) imply $f\rho = 0$ everywhere. We will assume that ψ_0 has no nodes, and hence $f = 0$ everywhere. The remaining content of the ground-state equation (4.8) is then the determination of the function g ,

$$g(q) = \int \frac{dq' \rho(q')}{(q-q')^2}. \quad (4.11)$$

We turn now to the commutator (3.2b). It is convenient to define the operator $\hat{\Pi}$, whose coordinate representation is

$$\Pi_{aa'} = \psi_0(q)\psi_0^{-1}(q') \frac{1}{i} \frac{\partial}{\partial q} \delta(q - q'). \quad (4.12)$$

It is easy to establish that

$$[\hat{q}, \hat{\Pi}] = i, \quad \hat{\Pi}|0\rangle = 0. \quad (4.13)$$

Furthermore, $\hat{\Pi}$ gives us a compact form for the Hamiltonian

$$\hat{H} = \hat{\Pi}\hat{p} + \hat{g}, \quad \hat{H}|0\rangle = 0, \quad (4.14)$$

where $\hat{g} = g(\hat{q})$, and g is given in Eq. (4.11). With the help of the identity

$$\frac{P}{z-a} \frac{P}{z-b} = \frac{1}{a-b} \left(\frac{P}{z-a} - \frac{P}{z-b} \right) + \pi^2 \delta(z-a)\delta(z-b), \quad (4.15)$$

it is straightforward to verify the operator identities

$$\begin{aligned} i[\hat{\Pi}, \hat{p}] &= -i[\hat{g}, \hat{p}] + \frac{\pi^2}{2} (\hat{\rho}^2)', \\ i[\hat{H}, \hat{p}] &= \frac{\pi^2}{2} (\hat{\rho}^2)', \end{aligned} \quad (4.16)$$

where $\hat{\rho} = \rho(\hat{q})$. Therefore, in the coordinate representation

$$\frac{\pi^2}{2} (\rho^2)' = -V'(q), \quad (4.17)$$

or

$$\rho(q) = \frac{1}{\pi} [2\{\epsilon - V(q)\}]^{1/2}. \quad (4.18)$$

The constant ϵ is determined from the normalization condition (4.3).

The system (3.2) and (3.3) has been successfully converted to a Schrödinger-equation form, and the ground state (ρ) has been determined in the bargain. Since there is no restriction on the phase of ψ_0 , we will set it to zero. It remains only to solve the excited-state Schrödinger equation, which follows from (3.3),

$$\oint dq' \rho(q') \frac{[\chi_n(q) - \chi_n(q')]}{(q - q')^2} = \omega_n \chi_n(q), \quad (4.19)$$

where the first term on the left can be written as $g(q)\chi_n(q)$. Here we have set $\psi_n = \psi_0\chi_n$. This equation was obtained in an entirely different manner by Marchesini and Onofri.⁵

We mention in summary the final forms

$$p_{aa'} = iP \frac{[\rho(q)\rho(q')]^{1/2}}{q - q'}, \quad (4.20a)$$

$$\begin{aligned} H_{aa'} &= -P \frac{[\rho(q)\rho(q')]^{1/2}}{(q - q')^2} \\ &+ \delta(q - q') \oint \frac{dq'' \rho(q'')}{(q - q'')^2}, \end{aligned} \quad (4.20b)$$

$$\begin{aligned} J_{aa'} &= \psi_0^{-1}(q) H_{aa'} \psi_0(q') \\ &= -P \frac{\rho(q')}{(q - q')^2} \\ &+ \delta(q - q') \oint \frac{dq'' \rho(q'')}{(q - q'')^2}. \end{aligned} \quad (4.20c)$$

The operator \hat{J} is the (doubly) reduced Hamiltonian

$$\hat{J} = \hat{D}\hat{p} + \hat{g}, \quad (D)_{aa'} = \frac{1}{i} \frac{\partial}{\partial q} \delta(q - q') \quad (4.21)$$

which governs the excited-state wave equation (4.19).

