A classical perturbation theory*

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A compact formula is found for the perturbation expansion of a general one-dimensional Hamiltonian system in classical mechanics. The technique is also applied to the mathematical problem of functional inversion.

PERTURBATION THEORY FOR CLASSICAL MECHANICS

We consider a system with one degree of freedom, with the Hamiltonian $H=H(p,q)$ being independent of the time t. We assume that we have bounded, periodic motion at some value E of the energy. That is, the equation $E=H(p,q)$ describes a single closed curve (the orbit) in the $p-q$ plane (phase space). Now consider the integral

$$T=\int\int dp\,dq\,\delta(E-H(p,q)), \tag{1}$$

involving the Dirac delta function; the domain of the integral is to include the orbit. We shall first show that this integral T is equal to the period of motion at energy E.

Doing the integral over p, we get

$$T=\int dq\sum_m\left(\left|\frac{\partial H(p,q)}{\partial p}\right|_{p=p_m}\right)^{-1}, \tag{2}$$

where p_m are all points satisfying $E=H(p_m,q)$ for fixed q. But we have from Hamilton's equation of motion

$$\dot{q}=\frac{\partial H}{\partial p}, \tag{3}$$

and we then see that the expression T is just

$$\oint_E\frac{dq}{\dot{q}}=\oint_E dt=T(E), \tag{4}$$

where the integral goes once around the orbit.

We can thus express the time average of any function F of the dynamical variables taken over the orbit at energy E, as

$$\langle F;E\rangle=\frac{1}{T(E)}\int\int dp\,dq\,\delta(E-H)F(p,q). \tag{5}$$

Now, for the perturbation theory, suppose that we are given $H=H_0+\lambda H_1$ and we seek an expansion in powers of λ. The basic step is to regard E as an independent variable and then write the Taylor series expansion,

$$\delta(E-H_0-\lambda H_1)=\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\lambda H_1\frac{d}{dE}\right)^n\delta(E-H_0), \tag{6}$$

$$T(E)=\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!}\left(\frac{d}{dE}\right)^n\int\int dp\,dq(H_1(p,q))^n\delta(E-H_0)$$

$$=\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!}\left(\frac{d}{dE}\right)^n T_0(E)\langle H_1^n;E\rangle_0. \tag{7}$$

Here, the subscript "0" means that the averages are performed over the orbit of the zeroth-order Hamiltonian H_0. This formula is very compact; its evaluation involves only the operation of integration over the unperturbed orbits, followed by differentiation with respect to the energy. For comparison, one may look at the formulas obtained in "canonical perturbation theory" (see, for example, Saletan and Cromer[1]). That analysis is based upon the action-angle formalism (our result can be reexpressed in terms of action-angle variables but there is no particular advantage in doing so) and appears as an expansion for the energy E, thought of as a dependent variable; the expansion formula is there worked out only to second order in λ, and the form is quite messy in appearance.

We also get the formula

$$T(E)\langle F;E\rangle=\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!}\left(\frac{d}{dE}\right)^n T_0(E)\langle H_1^nF;E\rangle_0. \tag{8}$$

We will compute some examples based upon the harmonic oscillator,

$$H_0=\frac{p^2}{2m}+\frac{k}{2}q^2, \tag{9}$$

which has the solutions (at energy E) given by

$$q=\sqrt{2E/k}\sin(\omega_0t+\phi),\quad \omega_0=\sqrt{k/m},$$
$$p=\sqrt{2Em}\cos(\omega_0t+\phi), \tag{10}$$
$$T_0(E)=2\pi/\omega_0\ \text{(independent of } E).$$

1. Consider the perturbation Hamiltonian $H_1=|q|^\sigma$

$$\langle H_1^n;E\rangle=\frac{1}{2\pi}\int_0^{2\pi}d\theta\left(\frac{2E}{k}\right)^{n\sigma/2}|\sin\theta|^{n\sigma}. \tag{11}$$

