Theory and Calculation of Scattering with the Bethe-Salpeter Equation*

CHARLES SCHWARTZ AND CHARLES ZEMACH
Department of Physics, University of California, Berkeley, California
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The Bethe-Salpeter equation studied in this paper describes the interaction of two scalar particles via the exchange of a third scalar particle in the ladder approximation. The properties of the Green's function and the potential in coordinate space are shown to permit a Wick rotation to an imaginary time variable, without appeal to information not contained in the original equation. The resulting four-dimensional (Euclidean) wave equation has a solution which grows exponentially for large time-like distances but behaves as an ordinary Schrödinger scattering wave for large space-like distances. A modification of the Schwinger variational principle is used to obtain, with a modest use of computing machinery, scattering phase shifts for various angular momenta and for energies below the inelastic threshold. The success of these calculations indicates that the Bethe-Salpeter equation can be accepted as a powerful and practical tool for the study of strong-interaction dynamics.

I. INTRODUCTION

THE calculation of scattering amplitudes and bound-state properties of systems of strongly interacting particles is a central problem in elementary-particle physics. Two techniques for defining such calculations when the forces are specified—equivalent in physical content though differing in form—have received serious attention. These are the Bethe-Salpeter equation[1] (BS equation), a relativistic analog of the Schrödinger wave equation, and S-matrix theory, which adopts the principles of analyticity and unitarity while avoiding the concept of wave function.

The S-matrix approach, in various approximate forms, has been applied to many practical problems in recent years and has contributed considerable physical insight to them. In contrast—apart from a recent calculation of some bound states[2]—work on the BS equation relevant to strong interactions has concentrated on its formal mathematical structure.

Now, when one attempts to solve a nonrelativistic potential problem, one finds that traditional methods, e.g., integration of the Schrödinger equation, variational principles, etc., are always easier and more accurate than S-matrix methods. Moreover, traditional methods have been successful in three-body problems which are wholly beyond the current scope of S-matrix techniques. In addition, present S-matrix methods for calculating phase shifts break down for states of angular momentum greater than one. We are thus motivated to attempt quantitative calculations with the BS equation and to take seriously the possibility that they may have real advantages over the relativistic S-matrix methods.

In order to present the BS equation in a fairly general way, let us start by writing down the familiar integral version of the Schrödinger equation:

$$\psi(\mathbf{r})=\psi_0(\mathbf{r})+\int d\mathbf{r}'G_0(\mathbf{r},\mathbf{r}')V(\mathbf{r}')\psi(\mathbf{r}'), \quad (1.1)$$

$$G_0(\mathbf{r},\mathbf{r}')=-\left(\frac{2m}{4\pi}\right)\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}. \quad (1.2)$$

With $\psi(\mathbf{r})$, $V(\mathbf{r})$, and $G_0(\mathbf{r},\mathbf{r}')$ interpreted, respectively, as the position probability amplitude, the interaction, and the free propagator, this equation is an expression of the basic quantum-mechanical rule for compounding such amplitudes. Thus the integral equation carries a direct intuitive meaning based on fundamental quantum-mechanical notions which are more generally valid than the nonrelativistic context in which (1.2) is written. It is then possible to write down a plausible relativistic version of (1.1) for two-body systems; this is the Bethe-Salpeter equation:

$$\Psi(x_1,x_2)=\Psi_0(x_1,x_2)+\int G_1(x_1,x_1')\times G_2(x_2,x_2')I(x_1'x_2';x_1''x_2'')\Psi(x_1'',x_2'')\times dx_1'dx_2'dx_1''dx_2'', \quad (1.3)$$

$$x_1=(\mathbf{r}_1,t_1), \quad x_2=(\mathbf{r}_2,t_2).$$

$\Psi(x_1,x_2)$ is the two-body wave function which determines the joint probability that the two particles may be found at the points $\mathbf{r}_1$, $\mathbf{r}_2$ at the times t_1 and t_2. For $t_1=t_2$, we have the "equal times wave function" which has the meaning of a Schrödinger wave function and from whose asymptotic form ($|\mathbf{r}_1-\mathbf{r}_2|\to\infty$) the scattering amplitude may be inferred. The Green's functions $G_1(x_1,x_1')$ and $G_2(x_2,x_2')$ describe free relativistic propagation with the causal boundary conditions.