V. QUASICLASSICAL DYNAMICS AND CLASSICALLY FORBIDDEN REGIONS

Because $\rho \geq 0$, it is clear from (4.18) that the spectrum of the operator \hat{q} is limited to the range

$$\epsilon - V(q) \geq 0. \quad (5.1)$$

In the simple case, for example, of the one-matrix oscillator $V = \frac{1}{2}q^2$, the normalization condition (4.3) fixes $\epsilon = 1$ and hence $|q| \leq 2^{1/2}$. In the case of a double well potential $V = -\frac{1}{2}q^2 + \lambda q^4$, q is restricted to the wells, and cannot penetrate the barrier.

This classical feature, no penetration of classically forbidden regions, is a striking property of our modified quantum mechanics, governed by the new equal-time algebra (3.2c). The feature properly belongs in the growing list of classical aspects of the large- N limit.

We should mention that, in general, the normalization condition ($\oint \rho dq = \hbar$, with $\hbar \neq 1$) may be written as

$$\oint dq [2(\epsilon - V)]^{1/2} = 2\pi\hbar. \quad (5.2)$$

Thus the constant ϵ may be identified as the energy of the first Bohr-Sommerfeld-Wilson orbit ($n=1$). For ordinary potentials then, the spectrum of \hat{q} is only out to the first Bohr-Sommerfeld-Wilson radius.

It is interesting to notice, too, that our modified one-dimensional quantum mechanics is capable of phase transitions. Qualitatively, what stops phase transitions in ordinary quantum mechanics is tunneling. Here, we have a partial return to classical physics; there is no tunneling. One expects, and finds,⁹ a qualitative change in the system as the coupling is raised so that ϵ just tops a potential barrier, as in the double well.

VI. RESULTS FROM THE GROUND-STATE WAVE FUNCTION

Our first observation is that, from Eq. (4.2b) and (4.18), the collective field is

$$\phi_0(x) = \frac{1}{\pi} \{2N[\epsilon - V(xN^{-1/2})]\}^{1/2} \quad (6.1)$$

in agreement with Ref. 6.

Consider next the ground-state energy, Eq. (4.2a), in the case $V = \lambda^{-1}v(q\lambda^{1/2})$. It is not hard to show then that

$$\epsilon(\lambda) = \Omega(\lambda)\lambda^{-1}, \quad (6.2)$$

where $\Omega(\lambda)$ is defined as in Ref. 5,

$$\frac{1}{\pi} \int dx \{2[\Omega(\lambda) - v(x)]\}^{1/2} = \lambda. \quad (6.3)$$

With additional algebra,¹⁰ we show that

$$E_0 = N^2 \lambda^{-2} \int_0^\lambda d\lambda' \Omega(\lambda'), \quad (6.4a)$$

$$(p^2)_0 = N^2 \lambda^{-2} \int_0^\lambda d\lambda' \lambda' \omega(\lambda'), \quad (6.4b)$$

$$\omega \equiv \frac{d\Omega}{d\lambda}, \quad (6.4c)$$

where $(p^2)_0$ is the true ground-state expectation value of \hat{p}^2 . The ground-state energy, Eq. (6.4a), is in agreement with Ref. 5.

We turn now to a discussion of the excited-state Schrödinger equation (4.19).

VII. EXCITED-STATE SOLUTIONS

The large- n spectrum (ω_{n0}) of the excited-state wave equation (4.19) was given in Ref. 5. To construct the large- N classical solution, we will also need the wave functions. We therefore present our own treatment of the equation.

