THREE-PARTICLE SCATTERING IN QUANTUM MECHANICS

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GENERALIZED BETHE-SALPETER EQUATIONS FOR COUPLED TWO- AND THREE-BODY AMPLITUDES*

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1. INTRODUCTION

The Bethe-Salpeter equation is a relativistic generalization of the Schrödinger wave equation. For two particles we would write the BS equation as follows

$$\Psi(1, 2) = \Psi_0(1, 2) + \int\int G(12, 1'2')V(1',2')\Psi(1',2') \ , \qquad (1)$$

where "1" stands for the 4-vector x_1 representing the space-time coordinate for particle number 1,

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V is some interaction, and the two-body Green's function factors as

$$G(12,\ 1'2') = G_1(1,1')G_2(2,2') \ . \qquad (2)$$

For scattering problems Ψ_0 is a product of two plane wave functions while for bound states we set Ψ_0 equal to zero. We will restrict our attention to particles with zero spin, so that relativistic covariance will be obtained with the requirement of invariance on the Green's functions and on V. We will assume the Feynman form for the Green's functions

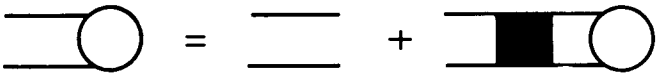
$$G_1(x_1,\ x_1') = -i\int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot(x_1 - x_1')}}{p^2 + m_1^2 - i\epsilon} \ ; \qquad (3)$$

and our metric is $A\cdot B = \underline{A}\cdot\underline{B} - A_0B_0$. (The reason for this choice is that we shall frequently work in the "Wick rotated" space where A_0 becomes iA_4 and we want a positive metric in this Euclidean 4-dimensional space.)

As a further shorthand we shall write equation (1) as

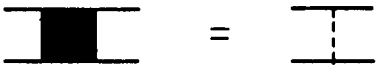
$$\Psi = \Psi_0 + G_1G_2V\Psi \qquad (4)$$

and represent it by the picture



(5)

Here the black box stands for the interaction V which we have now to talk about. The simplest approximation is the one-particle exchange (ladder approximation when we iterate the equation.[1,2])



(6)

For this the interaction is the local potential

$$V(x_1, x_2) = -\frac{i\lambda}{\pi^2}\int d^4q \frac{e^{iq\cdot(x_1 - x_2)}}{q^2 + m_3^2 - i\epsilon} \tag{7}$$

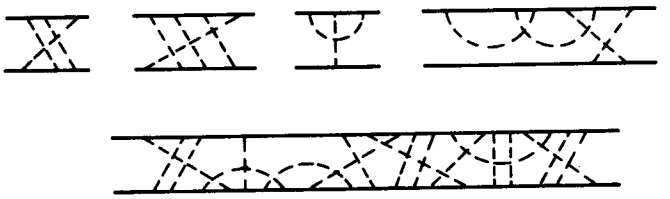
where m_3 is the mass of the spinless exchanged particle and λ is related to the coupling constants, $\lambda = g_1 g_2/16\pi^2$.

We are told that the Eq. (5) is exact (i.e. represents all of field theory) if the black box contains all irreducible diagrams and the Green's functions contain all the one-body renormalizations. We cannot hope to handle this ultimate theory; the main objective of our present work is to go beyond the ladder approximation as far as we can. The

first suggestion is to include a crossed graph, the simplest being

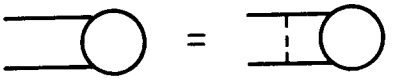
 (8)

We have found that in order to solve the equation with this non-local interaction we can go to coupled local equations involving a three-body wavefunction $\Phi(1, 2, 3)$ as well as the two-body function $\Psi(1, 2)$; and that once this step has been made we can easily add in a very large family of other crossed graphs, some of which are the following.

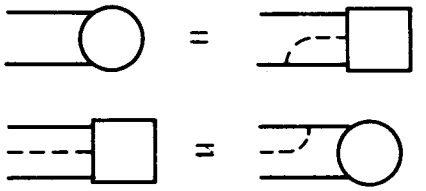
 (9)

2. THE COUPLED EQUATIONS IN GRAPHICAL FORM

Let us start with the two-body equation with single exchange potential and drop the plane wave term. Remember that  stands for $\Psi(1, 2)$, where 1 is the upper line and 2 the lower. The ladder equation is

 (10)

We will now cut this picture and introduce the symbol  for the three body function $\Phi(1, 2, 3)$. Eq. (10) is then written as



(11)

This is so far nothing more than a definition of Φ, and to make that definition more concrete we shall write the equations out:

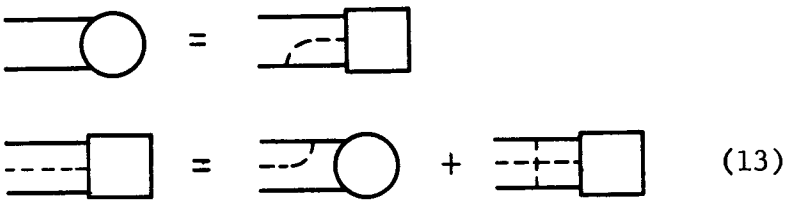
$$\Psi(1, 2) = g_2 \int G_2(2,2')\Phi(1, 2', 2') \qquad (12a)$$

$$\Phi(1, 2, 3) = g_1 \int G_3(3,1')G_1(1,1')\Psi(1', 2) \qquad (12b)$$

Here G_3 is the Green's function for the exchanged particle and the reader is encouraged to work out in detail that substituting (12b) into (12a) does yield exactly Eq. (1) with the potential (7). This exercise will give confidence for the later games played with the diagrams. The rule for going from the diagram to the equation is to put in a coupling constant g at a vertex with Green's functions for the legs which proceed to the left from the vertex, and integrate over the coordinate to the right of the

vertex.

Now we can add something new by putting in a single exchange potential in the second equation.

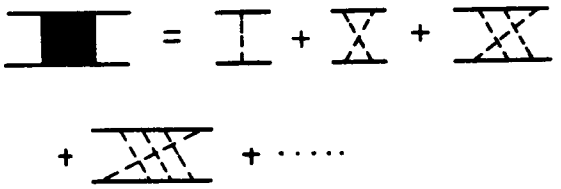


(13)

The added term is

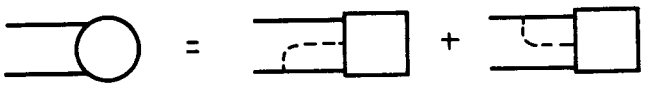
$$\int G_1(1, 1')G_2(2, 2')V(1', 2')\Phi(1', 2', 3) \quad (14)$$

where V is (7). We cannot write this as an equation in closed form for Ψ alone, but if we iterate the second equation for Φ we see that we get (5) with the black box replaced by the infinite series.