This integral may be evaluated and the derivatives with respect to E are likewise easily evaluated; the result is

$$T(E)=\frac{2\pi}{\omega_0}\sum_{n=0}^{\infty}\frac{(n\sigma/2-\frac{1}{2})!}{(-\frac{1}{2})!}\frac{1}{n!(n\sigma/2-n)!}\times\left[\frac{-2\lambda}{k}\left(\frac{2E}{k}\right)^{\sigma/2-1}\right]^n. \tag{12}$$

For $\sigma=2$ we get the familiar result,

$$T(E)=\frac{2\pi}{\omega}\sum_{n=0}^{\infty}\frac{(n-\frac{1}{2})!}{(-\frac{1}{2})!\,n!}\left(\frac{-2\lambda}{k}\right)^n=\frac{2\pi}{\omega_0}\left(1+\frac{2\lambda}{k}\right)^{-1/2}$$
$$=2\pi\left(\frac{m}{k+2\lambda}\right)^{1/2}.$$

Having this series explicitly given, we can ask about its radius of convergence. The ratio of successive terms, for large n, is

$$R_n\sim\frac{-\lambda\sigma}{k}\left(\frac{2E}{k}\frac{\sigma}{\sigma-2}\right)^{(\sigma-2)/2} \tag{13}$$

and thus we will have convergence up to that value of the energy E for which this ratio is 1. We now ask for the significance of this critical value of the energy,

$$E^* = \left| \frac{k}{2}\left(\frac{\sigma-2}{\sigma}\right)\left(\frac{-\lambda\sigma}{k}\right)^{2/2-\sigma} \right|. \qquad (14)$$

It can be readily shown that at this energy the orbit reaches an amplitude at which the total potential energy has a zero slope and the motion thereafter is qualitatively different. Thus we conclude, at least for this example, that the perturbation series will converge for all energies for which the period is a finite and continuous function of the energy. (In this we must allow for changing the sign of λ.)

2. Consider the perturbation

$$H_1 = \frac{a}{3}q^3 + \frac{b}{4}q^4 + \frac{c}{5}q^5 + \frac{d}{6}q^6 + \cdots. \qquad (15)$$

We calculate

$$T(E)\frac{\omega_0}{2\pi} = 1 + \frac{E}{k^2}\left(-\frac{3}{4}b + \frac{5}{6}\frac{a^2}{k}\right) + \frac{E^2}{k^3}\left(-\frac{5}{4}d + \frac{105}{64}\frac{b^2}{k} + \frac{7}{2}\frac{ac}{k} - \frac{105}{16}\frac{a^2b}{k^2} + \frac{385}{144}\frac{a^4}{k^3}\right) + O(E^3), \qquad (16)$$

which gives the leading energy dependent corrections to the period of a general nonlinear oscillator.

APPENDIX

The expansion technique used above finds application to some problems removed from Hamiltonian mechanics. Consider a given function F whose inverse is sought:

$$y = F(x), \quad x = F^{-1}(y).$$

We assume that F is a monotonic function, so that this inverse is unique. Now suppose we have $F = F_0 + \lambda F_1$, where λ is again a small parameter. We would expect to find an expansion

$$F^{-1}(y) = F_0^{-1}(y) + \sum_{n=1}^{\infty} \lambda^n G_n(y),$$

where the terms G_n could be found by a lenghy process of Taylor expansions. What is somewhat surprising is that we can find a compact formula for the general term in this series.

Again, starting from the integral of a delta function, we have

$$\int dx\, \delta(y - F(x)) = \frac{1}{F'(x)}\bigg|_{x=F^{-1}(y)} = \frac{dx}{dy} = \frac{d}{dy}F^{-1}(y).$$

Substituting $F = F_0 + \lambda F_1$, we make the Taylor series expansion of this same integral to get

$$\sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!}\left(\frac{d}{dy}\right)^n \int dx\, \delta(y - F_0(x)) F_1^n(x).$$

Equating these two expressions, and then performing one integral in y, we get

$$F^{-1}(y) = F_0^{-1}(y) + \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!}\left(\frac{d}{dy}\right)^{n-1} \frac{(F_1(x))^n}{F_0'(x)}\bigg|_{x=F_0^{-1}(y)}.$$

Again, the trick in finding this compact formula was to regard y, and not x, as the independent variable.