An exact solution of (4.19) is easily verified for the matrix oscillator $V = \frac{1}{2}q^2$, $\pi\rho = \{2(1 - q^2/2)\}^{1/2}$,

$$\chi_n = \frac{\sin(n+1)\theta}{\sin\theta}, \quad 2^{1/2} \cos\theta = q, \quad \omega_{n0} = n, \quad n=0, 1, \dots \quad (7.1)$$

The solutions are Chebyshev polynomials in q , and the spectrum is in agreement with Ref. 4. These states indeed correspond to the original matrix states $(\hat{a}^{\dagger n})_{rs} |0\rangle$ which saturate the adjoint channels of the ordered ground-state expectation values in the large- N limit.

In the case of more general V , we will limit ourselves to polynomial potentials for which the spectrum of \hat{q} is the single connected region R : $a \leq q \leq b$, and for which $\rho(q)$ vanishes as a square root as q approaches a turning point.

There is an important clue about analyticity (all complex q) in the oscillator solutions (7.1). Being polynomials in q , we may consider them as entire functions in the complex q plane. This is special for the oscillator, but a similar useful result can be established for the class of potentials specified above.

Consider a principal-value integral, like $g(q)$, of the form

$$F(q) = \int_a^b \frac{\rho(q') dq'}{q - q'}. \quad (7.2)$$

By our assumptions on ρ , considered as a function of complex q , it has branch points at a and b on the real axis, plus possible complex branch points. We choose to draw all cuts away from the region R . ρ is then certainly analytic near R , and the principal-value contours are taken just over and under R . We define an analytic continuation of F to complex q by continuously deforming the contours of the q' integral ahead of q . Analyticity for F is then established within the contour $C = \frac{1}{2}(C_+ + C_-)$ shown in Fig. 1(a). Because ρ has square root branch cuts from a and b , the horizontal portions Γ_1, Γ_2 of C_+ and C_- cancel, leaving the equivalent contour $C' = \frac{1}{2}(C'_+ + C'_-)$ shown in Fig. 1(b). Thus, our continuation of F is analytic about a and b . The same trick does not work for the complex branch points. Continuing in this fashion, we conclude that our analytic continuation of F has branch points only where

$$\tilde{\rho}(q) \equiv \rho(q)[(q-a)(b-q)]^{-1/2} \quad (7.3)$$

has branch points (i.e., only the "complex" branch points).

Consider now the excited-state integral equation (4.19). We have shown that $g(q)$ has branch points only where $\tilde{\rho}$ does. It is self-consistent then to

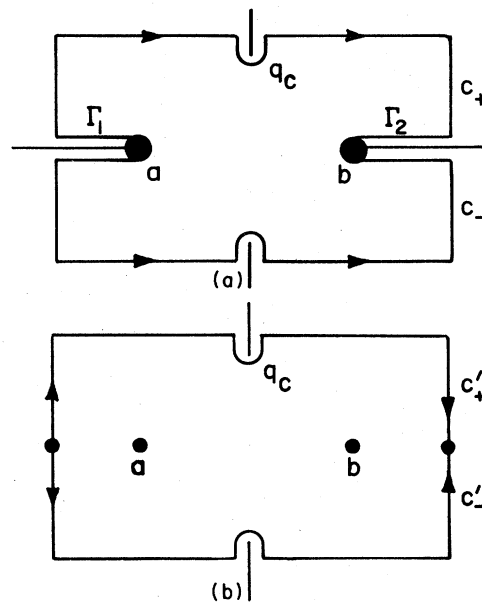


FIG. 1. (a) Region of analyticity of $F(q)$. (b) F is analytic about a and b .

search for solutions χ_n with those branch points. In the case of the oscillator, then, χ_n will be entire.