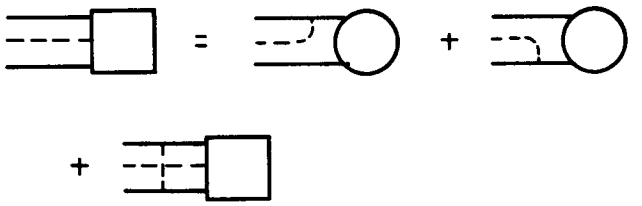


(15)

The obvious criticism of what we have so far is that it does not treat legs 1 and 2 symmetrically; this we fix by symmetrizing (13):



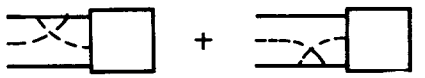
(16a)



(16b)

This step introduces another disease, the overcounting of diagrams. Both (6) and (8) appear twice when we write out the iterated series for the black box. At this point we also face the self-energy problem[3] with the graphs  . The first disease we must, and shall, fix by modifying the equations; the self energy we shall handle by standard renormalization procedures.

But before fixing the equations we shall include another pair of terms which make the theory even richer at very little extra cost; these are the graphs

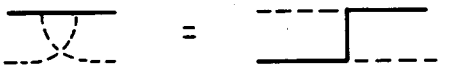


(17)

We shall use the word "exchange" for these interactions and describe the earlier picture  by the word "crossed." In the coordinate equation the first term of (17) is

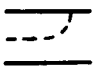
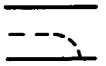
$$\int\int G_1(1, 1')G_3(3, 3')G_1(1', 3')\Phi(3', 2, 1') \quad (18)$$

and it is recognized that this is a two-body interaction via the exchange of the particle 1.

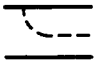
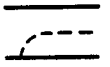


(19)

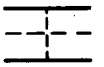
To return to the problem of graph counting we shall now pass from the graphical representation to a simple algebraic form. We will use E to mean 'emit' (a dashed line), A means 'absorb,' X means 'exchange,' and C means 'cross.' Thus the transition from the two-body function Ψ to the three-body function Φ is achieved by the E operators:

E_1  and E_2  (20)

and the reverse operations via the A's:

A_1  and A_2  (21)

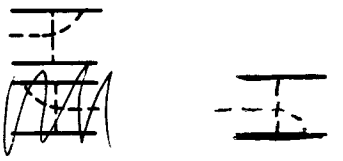
The 3-3 operations are

X_1  , X_2  , C  . (22)

The equations we wish to write down should contain as their formal iterated solution all possible products of these operators, which will give us the

large family of graphs we are interested in; but what double countings are possible? As far as the 3-3 operators X_1, X_2, C are concerned there is no trouble since these three are distinguishable and any sequential product of these operators represents a distinct graph. However if we look at 2-3 or 3-2 combinations we find the following identities:

$$X_1 E_2 = CE_1 \qquad X_2 E_1 = CE_2 \tag{23}$$



and their inverses,

$$A_1 X_2 = A_2 C \qquad A_2 X_1 = A_1 C \tag{24}$$

It appears that these are the only identities in the 2-3 and 3-2 operator products.

Now if there were no redundancies we could represent the sum of all products for the 2-3 operators by the equation,

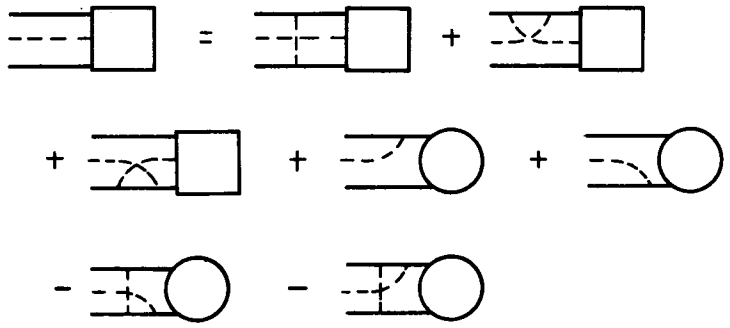
$$\Phi = (1 - C - X_1 - X_2)^{-1}(E_1 + E_2)\Psi \tag{25}$$

However (23) tells us that this will double count every term that has CE on the right, therefore we

write the corrected equation:

$$\Phi = [1 - C - X_1 - X_2]^{-1}(1 - C)(E_1 + E_2)\Psi \qquad (26)$$

Restored to graphical form this corrected equation is

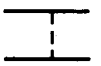


(27)

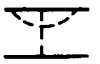
which differs from our earlier construction, (16b) plus (17), by the last two terms. Now to correct for over counting in Eq. (16a) we will write

$$\Psi = (A_1 + A_2)(1 - C)\Phi + \left[?\right]\Psi \ . \qquad (28)$$

The factor $(1 - C)$ takes care of the identities (24) in the 3-2 operators, and we must now find $\left[?\right]$ to correct for the identities in the 2-2 operator products. We find

$A_1E_2 = A_2E_1$ 

$A_2X_1E_2 = A_1X_2E_1$ 

$A_1X_1E_2 = A_2X_1E_1$ 

$$A_2X_2E_1 = A_1X_2E_2$$

$$A_1X_1X_2E_2 = A_2X_2X_1E_1 \qquad (29)$$

and believe that this is a complete list of identities, but have not proved this assertion. Substituting (26) into (28) we get

$$= (A_1 + A_2)(1 - C)(1 - C - X_1 - X_2)^{-1}$$

$$\times (1 - C)(E_1 + E_2) + [?] \qquad (30)$$

We now simply expand out the first few terms and see what has been miscounted, according to (29), and then correct by choosing:

$$[?] = -$$

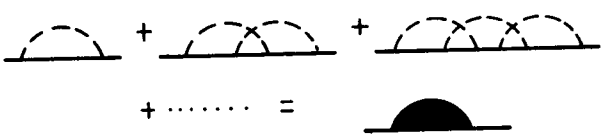
$$+$$

$$-$$

$$\qquad (31)$$

This counting of graphs may be verified by actually writing out the power series in g to higher order; This is how we first arrived at these results.