In field theory or in S-matrix theory, the interaction I is usually understood to represent a sum of irreducible Feynman graphs contributing to the interaction of the two particles. But owing to uncertainties about the completeness of these theories at the present time,

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[1] E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951); M. Gell-Mann and F. Low, *ibid.* **84**, 350 (1951); J. Schwinger, Proc. Natl. Acad. Sci. U. S. **37**, 452, 455 (1951).

[2] C. Schwartz, Phys. Rev. **137**, B717 (1965). For the special case where the exchanged particle is massless, the bound-state problem has been solved by R. E. Cutkosky, Phys. Rev. **96**, 1135 (1954).

one cannot guarantee that this specification of I is always correct or even meaningful.

By way of contrast, one may recall that in quantum electrodynamics the BS equation has been used to derive several specific observed effects: proton recoil correction to the fine structure[3] and to the hyperfine structure[4] in hydrogenic atoms, and similar contributions to the energy levels of positronium.[5] In these cases, since the electromagnetic coupling is inherently weak, only a few of the simplest Feynman diagrams contributing to I were needed, and their effects were calculated as small perturbations on the nonrelativistic Coulomb energy levels.

For the complete dynamics of strong interactions, more general methods are required. It behooves us, in inaugurating a quantitative investigation of the BS equation, to look first at the simplest system imaginable, namely, two spinless mesons which interact via the exchange of a third spinless meson of mass M. We shall represent I by the one-particle-exchange process (ladder approximation)[6]:

$$I(x_1x_2;\,x_1'x_2')=I(x_1-x_2)\delta(x_1-x_1')\delta(x_2-x_2'), \quad (1.4)$$

$$I(x_1-x_2)=\frac{-i\lambda}{\pi^2}\int\frac{d^4q e^{iq(x_1-x_2)}}{(q^2+M^2-i\epsilon)}. \quad (1.5)$$

The interaction strength parameter λ of (1.5) is $(4\pi)^{-2}$ times the product of the coupling constants which connect the exchanged meson to the incident particles, and it is inherently positive (attractive force) if the incident particles are identical.

The one-particle Green's functions obey the differential equations

$$\begin{aligned}&(p_1^2+m_1^2)G_1(x_1,x_1')=\delta^4(x_1-x_1'),\\&(p_2^2+m_2^2)G_2(x_2,x_2')=\delta^4(x_2-x_2'),\\&p=(-i(\partial/\partial\mathbf{r}),\,i(\partial/\partial t)),\quad p^2=\mathbf{p}^2-p_0^2.\end{aligned} \quad (1.6)$$

Applying the above forms, we obtain the starting equation in its differential form:

$$[(p_1^2+m_1^2)(p_2^2+m_2^2)-I(x_1-x_2)]\quad\Psi(x_1,x_2)=0. \quad (1.7)$$

The chief results to be reported in this paper are accurate numerical values for s-, p-, d-, f-, and g-wave phase shifts for a range of values of the coupling parameter λ and the mass M of the exchanged particle (but with the fixed relation $m_1=m_2$). The higher partial waves are shown to be well represented by their Born approximations. The calculations were done at several energies up to the threshold for production of the meson of mass M.[7]

In the next section, we examine some general properties of the equation and carry out in detail the "Wick rotation" which transforms the Lorentz metric into a Euclidean metric. Section III is devoted to the numerical solution of this transformed problem, and Sec. IV treats various related topics such as the nonrelativistic limit and the unitarity condition for a BS amplitude. Section IV includes a derivation of scattering variational principles of both the Schwinger and Kohn types. Some secondary mathematical details are given in Appendices which follow the summary of Sec. V.