It is convenient to take the solution in the form

$$\phi_n(q) \equiv \rho(q)\chi_n(q) \equiv \sin\Phi_n(q), \quad (7.4)$$

where ϕ_n and hence Φ_n have branch points where ρ does. χ_n is analytic at a and b , so $\phi_n(a) = \phi_n(b) = 0$. We specify the phase ambiguity by requiring for Φ_n

$$\Phi_n(a) = 0, \quad \Phi_n(b) = \pi(n+1). \quad (7.5)$$

Consider next the principal-value integral in the integral equation. It may be expressed as

$$\oint_a^b \frac{dq' \phi_n(q')}{(q-q')^2} = -\pi\Phi_n' \sin\Phi_n + \Delta_n, \quad (7.6a)$$

$$\Delta_n = \frac{1}{2i} \int_{C_+} \frac{dq'}{(q-q')^2} e^{i\Phi_n(q')} - \frac{1}{2i} \int_{C_-} \frac{dq'}{(q-q')^2} e^{-i\Phi_n(q')}. \quad (7.6b)$$

This time we swept both contours up in the $e^{i\Phi}$ term, and down for the $e^{-i\Phi}$ term, picking up the appropriate residues when we cross the pole at q . The contours C_{\pm} are again those shown in Fig. 1 (a). The horizontal portions (Γ_1, Γ_2) of C_+ and C_- may again be shown to cancel. The reason is that ρ and hence Φ_n has square root branch points at a and b , so $e^{i\Phi}$ on top of the cut equals $e^{-i\Phi}$ on the bottom. The contours C_{\pm} in Eq. (7.6b) may then be replaced by C'_{\pm} in Fig. 1(b).

For large n , we expect (and will find) that $e^{i\Phi}$ decreases rapidly in the upper half plane, and also will allow the vertical segments of Fig. 1(b) to be pushed to infinity. Our leading approximation is then to neglect Δ_n . The integral equation becomes

$$[-\omega_{n0} + g]\rho^{-1} \simeq -\pi\Phi_n', \quad (7.7)$$

which is easily integrated to

$$\Phi_n(q) \simeq \frac{1}{\pi} \int_a^q \frac{dq'}{\rho(q')} [\omega_{n0} - g(q')]. \quad (7.8)$$

With Eq. (7.5), the linear spectrum is obtained,

$$\omega_{n0} = \omega \left[(n+1) + \frac{1}{\pi^2} \int_a^b \frac{dq g(q)}{\rho(q)} \right], \quad (7.9a)$$

$$\omega \equiv \pi^2 \left[\int_a^b \frac{dq}{\rho(q)} \right]^{-1}. \quad (7.9b)$$

The constant ω is the same as that defined in Eq. (6.4c), and the linear spectrum is in agreement with Ref. 5.

Having the leading terms in the wave functions (7.8) and energies (7.9), the errors may be computed by insertion into Δ_n . It is easily checked that $e^{i\Phi}$ does decrease rapidly in the upper half

plane, and that the vertical segments of C'_{\pm} can be pushed to infinity. The wave-function error is then of order

$$\xi \equiv \exp[-|\text{Im}\Phi(q_c)|], \quad (7.10)$$

where, as in the figures, q_c is the location of the closest complex branch point of ρ . The solutions (7.8) and (7.9) are therefore exact for the oscillator. For the purely quartic potential, the error works out to be $\exp(-\pi n/2)$. For the energies, the error is at least this small, and probably $O(\xi^2)$. Our error estimates then disagree with the $O(n^{-1})$ of Ref. 5, although we are in qualitative agreement with their numerical results. The source of the discrepancy is presumably the cancellation of the horizontal segments in the figures.

A number of small points deserve mention. The approximate results (7.8) and (7.9) should be supplemented by the exact solution $\phi_0 = \rho$ (ground state for any potential, with $\omega_{00} = 0$). Also, we have noticed that, in the case $V = \frac{1}{2}q^2 + \lambda q^4$, the large- n wave functions may be expressed in terms of incomplete elliptic integrals. Finally, it is apparent, in comparison with Ref. 5, that our states are only the "ground-state family" of adjacent states. Presumably, as in the oscillator, these states saturate ordered ground-state expectation values in the large- N limit.