There is another type of correction which is still missing in our equations and should be added to $[?]$. This concerns the self-interaction diagrams, those which do not connect from leg 1 to leg 2. From our equations these are the following



(32)

and we shall use the symbols S_1 and S_2 to represent these. As it stands now we can write our black box as

$$= \mathcal{K} + S_1 G_2^{-1} S_2 G_1^{-1} \tag{33}$$

where $\mathcal{K}$ stands for all the diagrams which do connect legs 1 and 2. We want to fix this by adding the term $-S_1 S_2$ to [?] so that the equation finally looks like

$$\Psi = G_1 G_2 [\mathcal{K} + S_1 G_1^{-1} S_2 G_2^{-1} S_1 S_2] \Psi \tag{34}$$

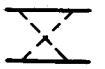
By algebraic manipulation this equation can be written as

$$\Psi = (G_1^{-1} - S_1)^{-1} (G_2^{-1} - S_2)^{-1} \mathcal{K} \Psi \tag{35}$$

which has the textbook form: the kernel is now irreducible and the propagators are dressed. This form will let us play the standard renormalization games on $(G^{-1} - S)$. We must return to this later.

A related problem, which we shall also postpone for now, concerns the effect of these manipulations with the one-body operators on the plane wave term we have been ignoring; this is tied up with the wave-function renormalization constants Z.

We shall now recess the formal analysis of these equations and turn to the practical work of rendering useful numerical answers. Whether we use numerical (mesh point formulas for differentiation or integration) methods or functional (variational) methods we will get finite matrix representations for the various operators E, A, X, C, S, and finite vector representations for the functions Ψ, Φ. Once these matrices have been constructed the solution of such formal equations as (26) is a straightforward job with a computing machine. The cost depends on the size of the matrices--multiplication or inversion of an $N \times N$ matrix takes about N^3 times a few microseconds--and so our objective is to get an accurate representation with a minimum basis.

It is interesting to note that we started out looking at just one crossed graph,  , and thought we got rid of its non-local character by

going to a local problem in more variables; but now we see this thing recurring in the correction term (31). What we now realize from the discussion of the previous paragraph is that we need not fear this since we will have the matrix representation

 $= A_2 X_1 E_2$

This is a very interesting indication that we can learn how to evaluate the complicated multi-dimensional integrals of many high order Feynman integrals by appropriate reduction to matrix algebraic calculations within a machine. This is of course not an exact result (as we cannot compute π exactly) but we hope to see that we can evaluate many of these things to some practical accuracy, say one percent or better.

3. NUMERICAL ANALYSIS

First let us count the variables: the two-body function Ψ has 8 variables and the three-body function Φ has 12, but these are considerably reduced by symmetry. We start with a 'center of mass' transformation to take care of the 4-dimensional

translational invariance

$$X = \mu_1 x_1 + \mu_2 x_2 \qquad \mu_1 + \mu_2 = 1$$

$$x = x_2 - x_1$$

$$y = x_3 - x_1 \tag{36}$$

$$\Psi(x_1, x_2) = e^{iP\cdot X}\psi(x)$$

$$\Phi(x_1, x_2, x_3) = e^{iP\cdot X}\phi(x, y) \tag{37}$$

P is the total energy-momentum of the system which will now appear as a parameter in the equations for ψ and ϕ. From Lorentz invariance we can say that for a given angular momentum of the state we will have the two functions depending only on scalar quantities. Thus ψ will have two arguments, e.g. x^2, $x\cdot P$, and ϕ will have five; x^2, $x\cdot y$, y^2, $x\cdot P$, $y\cdot P$. We usually work in the rest frame where $P = (E, 0, 0, 0)$, and E is the total energy or rest mass. If we make the further special choice $E = 0$, then we have ψ with only one variable and ϕ with only three; and so we shall work at first with this special case. It turns out that the most important problem concerns the behavior of the wavefunctions at small distances, and this does not depend on E.

Once we can solve the problem at $E = 0$ we expect it will be relatively straightforward to go on to $E > 0$.

We shall also write the equations in the Euclidean 4-space without bothering about the justification of the Wick rotation, and we start with the simplest equations (12). After removing the center-of-mass coordinate these can be written as the differential equations (before we set $P = 0$)

$$K_2(\mu_2 P + p)\psi(x) = g_2\phi(x, x)$$
$$K_1(\mu_1 P - p - q)K_3(q)\phi(x, y) = g_1\delta(y)\psi(x) \qquad (38)$$

where K is the Klein-Gordon operator

$$K_i(k) = k^2 + m_i^2 \qquad (39)$$

and p and q are the differential operators

$$p = -i\frac{\partial}{\partial x} \qquad q = -i\frac{\partial}{\partial y} . \qquad (40)$$

We will also want to look at these equations in momentum space.

$$\psi(x) = \int d^4p\, e^{ip\cdot x}\psi(p)$$

$$\phi(x, y) = \int d^4p \int d^4q \; e^{ip\cdot x} e^{iq\cdot y} \phi(p, q) \tag{41}$$

$$\begin{aligned} K_2(\mu_2 P + p)\psi(p) &= g_2 \int d^4q \phi(p - q, q) \\ K_1(\mu_1 P - p - q)K_3(q)\phi(p, q) &= \frac{g_1}{(2\pi)^4}\psi(p) \end{aligned} \tag{42}$$

It is of course clear that we could now solve for ϕ and get just the old ladder approximation equation for ψ

$$K_1(\mu_1 P - p)K_2(\mu_2 P + p)\psi(p)$$

$$= \frac{g_1 g_2}{(2\pi)^4} \int d^4p' \frac{1}{K_3(p')}\psi(p - p') \tag{43}$$

but this would be defeating our aim of learning to deal with the three-body function so that we can put in the other new interactions. So we will attack the equations (38) or (42) as they stand.

Our first numerical attempt used a variational method in the simplest way, with the three-variable function ϕ represented by a sum of product functions in the separate variables. This made it easy to evaluate the integrals needed for the matrix elements, but the results were poor. (Poor results

mean that as we increased systematically the number of basis functions used the calculated result converged too slowly.) In setting up this calculation we were easily able to learn from the differential equation what should be the leading behavior of the functions as $|x| \to 0$ and as $|y| \to 0$. We finally decided that what we had neglected was the rather tricky behavior as $|x - y| \to 0$, and this required behavior was not easily represented by the expansion in products of functions of x times functions of y.

This point $x = y$ is certainly important (see Eq. (38)), and in the Fourier transform variables concerns the region $(p + q)$ large. Our second hope was to try mesh point solution of the integral equations in momentum space, since it appeared that such factors as $K_1(p + q)$ in the second of Eqs. (42) would take care of this correlation problem for us. After a long sojourn into the art and science of numerical quadrature we abandoned this hope. We will forego a discussion of why we failed here since if a more clever person, unencumbered by our doubts, can find a way to make this approach succeed, it would be very nice.