For one simple example of application of this formula, consider

$$y = x^\alpha + \lambda x^\beta;$$

we find the inversion:

$$x = y^{1/\alpha} + \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!}\left(\frac{d}{dy}\right)^{n-1} \frac{x^{n\beta}}{\alpha x^{\alpha-1}}\bigg|_{x=y^{1/\alpha}}$$
$$= y^{1/\alpha}\sum_{n=0}^{\infty} (-\lambda y^{(\beta-\alpha)/\alpha})^n \frac{(n\beta/\alpha + 1/\alpha - 1)!}{n!\,\alpha\{n[(\beta-\alpha)/\alpha] + 1/\alpha\}!}.$$

This series is readily seen to be convergent up to that point at which dy/dx becomes zero for either sign of λ.

Now consider extending this technique to functions of several variables:

$$y_i = F_{0_i}(\mathbf{x}) + \lambda F_{1_i}(\mathbf{x}), \quad \mathbf{x} = (x_1, x_2, \ldots, x_N), \quad i = 1, N,$$

where we wish to solve for x_i in terms of the y_j. For simplicity we take the function F_0 to be the identity function:

$$y_i = x_i + \lambda\phi_i(\mathbf{x})$$

[later, one can set $x_i = F_{0_i}(\mathbf{z})$ to recover the more general case]. Now consider the following integral:

$$\iint d\xi_1 \cdots d\xi_N\, \delta(y_1 - \xi_1 - \lambda\phi_1(\boldsymbol{\xi})) \cdots \delta(y_N - \xi_N - \lambda\phi_N(\boldsymbol{\xi})) \cdot \det\left|\delta_{ij} + \lambda\frac{\partial\phi_i(\boldsymbol{\xi})}{\partial\xi_j}\right| \cdot G(\boldsymbol{\xi}).$$

by changing integration variables from the ξ_i's to

$$\eta_i = \xi_i + \lambda\phi_i(\boldsymbol{\xi}),$$

we see that this integral has just the value $G(\mathbf{x})$ where x_i are related to y_i by the equations given above. (The determinant is the Jacobian which is needed to make this transformation work out simply.)

Now we use the Taylor series expansion, as before, using the variables y_i to expand the arguments of the delta functions in power series in λ:

$$G(\mathbf{x}) = \iint d\xi_1 \cdots d\xi_N \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!}\left(\sum_{i=1}^{N}\frac{\partial}{\partial y_i}\phi_i(\boldsymbol{\xi})\right)^n \times \delta(y_1 - \xi_1)\cdots\delta(y_N - \xi_N)\det\left|\delta_{ij} + \lambda\frac{\partial\phi_i(\boldsymbol{\xi})}{\partial\xi_j}\right| \cdot G(\boldsymbol{\xi})$$
$$= \sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!}\sum_{i_1, i_2, \ldots, i_n=1}^{N}\frac{\partial}{\partial y_{i_1}}\cdots\frac{\partial}{\partial y_{i_n}} \times \phi_{i_1}(\mathbf{y})\cdots\phi_{i_n}(\mathbf{y})\det\left|\delta_{ij} + \lambda\frac{\partial\phi_i(\mathbf{y})}{\partial y_j}\right| G(\mathbf{y}).$$

This general formula is not exactly in the form of a power series in λ because the determinant is an Nth degree polynomial in λ; but it is the most compact form of the functional inversion problem for several variables. Using the equation that results from setting $G(\mathbf{x}) = 1$, we can rewrite the general formula as

$$G(\mathbf{x}) = G(\mathbf{y}) + \sum_{n=1}^{\infty}\frac{(-\lambda)^n}{n!}\sum_{i_1\cdots i_n=1}^{N} \times\left[\frac{\partial}{\partial y_{i_1}}\cdots\frac{\partial}{\partial y_{i_n}}, G(\mathbf{y})\right]\phi_{i_1}(\mathbf{y})\cdots\phi_{i_n}(\mathbf{y})$$

$$\times \det \left| \delta_{ij} + \lambda \frac{\partial \phi_i(\mathbf{y})}{\partial y_j} \right|,$$

involving the commutator of G with the derivative operators.