II. BS EQUATION

A. Kinematics

The coordinates and momenta of the two scattering particles have already been denoted by x_1, x_2 and p_1, p_2. We now wish to introduce the total-momentum operator P and the relative coordinate x. A complete canonical transformation is given by

$$\begin{aligned}&P=p_1+p_2, &&X=\mu_1x_1+\mu_2x_2,\\&p=\mu_2p_1-\mu_1p_2\equiv(\mathbf{p},p_0), &&x=x_1-x_2\equiv(\mathbf{r},t),\end{aligned} \quad (2.1)$$

where μ_1, μ_2 are constants restricted by

$$\mu_1+\mu_2=1. \quad (2.2)$$

Condition (2.2) assures that $dPdp=dp_1dp_2$ and $dXdx=dx_1dx_2$.

Let P have the eigenvalue K; then the BS wave function can be written

$$\Psi(x_1,x_2)=e^{iKX}\psi(x), \quad (2.3)$$

and the scattering proceeds from the initial plane-wave state

$$\Psi_0(x_1,x_2)=e^{ik_1x_1}e^{ik_2x_2}=e^{iKX}\psi_0(x). \quad (2.4)$$

Working in the center-of-mass coordinate frame we can write

$$\begin{aligned}&k_1=(\mathbf{k},\omega_1), &&k_2=(-\mathbf{k},\omega_2),\\&\omega_i=+(\mathbf{k}^2+m_i^2)^{1/2}, &&i=1,2,\end{aligned} \quad (2.5)$$

so that

$$K=k_1+k_2=(\mathbf{0},E),\quad E=\omega_1+\omega_2. \quad (2.6)$$

Then the incident wave in the relative coordinate is

$$\psi_0(\mathbf{r},t)=e^{i(\mathbf{k}\cdot\mathbf{r}-\nu t)}, \quad (2.7)$$

where

$$\nu=\mu_2\omega_1-\mu_1\omega_2. \quad (2.8)$$

[3] E. E. Salpeter, Phys. Rev. **87**, 328 (1952).

[4] R. A. Arnowitt, Phys. Rev. **92**, 1002 (1953); W. A. Newcomb and E. E. Salpeter, *ibid.* **97**, 1146 (1955).

[5] R. Karplus and A. Klein, Phys. Rev. **87**, 848 (1952); T. Fulton and P. C. Martin, *ibid.* **95**, 811 (1954).

[6] The Lorentz scalar product is $AB=\mathbf{A}\cdot\mathbf{B}-A_0B_0$, and we use units $\hbar=c=1$.

[7] There are a number of reasons for limiting ourselves to low energies: (a) At higher energies the inelastic channels must also be considered in a consistent manner; a related circumstance is that our method of calculation—via the Wick rotation—is valid only below the inelastic threshold. (b) The omitted higher order contributions to I involve the exchange of higher mass systems and hence give forces of shorter range. The ordering of forces by their ranges and the use of the longest range force in a first approximation is commonly (and plausibly) thought to be a good rule for scattering at low energies.

Last, we make the transformation

$$\psi \to e^{i\nu t}\psi, \tag{2.9}$$

so that all time dependence has been removed from the incident wave.[8]

B. The Green's Function

The differential equation for the transformed relative wave function now reads

$$[\mathbf{p}^2-(p_0+\omega_1)^2+m_1^2][\mathbf{p}^2-(p_0-\omega_2)^2+m_2^2]\psi(x) = I(x)\psi(x). \tag{2.10}$$

(Note that the undetermined parameters μ_1, μ_2 have now disappeared.) The corresponding integral equation is

$$\psi(x)=\psi_0(x)+\int d^4x' G(x,x')I(x')\psi(x'), \tag{2.11}$$

where

$$\psi_0(x)=e^{i\mathbf{k}\cdot\mathbf{r}},$$

and the Green's function, with causal boundary conditions, is

$$G(x,x')=\int \frac{d^4p}{(2\pi)^4}\frac{e^{ip(x-x')}}{[\mathbf{p}^2-(p_0+\omega_1)^2+m_1^2-i\epsilon][\mathbf{p}^2-(p_0-\omega_2)^2+m_2^2-i\epsilon]}. \tag{2.12}$$