Now we return to try variational methods with correlated basis functions. The task seems very hard since we will have to evaluate integrals like

$$\int d^4p \int d^4q \; g_1(p) g_2(q) g_3(p-q) \; . \tag{44}$$

But this becomes much easier if we have the Fourier transforms

$$f(x) = \int d^4p \; e^{ip\cdot x} g(p) \; ,$$

for then (44) is just the single integral

$$(2\pi)^{-4} \int d^4x \; f_1(-x) f_2(x) f_3(x) \; . \tag{45}$$

A nice set of functions is the following:

$$G_{kn\ell m}(p) = \frac{\alpha^{n+2k+2} \, p^n}{(\hat{p}^2 + \alpha^2)^{n+k+2}} \mathcal{Y}_{n\ell m}(\hat{p}) \tag{46}$$

where $\mathcal{Y}_{n\ell m}$ is the 4-dimensional spherical harmonic; on the left side of (46) p means the 4-vector, while on the right side $\hat{p}$ stands for the three angular coordinates and p stands for the magnitude of the 4-vector. The Fourier transform of (46) is

$$F_{kn\ell m}(x) = \frac{2\pi^2 i^n \alpha^2 (\alpha R)^{n+k} K_k(\alpha R)}{2^{n+k}(n+k+1)!} \mathcal{Y}_{n\ell m}(\hat{x})$$

$$= 2\pi^2 i^n \alpha^2 F_{kn}(\alpha R)\, \mathcal{Y}_{n\ell m}(\hat{x}) \qquad (47)$$

where now we use R for the magnitude of the 4-vector x, and K is the modified Bessel function.[4]

For the trick from (44) to (45) to work we cannot have more products of functions g than shown in (44), yet the variational principle brings the trial function in twice along with the operators. This will be resolved by the fact that the product of two functions of the type (46) can be easily expanded in a finite series of functions of the same type.

$$\mathcal{Y}_{n\ell m}(\hat{x})\, \mathcal{Y}_{n'\ell'm'}(\hat{x})$$

$$= \sum_{NLM} A^{NLM}_{n\ell mn'\ell'm'}\, \mathcal{Y}_{NLM}(\hat{x}) \qquad (48)$$

$$(q^2+1)\frac{q^n}{(q^2+1)^{n+j+2}}\,\frac{q^{n'}}{(q^2+1)^{n'+j'+2}}$$

$$= \sum_J D^{J,N}_{n+n',j+j'}\,\frac{q^N}{(q^2+1)^{N+J+2}} \qquad (49)$$

where

$$D^{J,N}_{n+n',j+j'} = (-1)^r \frac{\lambda!}{r!(\lambda - r)!} \tag{50}$$

with

$$\lambda = \frac{n + n' - N}{2} \qquad r = J - \lambda - j - j' - 1 \ .$$

We will need the addition formula

$$C_n^{(1)}(\hat{x}\cdot\hat{y}) = \frac{2\pi^2}{n+1} \sum_{\ell m} \mathcal{Y}^*_{n\ell m}(\hat{x})\, \mathcal{Y}_{n\ell m}(\hat{y}) \tag{51}$$

which involves the Gegenbauer polynomial[4] $C_n^{(1)} = \sqrt{2\pi^2}\,\mathcal{Y}_{n00}$. Some other useful formulas are

$$\mathcal{Y}_{n\ell m}(-\hat{x}) = (-1)^n \mathcal{Y}_{n\ell m}(\hat{x})$$

$$\mathcal{Y}^*_{n\ell m} = \mathcal{Y}_{n\ell -m}(-1)^m \tag{52}$$

$$\int d\Omega\, \mathcal{Y}^*_{n\ell m} \mathcal{Y}_{n'\ell'm'} = \delta_{nn'}\delta_{\ell\ell'}\delta_{mm'} \tag{53}$$

$$\int d^4x = \int d\Omega \int_0^\infty R^3 dR, \qquad \int d\Omega = 2\pi^2,$$

$$\Sigma_{\ell m} = (n+1)^2 \tag{54}$$

and

$$C_n^{(1)}(t)C_{n'}^{(1)}(t) = \sum_N B_{nn'}^N C_N^{(1)}(t) \tag{55}$$

where

$$B_{nn'}^N = \frac{2\pi^2}{(n+1)(n'+1)(N+1)} \times \sum_{LM\ell m\ell' m'} |A_{n\ell mn'\ell' m'}^{NLM}|^2 \tag{56}$$

These coefficients $B_{nn'}^N$ are related to some Clebsch-Gordan coefficients for the 4-dimensional rotation group, and they turn out to be given by the very simple formula

$$B_{nn'}^N = \begin{cases} 1 & \text{if } N = n+n',\ n+n'-2, \\ & \qquad n+n'-4, \ldots, |n-n'| \\ 0 & \text{otherwise} \end{cases} \tag{57}$$

Now we return to Eqs. (42) to construct a variational principle. Since the Eqs. (12) which we are now working from do not treat the coordinates x_1 and x_2 in a symmetrical manner the equations we have are not self-adjoint and the variational principle must allow for this. We start by writing down the expression

$$\mathcal{J} = \int dp\ \tilde{\psi}(p)K_1(\mu_1 P - p)K_2(\mu_2 P + p)\psi(p)$$
$$-g_2\int dp\int dq\ \tilde{\psi}(p)K_1(\mu_1 P - p)\phi(p - q, q)$$
$$+(2\pi)^4\int dp\int dq\ \tilde{\phi}(p, q)K_1(\mu_1 P - p - q)$$
$$\times K_2(\mu_2 P + p)K_3(q)\phi(p, q)$$
$$-g_1\int dp\int dq\ \tilde{\phi}(p, q)K_2(\mu_2 P + p)\psi(p) \qquad (58)$$

which directly reproduces Eqs. (42) when we carry out variations of the functions $\tilde{\psi}(p)$ and $\tilde{\phi}(p, q)$. If we now vary the functions $\psi(p)$ and $\phi(p, q)$ we get

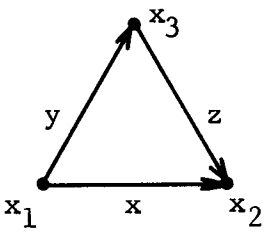
$$K_1(\mu_1 P - p)\tilde{\psi}(p) = g_1\int dq\ \tilde{\phi}(p, q)\ ,$$
$$K_2(\mu_2 P + p)K_3(q)\tilde{\phi}(p, q)$$
$$= \frac{g_2}{(2\pi)^4}\ \tilde{\psi}(p + q) \qquad (59)$$

which equations are not the same as (42). In order to find the correspondence we make the following definitions:

$$\tilde{\psi}(p) = \tilde{\psi}^{\dagger}(p)$$

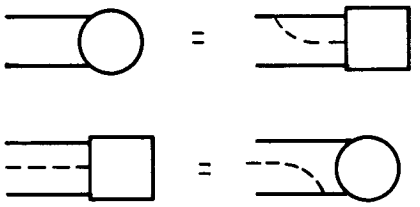
$$\tilde{\phi}(p, q) = \frac{g_2}{g_1} \frac{K_1(\mu_1 P - p)}{K_2(\mu_2 P + p)} \phi^{\dagger}(p + q, -q) \qquad (60)$$

Putting (60) into (59) we recover exactly the form of the original Eqs. (42). The change of variables in (60) reflects just the interchange of coordinates 1 and 2 since if $\phi(x, y)$ is the Fourier transform of $\phi(p, q)$ the Fourier transform of $\phi(p + q, -q)$ is $\phi(x, z)$ where $z = x - y$.