For the case $N=1$ we have

$$G(x) = G(y) + \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} \left[\left(\frac{\partial}{\partial y} \right)^n, G(y) \right] \phi^n(y)$$

$$\times \left[1 + \lambda \frac{\partial \phi(y)}{\partial y} \right], \quad y = x + \lambda \phi(x),$$

and this can be rearranged into a strict power series in λ, yielding

$$G(x) = G(y) + \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} \left(\frac{\partial}{\partial y} \right)^{n-1} \phi^n(y) \frac{\partial G(y)}{\partial y}.$$

This formula is equivalent to our earlier result on function inversion with one variable. This formula was first published by Lagrange in 1770 (see Whittaker and Watson[2]; the derivation given there does not use delta functions and has the added virtue that one can more readily see what the radius of convergence of the series will be). The formula we have given above for several variables is, as far as we know, new. For $N=2$ the series can be rearranged and, with some care, we obtain

$$y_1 = x_1 + \lambda \phi_1(x_1, x_2), \quad y_2 = x_2 + \lambda \phi_2(x_1, x_2),$$

$$G(x_1, x_2) = G(y_1, y_2) + \sum_{n=1}^{\infty} (-\lambda)^n \sum_{l_1 + l_2 = n} \frac{1}{l_1!} \frac{1}{l_2!} \left(\frac{\partial}{\partial y_1} \right)^{l_1 - 1} \left(\frac{\partial}{\partial y_2} \right)^{l_2 - 1}$$

$$\times \left[\frac{\partial^2 G(y_1, y_2)}{\partial y_1 \partial y_2} + \frac{\partial G(y_1, y_2)}{\partial y_1} l_1 \frac{\partial \ln \phi_1(y_1, y_2)}{\partial y_2} + \frac{\partial G(y_1, y_2)}{\partial y_2} l_2 \frac{\partial \ln \phi_2(y_1, y_2)}{\partial y_1} \right] \phi_1^{l_1}(y_1, y_2) \phi_2^{l_2}(y_1, y_2)$$

of which a special case is

$$x_1 = y_1 + \sum_{n=1}^{\infty} (-\lambda)^n \sum_{l} \frac{1}{l!\,(n-l)!} \left(\frac{\partial}{\partial y_1} \right)^{n-l-1} \left(\frac{\partial}{\partial y_2} \right)^{l-1} \phi_2^l \frac{\partial \phi_1^{n-l}}{\partial y_2}.$$

For another special case consider the linear forms

$$\phi_i(\mathbf{x}) = \sum_{j=1}^{N} A_{ij} x_j \quad \text{for any } N.$$

Then, taking $G=1$, we find the formula

$$\frac{1}{\det(1 + \lambda A)} = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \sum_{l_1, l_2, \ldots, l_n = 1}^{N} \sum_{k_1, k_2, \ldots, k_n}$$

$$\times A_{l_1 k_1}, A_{l_2 k_2}, \ldots, A_{l_n k_n},$$

where the set of labels $(k_1, k_2, \ldots, k_n)$ goes over each permutation of the set $(l_1, l_2, \ldots, l_n)$.

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[1]E. J. Saletin and A. H. Cromer, *Theoretical Mechanics* (Wiley, New York, 1971), p. 241.
[2]E. T. Whittaker and G. N. Watson, *A Course in Modern Analysis* (Cambridge U. P., Cambridge, 1958), p. 133.