Combining the denominators of (2.12) by the Feynman parameter technique and then translating the p_0 integration variable, we get

$$G(x,x')=\int_{-\omega_2}^{\omega_1}\frac{d\beta}{E}e^{i\beta(t-t')} \times\int \frac{d^4p}{(2\pi)^4}\frac{e^{ip(x-x')}}{[\mathbf{p}^2-p_0^2+\beta^2-k^2-i\epsilon]^2}. \tag{2.13}$$

Using the results of Appendix A, Eq. (A5), we have

$$G(x,x')=\frac{i}{8\pi^2}\int_{-\omega_2}^{\omega_1}\frac{d\beta}{E}e^{i\beta(t-t')}K_0(QR), \tag{2.14}$$

where

$$R=[(\mathbf{r}-\mathbf{r}')^2-(t-t')^2]^{1/2},\quad Q=(\beta^2-k^2)^{1/2}. \tag{2.15}$$

As is explained in Appendix A, Q is taken to lie in the fourth quadrant and R in the first, so regardless of whether Q and R are real or imaginary, their product has a positive real part. Now since

$$K_0(QR)\to(\pi/2QR)^{1/2}e^{-QR}\quad\text{as}\quad |QR|\to\infty, \tag{2.16}$$

we see that $G(x,x')$ is well behaved at large space-like or time-like intervals.

Further insight into the structure of $G(x,x')$ is obtained by asking for the nonrelativistic limit of the Green's function. Looking at (2.13) or (2.14), we are led to consider the infinite integral over the parameter β. Thus, making the separation

$$\int_{-\omega_2}^{\omega_1}d\beta=\int_{-\infty}^{\infty}d\beta-\left(\int_{-\infty}^{-\omega_2}+\int_{\omega_1}^{\infty}\right)d\beta \tag{2.17}$$

and applying formula (A6) to the first part, we obtain[9]

$$G(x,x')=\frac{i}{8\pi E}\left\{\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}-\left(\int_{-\infty}^{-\omega_2}+\int_{\omega_1}^{\infty}\right)\frac{d\beta}{\pi} \times e^{i\beta(t-t')}K_0(QR)\right\}. \tag{2.18}$$

Now when β is in one of the intervals $(-\infty,-\omega_2)$ or (ω_1,∞), Q is real, positive and, in fact, larger than m_2 or m_1. Thus if we consider large space-like separations, $R\to\infty$ (real), we have, using (2.18) and (2.16),

$$G(x,x')\to\frac{i}{8\pi E}\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}, \tag{2.19}$$

which, apart from a constant factor, is just the nonrelativistic free-particle Green's function (independent of the relative time!).

Taking this limit in the integral equation (2.11), we see

$$\psi(x)\underset{|\mathbf{r}|\to\infty}{\longrightarrow}e^{i\mathbf{k}\cdot\mathbf{r}}+\frac{i}{8\pi E}\int d^4x'\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}I(x')\psi(x'), \tag{2.20}$$

from which we can pick out the scattering amplitude as

$$f(\mathbf{k}'\leftarrow\mathbf{k})=\frac{i}{8\pi E}\int d^4x' e^{-i\mathbf{k}'\cdot\mathbf{r}'}I(x')\psi(x'). \tag{2.21}$$

[That (2.21) gives the correct expression for the scattering amplitude is known from field theory. We have not shown, wholly within the context of the BS equation, that our prescription is the correct one to associate with actual scattering experiments. This presumably would require a careful discussion of particle fluxes and wave packets, which we have not undertaken.]