(61)

Thus the "adjoint" Eqs. (59) are represented by the diagrammatic equations



(62)

which should be compared to (11) which represent (42).

Now we make a linear expansion of the variational functions:

$$\psi(p) = \sum_i c_i \psi_i(p)$$
$$\tilde{\psi}(p) = \sum_i \tilde{c}_i \psi_i(p) \tag{63}$$

$$\phi(p, q) = \sum_i d_i \chi_i(p, q) K_1^{-1}(\mu_1 P - p - q)$$
$$\tilde{\phi}(p, q) = \sum_i \tilde{d}_i \chi_i(p + q, -q) K_2^{-1}(\mu_2 P + p) , \tag{64}$$

where the relations (60) have been built in. The reasoning behind this construction--the way in which the K factor appears in (64)--should be apparent if one looks at the second of equations (42):

$$\phi(p, q) = \frac{g_1}{(2\pi)^4} K_1^{-1}(\mu_1 P - p - q)$$
$$\times [K_3^{-1}(q)][\psi(p)] . \tag{65}$$

The $(p + q)$ dependence in K_1 here is precisely the correlation which caused difficulty in the first try discussed earlier; furthermore we are led to use a separable form for the remaining function which is called χ. Thus for $P = 0$ and for s-waves we take

$$\psi_i(p) = G_{k000}(p) \tag{66}$$

$$\chi_i(p, q) = \frac{(-1)^n}{(32\pi^4)^{1/2}} \frac{1}{(n+1)} \sum_{\ell m} G_{kn\ell m}(p) \times G_{jn\ell -m}(q)(-1)^m \tag{67}$$

where the index i stands for the single label (k) in the case of ψ, and stands for the triple label (k, j, n) in the case of χ. In what follows we take all the masses, m_1, m_2, m_3, as well as the scale parameter α equal to unity.

Variation of the expansion constants $\tilde{c}_i$, $\tilde{d}_i$ in $\mathcal{J}$ of (58) leaves us with the matrix equation

$$\begin{vmatrix} \mathcal{J}^{22} & \mathcal{J}^{23} \\ \mathcal{J}^{32} & \mathcal{J}^{33} \end{vmatrix} \begin{vmatrix} C \\ D \end{vmatrix} = 0 \tag{68}$$

where C and D are the vectors of coefficients c_i and d_i; and the matrix elements are now worked out.

$$(\mathcal{J}^{22})_{i'i} = \int d^4p \; \psi_{i'}(p) K^2(p) \psi_i(p)$$
$$= \int_0^\infty p^3 dp \; \frac{1}{(p^2+1)^{k+k'+2}}$$
$$= \frac{1}{2(k+k')(k+k'+1)} \tag{69}$$

$$(\mathcal{J}^{23})_{i'i} = -g_2\int d^4p\int d^4q\ \psi_{i'}(p)\chi_i(p-q,\ q)$$

$$= -g_2\int d^4p\int d^4q\ G_{k'000}(p)\sum_{\ell m} G_{kn\ell m}(p-q)$$

$$G_{jn\ell -m}(q)\frac{(-1)^{n+m}}{(n+1)(32\pi^4)^{1/2}}$$

$$= -g_2(2\pi)^{-4}\int d^4x\ F_{k'000}(-x)\sum_{\ell m} F_{kn\ell m}(x)$$

$$F_{jn\ell -m}(x)\frac{(-1)^{n+m}}{(n+1)(32\pi^4)^{1/2}}$$

$$= \frac{-g_2(n+1)}{16\pi}\int_0^\infty R^3dR\ F_{k'0}(R)F_{kn}(R)F_{jn}(R) \tag{70}$$

$$(\mathcal{J}^{32})_{i'i} = \frac{-g_1(n'+1)}{16\pi}\int_0^\infty R^3dR$$

$$F_{k'n'}(R)F_{j'n'}(R)F_{k0}(R) \tag{71}$$

$$(\mathcal{J}^{33})_{i'i} = (2\pi)^4\int d^4p\int d^4q\ \chi_{i'}(p+q,\ -q)K(q)\chi_i(p,\ q)$$

$$= \int d^4p\int d^4q\ \frac{(2\pi)^4(-1)^{n+n'}}{(n+1)(n'+1)32\pi^4}\sum_{\ell m}\sum_{\ell' m'}$$

$$G_{k'n'\ell' m'}(p+q)G_{j'n'\ell' -m'}(q)(-1)^{n'}$$

$$\times\ K(q)G_{kn\ell m}(p)G_{jn\ell -m}(q)$$

$$= \int d^4p \int d^4q \, \frac{(-1)^n}{2(n+1)(n'+1)} \sum_{\ell m \ell' m'}$$

$$G_{k'n'\ell'm'}(p+q) G_{kn\ell m}(p) \sum_{NLMJ}$$

$$A^{*NLM}_{n\ell mn'\ell'm'} D^{J,N}_{n+n',j+j'} G_{JNL-M}(q)(-1)^M$$

$$= \frac{(2\pi)^{-4}}{2(n+1)(n'+1)} \sum_{\substack{\ell m \ell' m' \\ NLMJ}} A^* D (-1)^N \int d^4x$$

$$\times F_{kn\ell m}(x) F_{k'n'\ell'm'}(x) F_{JNL-M}(x)(-1)^M$$

$$= \frac{(2\pi)^{-4}(-1)^N}{2(n+1)(n'+1)} \sum_{\substack{\ell m \ell' m' \\ NLMJ}} |A^{NLM}_{n\ell mn'\ell'm'}|^2$$

$$\times D^{JN}_{n+n',j+j'} (2\pi^2)^3 i^{n+n'+N} \int_0^\infty R^3 dR$$

$$\times F_{kn}(R) F_{k'n'}(R) F_{JN}(R)$$

$$= \sum_{NJ} B^N_{nn'} D^{JN}_{n+n',j+j'} \frac{(N+1)}{8} (-1)^{\frac{N-n-n'}{2}}$$