VIII. THE LARGE- N CLASSICAL SOLUTION

Here we summarize our results for the large- N classical solution for the one-matrix model, Eq. (2.9). We want

$$\begin{aligned} M_{mn}(t) &= N^{1/2} \langle m | \hat{q}(t) | n \rangle \\ &= N^{1/2} e^{i\omega_{mn}t} \int \frac{dq}{\rho} \phi_m \phi_n q. \end{aligned} \quad (8.1)$$

To bring out the classical structure of the result, we define

$$\tau(q) \equiv \int_a^q \frac{dq'}{\pi\rho(q')}, \quad T \equiv 2\tau(b), \quad (8.2)$$

where a and b are the left and right turning points, respectively. $\tau(q)$ is then the classical transit time from the left turning point to q , while T is the classical period of the motion. Further,

$$\omega T = 2\pi, \quad \dot{q} = \pi\rho, \quad (8.3)$$

where the overdot denotes derivative with respect to τ . Then,

$$M_{mn}(t) = N^{1/2} e^{i\omega_{mn}t} \pi \int_0^{T/2} d\tau q \phi_m \phi_n. \quad (8.4)$$

We know from our previous work

$$\begin{aligned}\phi_0 &= \rho = \frac{1}{\pi} q, \\ \phi_n &= 2(\pi T)^{-1/2} \sin \frac{2\pi n\tau}{T}, \quad n \text{ large}\end{aligned}\quad (8.5)$$

where, as written, the functions ϕ_n are orthonormal.

Our results for the large- N classical solution are then (m, n large),

$$M_{00} = N^{1/2} \int dq q \rho = N^{1/2} \int_0^{T/2} \frac{d\tau}{\pi} \dot{q}^2 q, \quad (8.6a)$$

$$\begin{aligned}M_{n0}(t) &= 2 \left(\frac{N}{\pi T} \right)^{1/2} \exp \left(\frac{2\pi i n t}{T} \right) \\ &\times \int_0^{T/2} d\tau q \dot{q} \sin \left(\frac{2\pi n \tau}{T} \right), \quad (8.6b)\end{aligned}$$

$$\begin{aligned}M_{mn}(t) &= \frac{4}{T} N^{1/2} \exp \left\{ \frac{2\pi i t}{T} (m-n) \right\} \\ &\times \int_0^{T/2} d\tau q \sin \left(\frac{2\pi m \tau}{T} \right) \sin \left(\frac{2\pi n \tau}{T} \right).\end{aligned}\quad (8.6c)$$

Note added in proof: The following work has been brought to our attention: O. Haan, Z. Phys. C6, 345 (1980). In a Euclidean formulation, this author has independently obtained the quasiclassical commutation relation (1.2), and used it to solve the zero-dimensional model.

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⁷The "constraint" derives from the conserved $SU(N)$ generators of the theory, and the equal-time commutators (see Refs. 3 and 4).

⁸It is not hard to guess \hat{H} for the matrix oscillator $V = \frac{1}{2} q^2$. Define $\hat{q} = 2^{-1/2}(\hat{a}^\dagger + \hat{a})$, $\hat{p} = i2^{-1/2}(\hat{a}^\dagger - \hat{a})$, $[\hat{a}, \hat{a}^\dagger] = |0\rangle\langle 0|$, $\hat{a}|0\rangle = 0$. The Hamiltonian is then seen to be $\hat{H} = \sum_{n=1}^{\infty} \hat{a}^\dagger n \hat{a}^n$. The normalized states $|n\rangle = \hat{a}^\dagger n |0\rangle$ have $\omega_{n0} = n$.

⁹H. Neuberger, Berkeley report, 1980 (unpublished).

¹⁰Some useful intermediate identities are

$$\int dx x v'(x) \{2[\Omega(\lambda) - v(x)]\}^{-1/2} = \lambda \pi, \text{ and}$$

$$\int dx v(x) \{2[\Omega(\lambda) - v(x)]\}^{-1/2} = \pi(\Omega \omega^{-1} - \lambda 2^{-1}).$$