[8] In a discussion of two free particles with momenta as in (2.5), a natural choice is $\mu_1=\omega_1/(\omega_1+\omega_2)$. This implies that $\nu=0$ and that the relative momentum $k=\mu_2k_1-\mu_1k_2$ is orthogonal to $K=k_1+k_2$ and is purely space-like in the two-particle center-of-mass frame. However, as the text shows, there is no need to specify μ_1, μ_2 so long as (2.2) is obeyed.

[9] A simple way to understand why the first term of (2.18) is time independent is to note that the corresponding integral over $(d\beta d^4p)$ is invariant in a five-dimensional space, with β a fourth spacelike dimension. The integral must then depend only on the invariant $|\mathbf{r}-\mathbf{r}'|^2+(t-t')^2-(t-t')^2$.

The interaction (1.5) can be expressed, using (A7), as

$$I(x')=(4\lambda M/|x'|)K_1(M|x'|),\quad |x'|=(\mathbf{r}'^2-t'^2)^{1/2}, \tag{2.22}$$

with $|x'|$ in the first quadrant. For large x', we have

$$I(x')\sim\lambda(8\pi M)^{1/2}|x'|^{-3/2}e^{-M|x'|}, \tag{2.23}$$

showing exponential decay for space-like distances.

The Born approximation to (2.21) is obtained by setting

$$\psi(x')\to\psi_0(x')=e^{i\mathbf{k}\cdot\mathbf{r}'}. \tag{2.24}$$

This gives

$$f^B(\mathbf{k}'\leftarrow\mathbf{k})=\frac{2\pi\lambda/E}{(\mathbf{k}-\mathbf{k}')^2+M^2}, \tag{2.25}$$

which agrees, of course, with the one-particle exchange amplitude calculated directly from Feynman's rules. It also agrees, in the limit $E\to m_1+m_2$, with the Born amplitude for scattering by the Yukawa potential

$$V(r)=(-\pi\lambda/m_1m_2)(e^{-Mr}/r) \tag{2.26}$$

in the two-body Schrödinger equation.

C. The Wick Rotation

Wick[10] has shown that, for the bound-state problem, the Lorentz metric can be transformed into a Euclidean metric by a rotation of the relative time variable, $t\to -it$. He found that no additional singularities were encountered in the BS wave function or the Green's function, and the subsequent mathematical studies were much simpler in the Euclidean metric. However, certain assumptions made in that original work—such as the presumed spectrum of stable particles—appear now to be unnecessary; all our results follow simply from the $i\epsilon$ prescription in (2.12).

For the general problem of bound states ($E<m_1+m_2$) or scattering ($E\geqq m_1+m_2$), we consider the Wick rotation applied to the integral equation (2.11) with the Green's function (2.18). Set

$$t=\tau e^{-i\varphi},\quad t'=\tau' e^{-i\varphi}, \tag{2.27}$$

and let the angle φ go from zero to $\frac{1}{2}\pi$. This is similar to the continuation process used in Appendix A to evaluate some important integrals; and, in particular, we note that the distance function R remains in the first quadrant of the complex plane. The variable t appears in the integral equation as an argument of the analytic function $G(x,x')$. Then the justification of the analytic continuation requires a demonstration that the t' integral converges. As $|\tau-\tau'|\to\infty$, the quantity $e^{i\beta(t-t')}K_0(QR)$ in the Green's function has a magnitude proportional to the real exponential factor

$$e^{(\tau-\tau')\beta\sin\varphi}e^{-|\tau-\tau'|(\beta^2-k^2)^{1/2}\sin\varphi}, \tag{2.28}$$

which can be a *growing* function. This exponential growth under the t' integral will be overcome by the interaction I, which, by (2.23), is proportional to $e^{-M|\tau'|\sin\varphi}$ for $|\tau'|\to\infty$.