$$\times \int_0^\infty R^3 dR \; F_{kn}(R) F_{k'n'}(R) F_{JN}(R) \qquad (72)$$

The coefficients B and D have already been given, (50), (57), and we need only the family of one dimensional integrals

$$\frac{1}{8}\int_0^\infty R^3 dR\ F_{kn}(R)F_{k'n'}(R)F_{JN}(R)$$

$$= [(n+k+1)!(n'+k'+1)!(N+J+1)!]^{-1}$$

$$\times \int_0^\infty dR\left(\frac{R}{2}\right)^{n+n'+N+k+k'+J+3}$$

$$\times K_k(R)K_{k'}(R)K_J(R) \qquad (73)$$

If we define

$$f_k(R) = \left(\frac{R}{2}\right)^k K_k(R) \qquad (74)$$

we can use the recursion formula

$$f_k = (k-1)f_{k-1} + \left(\frac{R}{2}\right)^2 f_{k-2} \qquad (75)$$

to generate all of (73) from the small number of integrals

$$\int_0^\infty dR\left(\frac{R}{2}\right)^s f_\sigma f_\mu f_\nu \qquad (76)$$

where σ, μ, ν take on the values 0 and 1. Efficient computer programs have been constructed to evaluate the functions K_0 and K_1, the integrals (76) are done numerically, and the larger table of integrals is built up all in a couple of seconds.

The range of the indices used in the trial functions, (66), (67) must be specified. From (43) we see that

$$\psi(p) \sim p^{-6} \quad \text{as} \quad p \to \infty \tag{77}$$

and so we choose $k \geq 1$ in (66). Then looking at the large p and q behavior of the second Eq. (42) we choose

$$n \geq 0, \quad j \geq -1, \quad k \geq 1 \tag{78}$$

in (67). The sequences of terms were ordered by the inequalities

$$\begin{aligned} 1 &\leq k \leq L \\ 0 &\leq n \leq L - 1 \\ -1 &\leq j \leq L - 2 \\ n + k &\leq L \\ n + j &\leq L - 2 \end{aligned} \tag{79}$$

and thus as the limit L was stepped up, 1, 2, 3, 4, 5, 6, the dimension of the basis for ϕ increased as 1, 5, 14, 30, 55, 91. The further algebra of computing the matrix elements and solving the algebraic equations (68) up to this size for the

eigenvalue $\lambda = g_1 g_2/16\pi^2$ could be done in a fraction of a minute. The numerical results for this simple exercise are not interesting since this is just the old ladder approximation. (We see from (65) that merely the terms $n = 0$, $j = -1$ give the exact solution for ϕ in terms of ψ.) We have solved this problem before and so we just get a check on the new method and all the details of this calculation. Actually the results here converge faster than those of ref. 1 since the basis of functions used here gives a better representation of the analytic properties of the wave function at small distances.

We now advance one step and attack Eqs. (13), which means adding the term

$$K_2^{-1}(\mu_2 P + p)K_3(q)V(x)\phi(x,y) \tag{80}$$

to the right side of the second equation (38). This gives the additional term in the variational expression (58) as

$$-g_1 g_2 \int dp \int dq \int dp' \tilde{\phi}(p, q) \frac{K_3(q)}{K_3(p - p')} \phi(p', q) \tag{81}$$

and with our choice of trial functions (64) this

multi-dimensional integral cannot be reduced, as (44), to a simpler form. We shall make another approximation now,[5] but one which we will be able to systematically improve on; what we shall lose by this approximation should not be worse than what we already give up by limiting our trial function expansions (63), (64) to a finite number of terms.

In a shorthand notation we write the equations as

$$K_1 K_2 \psi = K_1 \phi$$
$$K_1 K_2 K_3 \phi = \delta_{13} K_2 \psi + V_{12} K_3 \phi \tag{82}$$

which we then rewrite in the form

$$K_1 K_2 \psi = K_1 \phi$$
$$K_3 K_1 K_2 \phi = \delta_{13} K_2 \psi + K_2 \Omega$$
$$K_1 K_3^{-1} V_{12}^{-1} K_2 \Omega = K_1 \phi \tag{83}$$

involving a new three-body function Ω. For this function Ω we shall expand in a separable basis as we did for the function χ and the resulting solution of (83) can be described as follows. We use the symbol $\langle X \rangle$ to mean the finite matrix representation in a separable basis of the operator X.

What is needed in (81) is

$$\langle K_2^{-1} K_3 V_{12} K_1^{-1} \rangle \tag{84}$$

and what we shall calculate with (83) is

$$\langle K_1 K_3^{-1} V_{12}^{-1} K_2 \rangle^{-1} \quad . \tag{85}$$

If our basis were complete, these two expressions would be identical; with the finite basis we hope that the loss of accuracy here will not be worse than the loss of accuracy in the whole calculation due to the finite size of the basis for ϕ.

For the function Ω we take the expansion

$$\Omega(p, q) = \Sigma\, e_i \chi_i(p, q) \tag{86}$$

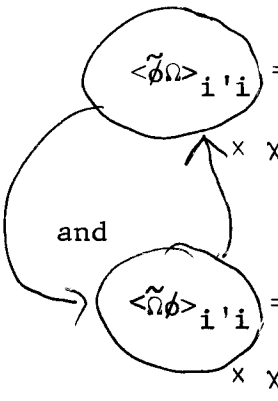
and for its adjoint,

$$\tilde{\Omega}(p, q) = \Sigma\, \tilde{e}_i \chi_i(p + q, -q) \quad .$$

To the matrix $\mathcal{G}^{33}$ we add the new term

$$\langle \tilde{\phi}\Omega \rangle \langle \tilde{\Omega}\Omega \rangle^{-1} \langle \tilde{\Omega}\phi \rangle \tag{87}$$

where



$$\langle\tilde{\phi}\Omega\rangle_{i'i} = (2\pi)^4\int d^4p\int d^4q\ \chi_{i'}(p+q,\ -q)\times\chi_i(p,\ q) \tag{88}$$

and

$$\langle\tilde{\Omega}\phi\rangle_{i'i} = (2\pi)^4\int d^4p\int d^4q\ \chi_{i'}(p+q,\ -q)\times\chi_i(p,\ q) \tag{89}$$

