ANALYTICITY AS A USEFUL COMPUTATION TOOL*

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Although the analytic properties of the Schrödinger equation have been extensively studied, they have rarely been put to use in any actual calculational problem. In this paper we report the solution of a potential scattering problem for negative (unphysical) values of the energy E, and the analytic continuation of these numerical results to $E>0$, which yields the actual phase shifts. While this method has some advantages over the usual methods for solving the Schrödinger equation, we anticipate that the real value of the new approach will be realized in the study of three-body scattering, and in relativistic (Bethe-Salpeter) equations.

We start by considering the Schrödinger equation

$$(E-H)\psi_{\vec{k}}=0 \tag{1}$$

with the decomposition

$$H=H_0+V, \tag{2}$$

and the "plane-wave" reference states $\varphi_{\vec{k}}$ which satisfy

$$(E-H_0)\varphi_{\vec{k}}=0. \tag{3}$$

The T matrix is defined, as a function of a complex energy variable w, as

$$T(w)=V+V(w-H_0)^{-1}T(w), \tag{4}$$

and the scattering amplitude is the matrix element of this T between plane-wave states at $w=E+i\epsilon$. The analytic structure of T in which we are interested concerns only the variable w, and not the momentum k of the plane waves.

In order to deduce the analytic properties of $T(w)$ we write the formal solution of Eq. (4):

$$T(w)=V+V(w-H)^{-1}V. \tag{5}$$

Knowing the spectrum of H we conclude that $T(w)$ may have simple poles at negative real w (bound states), and it has a line of discontinuity along the positive real axis (scattering cut); everywhere else in the complex w plane $T(w)$ is analytic. Further examination of this cut leads us to conclude that we have simply a square-root branch point at $w=0$.

Our plan is this: First calculate $T(w)$ at several real negative values of w; then extrapolate from these numbers to get $T(k^2/2m)$.

The calculation of the first step is really much easier than the direct solution of the Schrödinger equation at $w=k^2/2m$. Looking at Eq. (4) one sees that our method involves a nonsingular integral equation, while the usual method leads to a singular integral equation. Alternatively, in coordinate space, we must solve the inhomogeneous Schrödinger equation,

$$(w-H)\psi_{\vec{k}}^{w}=V\varphi_{\vec{k}}, \tag{6a}$$

to get T as

$$T_{\vec{k}'\vec{k}}(w)=(\varphi_{\vec{k}'},V\varphi_{\vec{k}})+(\varphi_{\vec{k}'},V\psi_{\vec{k}}^{w}). \tag{6b}$$

For real negative w the asymptotic behavior of ψ^w at large distances is that of a decaying exponential; and the entire calculation is thus confined to a finite region, as for real bound-state calculations. The example we studied was that of s waves in the potential

$$V=\frac{\hbar^2}{2m}g\frac{e^{-r}}{r};$$

and we solved Eqs. (6) by a variational method essentially that of Kohn and Hulthén,[1] but simplified by the neglect of the scattered-wave terms.

In order to continue T to $w>0$, we fit the computed values of $T(w)$ at several negative values of w to a ratio of polynomials in $\sqrt{-w}$. From the integral equation for T, it can be seen that $T-V\to O(1/w)$ as $w\to\infty$. This "global constraint" was incorporated in the fitting procedure by representing $T-V$ as $P_N(\sqrt{-w})/Q_{N+2}(\sqrt{-w})$ where[2]

$$P_N=\sum_{j=0}^{N}a_j(\sqrt{-w})^j,\quad Q_{N+2}=1+\sum_{j=1}^{N+2}b_j(\sqrt{-w})^j. \tag{7}$$

The constraint is essential for the success of the continuation, since a function small at $w<0$ could be large for $w>0$ and dominate the behavior at physical $w=k^2/2m$. Evaluating P_N/Q_{N+2} at $w=k^2/2m$ we obtain T in the form $-1/(\alpha-i\beta)$ where α should equal $k\cot\delta$ and β/k

Table I. Results of analytic continuation of the off-shell T matrix for the Yukawa potential of strength $g = -2$ at $k^2 = 0$.

Order of fitting N	$\alpha^{-1} = \tan\delta/k$	β/k
Born approximation	+2.0	0.0
1	−8.0605	0.9577
2	−8.3048	0.9351
3	−7.9701	0.9865
4	−7.9432	0.9928
5	−7.9870	0.9833
Exact	−7.9114	1.0000

should equal +1 (unitarity!). In Table I are shown results of this extrapolation for the potential of strength $g = -2$ at $k^2 = 0$. For this value of g there is a bound-state pole in $T(w)$ at very small negative w; and our extrapolation around this pole represents a very tough test of the present method. We achieved similarly good results at several values of g and k^2, the agreement between calculated and known results being 1% or better. (The unitarity condition $\beta/k = 1$ can serve as a measure of the accuracy in cases where the exact answer is not known.)

Still lacking is some firm mathematical understanding of the convergence of this analytic continuation, and the dependence on the location and accuracy of the input numbers. Nevertheless, the success we have found with our first crude attempts convinces us that the method is basically sound; and we look forward to a broad range of applications as well as a refinement of the numerical techniques.

[1]See T.-Y. Wu and T. Ohmura, Quantum Theory of Scattering (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1962), Sec. D.

[2]This construction is reminiscent of the Padé method; see G. A. Baker's review article in Advances in Theoretical Physics (Academic Press, Inc., New York, 1965), Vol. 1. However, that approach relies entirely on the Born series, while we have no such bias.

CURRENT-COMMUTATOR CONSTRAINTS ON THREE- AND FOUR-POINT FUNCTIONS*

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The equal-time commutator algebra of quark currents,[1] combined with a causal representation for vertex functions or forward scattering amplitudes such as the Deser-Gilbert-Sudarshan[2] (DGS) formula, provides constraints on these amplitudes.[3] There are numerous applications of these constraints, some of them studied independently by Bjorken[4] especially in connection with electromagnetic renormalization. In this note, we discuss the general theory briefly, and give the constraints in the form of sum rules in the vertex function VVP (V = vector particle, P = pseudoscalar).

The DGS representation for causal commutators reads

$$(2E)^{1/2}\langle p|[J(x),J'(y)]|0\rangle = \int d\lambda^2 d\beta d^4q \frac{\epsilon(q_0)}{(2\pi)^3} e^{-iq(x-y)-ipy+i\beta p(x-y)} \times H(\lambda^2,\beta)\delta(q^2-\lambda^2), \quad (1)$$

where p refers to a single-particle state of momentum p (energy E). The same spectral function H appears in the matrix element of the time-ordered product:

$$(2E)^{1/2}\langle p|T(J(x)J'(y))|0\rangle = \int \frac{d\lambda^2 d\beta d^4q}{i(2\pi)^4} \frac{e^{-iq(x-y)-ipy+i\beta p(x-y)}}{q^2-\lambda^2+i\epsilon} \times H(\lambda^2,\beta). \quad (2)$$

These are support conditions on H: $0 \geq \beta \geq -1$, $\lambda^2 > 0$ (in the absence of zero-mass intermediate states). H may also depend on p^2, but this dependence will not be made explicit. It is understood that the integral over d^4q is to be done first.

Now consider the VVP three-point function for a pion of momentum p. We write for the invariant amplitude

$$M(k,p) = \epsilon^{\mu}(k)\epsilon^{\nu}(p-k)M_{\mu\nu}(k,p),$$

$$M_{\mu\nu}(k,p) = \epsilon_{\mu\nu\alpha\beta}p^{\alpha}k^{\beta}M(k,p). \quad (3)$$