If we compare (88) and (89) with (72), and notice that

$$K(q)G_{kn\ell m}(q) = G_{k-1n\ell m}(q) \tag{90}$$

then we see that (88) is given by the final formula of (72) with the index $j+j'$ on the D coefficient increased by one. Eq. (89) is of the same form with the primed and unprimed indices interchanged. Finally we work out

$$\langle\tilde{\Omega}\Omega\rangle_{i'i} = (2\pi)^4\int d^4p\int d^4q\ \chi_{i'}(p+q,\ -q)K(p+q)$$
$$K^{-1}(q)\int d^4p'\int\frac{d^4x}{(2\pi)^4}\ V^{-1}(x)e^{i(p-p')\cdot x}$$
$$\times K(p')\chi_i(p',\ q)\ , \tag{91}$$

and following the plan of (72) we find the result

is given by the final formula of (72) with the following modifications:

> reduce the indices k and k' each by one, increase the index $j + j'$ by two, and put the extra factor $V^{-1}(R)$ under the integral. (92)

Here is the nice result that all the matrix elements are of the very same type as we have already dealt with, only the labels are shifted. The one-dimensional integrals with the $V^{-1}(R)$ factor are trivial added work

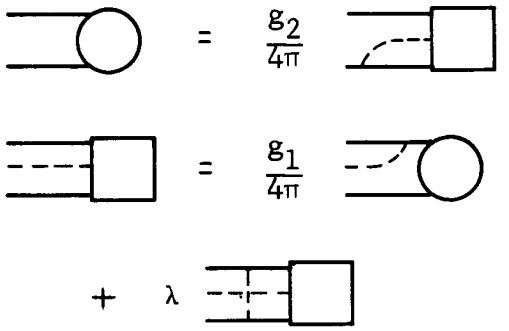
$$V(R) = \lambda \frac{4}{R} K_1(R) \quad . \tag{93}$$

We must also specify the range of the indices for the function Ω. From a study of the last Eq. (83) we conclude that we want

$$k \geq 0 \qquad n \geq 0 \qquad j \geq -1 \quad , \tag{94}$$

or in other words, the k index is lowered one from what we used for ϕ.

Now we shall report some numerical results of the calculation we have just set up. To repeat, the graphical equations are



(95)

where we have put in the coupling constants which are the eigenvalue parameter we are solving for (at $E = 0$). The correct relation is $\lambda = g_1 g_2/16\pi^2$, but in these calculations we fixed λ and then solved for the eigenvalue $\bar{\lambda} = g_1 g_2/16\pi^2$. Some numerical results are given in Table 1 below:

Table 1

L	$\frac{\lambda}{\bar{\lambda}} = 0$	$\frac{\lambda}{\bar{\lambda}} = 1.0$	$\frac{\lambda}{\bar{\lambda}} = 2.0$
1	3.693737	2.9766	2.2594
2	3.445500	2.7001	1.9560
3	3.421062	2.7114	2.0002
4	3.418688	2.7042	1.9792
5	3.418501	2.7084	1.9855
6	3.418492		1.9815
extrap.	3.418491	2.7070 ±4	1.9828 ±3

The correct eigenvalue is at $\lambda = \bar{\lambda} = 1.991$, as may be seen from interpolation on the above data. All the data in this table was produced in less than two minutes time on a CDC 6400 machine, with programs that were not optimally efficient. The convergence of the outputs--as the numerical limit L is advanced--looks quite good. We conclude that we have successfully solved the problem up to this point.

4. FURTHER WORK

In the present paper we shall not discuss in detail the several remaining steps needed to complete the analysis of the problem undertaken. Some of these steps have been worked out in detail, others only sketched so far. We will now only mention what these other problems are and how we expect to deal with them.

After solving (95) we should return to (16) to get the symmetric emission and absorption from both legs 1 and 2. This will involve us with the self-interaction terms (32), the first one of which is divergent. We then say that the masses appearing

in the equations are the unrenormalized ones and use these to formally cancel out the infinity. A preliminary version of this program has been successfully carried out on the computer. Next the trial function must be taken as a symmetric form: the sum of the functions ϕ and $\tilde{\phi}$ of (64). This generates more integrals but they also turn out to be of the same form as those we already have. Finally we should add in the terms (17), and again the treatment of the integrals looks familiar, although we have not carried this out on the computer yet.

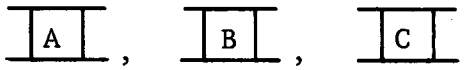
To get away from $E = 0$ we should be able to simply put factors of

$$\left(\frac{p_4}{p^2 + 1}\right) , \qquad \left(\frac{q_4}{q^2 + 1}\right) \tag{96}$$

into the trial functions. Again this has not been worked out in detail and there may be pitfalls, but we guess not. As we get to very weakly bound states (near the elastic scattering threshold) it may be necessary to look carefully at the asymptotic (in x-space) form of the trial functions; and of course we have not yet looked at the equations for elastic

and inelastic scattering processes, but this we also expect to be able to manage.

In considering scattering problems we are very interested in seeing that we can calculate scattering amplitudes that are guaranteed to satisfy the unitarity condition. As a first orientation to this problem we consider the game of cutting and sewing graphs. As long as the energy is limited to the two-body (elastic) region this is rather simple. If we have all sequences of graphs



etc. (97)

where each box has no two-body intermediate state, then cutting can take place only between the boxes to have at most the two particles on the mass shell, and the resewing reproduces the same set; thus elastic unitarity is easy to assure.

In the three body sector, $m_1 + m_2 + m_3 \leq E < m_1 + m_2 + 2m_3$, things are more complicated since we allow ourselves to cut the two outer legs plus one inner leg for particle 3 (we have systematically ignored all closed loops of the outer particles 1 and 2). It is an interesting graphical exercise to show (and we shall not reproduce this exercise here)

that the family of graphs generated by our Eqs. (16) plus (17) is closed under this three-body cutting and sewing game. Furthermore since we have learned to construct the equations so that the counting of graphs is correct, as compared with the series of Feynman graphs, we are led to say that we will get a unitary scattering amplitude up to the four-body threshold.

This argument is incomplete in that it ignores the contribution of the self-interaction graphs. It is our hope that, following Saenger,[6] we can find a set of wavefunction renormalization Z factors which will sort this out.

5. CONCLUSION

The four cardinal principles of elementary particle theory are relativistic invariance, analyticity, unitarity and crossing symmetry. The line of approach we are taking aims to incorporate the first three. Our equations come from field theory, which can be represented as an infinite set of coupled equations for N-body amplitudes. We truncate first keeping only two-body functions to get the ladder

approximation, and at the present stage we include three-body terms. We hope to go on to four-body[7] terms some day and hope to see if this N-body sequence of approximations is converging. If this sequence does in fact converge well then we may assume that the crossing symmetry, while not exactly built in at any finite approximation, will be well represented in our final answers.

In the present paper we have seen how to overcome the problem of miscounting graphs and, most encouraging, we have found that the numerical aspects of the three-body equations are thoroughly tractable. The details of the renormalization and the proof of the unitarity in the three-body sector remain to be completed, as does the actual extension of the computations to E above zero.

APPENDIX

In considering asymptotic states of the interacting equations we are used to saying that when the two interacting particles are far apart the potential terms can be dropped from the equations. This is obvious in the Schrödinger equation when we have

$V(|\underline{r}_1 - \underline{r}_2|)$ and this function falls to zero rapidly as $\underline{r}_1 - \underline{r}_2$ goes to infinity. However in dealing with the covariant Bethe-Salpeter equation the potential depends on the relative time as well as the relative space distance of the two particles

$$x = x_1 - x_2 = (t_1 - t_2, \underline{r}_1 - \underline{r}_2) = (t, \underline{r})$$

and the potential is an invariant function

$$V = V(\sqrt{r^2 - t^2})$$

as for example the function given by one particle exchange

$$V = \int d^3p \int dp_0 \frac{e^{i\underline{p}\cdot\underline{r}} e^{-ip_0 t}}{\underline{p}^2 - p_0^2 + \mu^2 - i\epsilon}$$

If we let r go to infinity with $t = 0$, then indeed we find $V \sim e^{-\mu r}$. However if the interval is timelike rather than spacelike, i.e. if $t_1 - t_2$ is greater than $|\underline{r}_1 - \underline{r}_2|/c$, then V will be oscillatory and not exponentially decaying and hence we cannot ignore it. This is a ridiculous conclusion for it suggests that if the two particles are found in two detectors 3 meters apart at times which

differed by more than 10^{-8} seconds, then they are still interacting.

To straighten this out we must recognize that the asymptotic states are finite packets and that the process of detection takes place over a finite time according to some smooth time-packet function.

Let's assume that the detection is monitored by some time-gating function

$$f(t) \quad \text{such that} \quad f(0) = 1,$$

$$f \to 0 \quad \text{for} \quad |t| \gg T \ ,$$

and we need the potential averaged as

$$V_T(r) = \int_{-\infty}^{\infty} dt \; V(r, t) \, f(t)$$

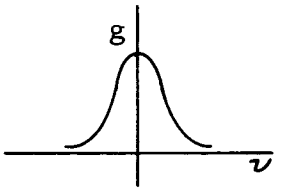
Putting in the explicit form of the one-particle-exchange potential we get

$$V_T(r) = \int d\nu g(\nu) \frac{e^{-r\sqrt{\mu^2 - \nu^2 - i\epsilon}}}{4\pi r}$$

where

$$f(t) = \int d\nu \, g(\nu) e^{i\nu t}$$

and the spectral function g has unit area and width T^{-1}:



In the limit of f = constant, we recover the Yukawa potential $V_\infty = \frac{e^{-\mu r}}{4\pi r}$, and while the integral above depends on the precise form of $g(\nu)$ we might represent the character of the result as

$$V_T \sim \frac{e^{-r\sqrt{\mu^2 - T^{-2}}}}{4\pi r}$$

Thus for any reasonable time-gate we do have effectively the exponential decrease of the interaction with distance, which we expected on physical grounds. However if T is so short that the uncertainty principle leads to an energy fluctuation as great as μ then this particle may actually be created, and can travel over to the other particle and cause an interaction, and this is what is implied by the oscillating phase condition in the potential.

References

1. Bound state solutions were first obtained in C. Schwartz, Phys. Rev. 137, B717 (1965).

2. Elastic scattering calculations were first achieved by C. Schwartz and C. Zemach, Phys. Rev. 141, 1454 (1966).

3. These bubbles were first included with the ladders in calculations by M. Levine, T. Tjon and J. Wright, Phys. Rev. 157, 1416 (1967).

4. For the definitions and properties of the special functions we use the reference W. Magnus and F. Oberhettinger, "Formulas and Theorems for the Functions of Mathematical Physics." (Chelsea Publishing Company, New York, 1949).

5. See C. Schwartz, Journal of Computational Physics, 2, 90 (1967).

6. R. Saenger, Phys. Rev. 159, 1433 (1967).

7. Saenger, ref. 6, states that at the four-body level we cannot obtain unitarity with any subset of graphs but must include them all; this will certainly make things more complicated than what we have at the three-body level.

DISCUSSION OF DR. SCHWARTZ'S LECTURE

Pagnamenta: If you have a bound state, then the uniqueness theorem for such equations tells you that the inhomogeneous equation even below threshold has no solution; your homogeneous

equation has a solution. You may have a solution near that bound state pole. Does your method still work?

Schwartz: Yes. You do the calculation of your input at any point except precisely at a bound state position.

Pagnamenta: You need a finite region where you can fit your function.

Schwartz: Yes. You have the whole real line to fit the function; all that is excluded is a set of points.

Pagnamenta: An important case is where you have a weakly bound state.

Schwartz: We treated such an example with a Yukawa potential and showed that the extrapolation procedure works.

Omnès: In terms of Padé techniques, I think the logarithmic behavior at threshold creates no particular problems.

Nuttall: The difficulty is that there are other singularities beyond the three-body threshold, like the sub-energy normal thresholds. You have to eliminate them before applying the Padé method. For instance in the two-body region

you continue the R-matrix which has no singularities. In the same spirit you should extract all the singularities in the three-body region before you can apply such a method. The Padé method does not know which side of the cut to go on and so it cannot possibly give the right answer if a cut is left. You can continue by this method only if the function is analytic.

Schwartz: If you confine yourself to the standard Padé form, that is true. I would say that if your function has square root and logarithmic singularities, then you can write that kind of function into a Padé form. Then you will have a function which will mock up the multisheeted structure that you need. You can now adjust the parameters to fit the input and this might give you a better result.

Nuttall: If you are talking of the T-matrix, then there are sub-energy normal thresholds besides the three-body thresholds.

Schwartz: If the two-body systems have a finite number of bound states then there will be a

certain number of two body thresholds. These will be put in the calculation.

Nuttall: The difficulty is that there are more variables in the problem when you go to break-up.

Schwartz: I am talking of the T-matrix as a function of one variable, the total energy. Here the only singularities on this sheet are the physical ones, namely the physical thresholds. But I agree that there may be some strange things happening in the other sheets.

Sugar: You are off the energy shell with your external particles and this is the reason why you have singularities only in the total energy. Have you tried calculations where you have more than one two-particle channel to see what happens to the second threshold?

Schwartz: Schlessinger has an earlier paper on two-body scattering and there he has discussed a two-channel case and it seems to work all right.