We must also see how the wave function $\psi(\tau')$ behaves for large τ', and we can learn this from the integral equation. Writing (2.11) in a very schematic form, appropriate to $|\tau|\to\infty$, we see

$$\psi(\tau)\sim e^{i\mathbf{k}\cdot\mathbf{r}}+\frac{e^{ikr}}{r}\int d\tau' e^{-M|\tau'|\sin\varphi}\psi(\tau')$$
$$+\left(\int_{-\infty}^{-\omega_2}+\int_{\omega_1}^{\infty}\right)d\beta\int d\tau' e^{(\tau-\tau')\beta\sin\varphi}$$
$$\times e^{-|\tau-\tau'|(\beta^2-k^2)^{1/2}\sin\varphi}e^{-M|\tau'|\sin\varphi}\psi(\tau'). \tag{2.29}$$

Thus, assuming for the moment that the τ' integral does converge, the dominant terms (coming from β near ω_1 and $-\omega_2$) show that

$$\psi(\tau)\sim e^{|\tau|(\omega_2-m_2)\sin\varphi},\quad \tau\to-\infty$$
$$\sim e^{|\tau|(\omega_1-m_1)\sin\varphi},\quad \tau\to+\infty. \tag{2.30}$$

Finally, if this behavior is inserted under the τ' integrals in (2.29), it is seen that the integrals *will* converge, both for $\tau\to+\infty$ and $\tau\to-\infty$, provided that

$$E=\omega_1+\omega_2<m_1+m_2+M. \tag{2.31}$$

Equation (2.31) may be read as the condition that the energy of the two-particle system is less than the threshold for the production of a real meson of mass M; all further work in this paper will be subject to this restriction. The Wick rotation is now completed by setting $\varphi=\frac{1}{2}\pi$ so that $t=-i\tau$, $t'=-i\tau'$. The rotated wave function is well defined in coordinate space, but it does have an exponential growth for $|\tau|\to\infty$. This function in momentum space will then have special singularities which make it hard to analyze.[11]

D. The Euclidean Problem

Having justified the Wick rotation, subject to (2.31), we restate the problem in notation appropriate to the Euclidean metric. The BS equation now reads

$$\varphi(x)=\varphi_0(x)+\int d^4x'H(x,x')V(x')\varphi(x'), \tag{2.32}$$

where

$$x=(x_1,x_2,x_3,\tau)=(\mathbf{r},\tau), \tag{2.33}$$

$$\varphi_0=e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{2.34}$$

[10] G. C. Wick, Phys. Rev. **96**, 1124 (1954). A general Euclidean form for field theory has been considered by J. Schwinger, Proc. Natl. Acad. Sci. U. S. **44**, 956 (1958).

[11] Such singularities have been noted by N. Kemmer and A. Salam, Proc. Roy. Soc. (London) **A230**, 266 (1955), who first applied Wick's trick to the scattering problem.

$$V(x)=(4M\lambda/R)K_1(MR), \quad R=(\mathbf{r}^2+\tau^2)^{1/2}, \quad (2.35)$$

$$H(x,x')=\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{8\pi E|\mathbf{r}-\mathbf{r}'|}-\frac{1}{8\pi^2 E}\left(\int_{-\infty}^{-\omega_2}+\int_{\omega_1}^{\infty}\right)d\beta \times e^{\beta(\tau-\tau')}K_0(Q|R-R'|), \quad (2.36)$$

$$|R-R'|=[|\mathbf{r}-\mathbf{r}'|^2+(\tau-\tau')^2]^{1/2}, \; Q=(\beta^2-k^2)^{1/2}. \quad (2.37)$$

The differential form of this equation is

$$[p^2+2i\omega_1 p_4-k^2][p^2-2i\omega_2 p_4-k^2]\varphi(x) = V(x)\varphi(x), \quad (2.38)$$

where

$$p_4=-i(\partial/\partial\tau), \quad p^2=\mathbf{p}^2+p_4^2. \quad (2.39)$$

This is a fourth-order partial differential equation. The boundary conditions at the origin $R=0$ are the usual ones that the singular solutions are to be discarded. The boundary conditions at infinity are given in the integral equation (2.32), and we may read them off from the asymptotic behavior of the Green's function $H(x,x')$. For $\tau-\tau'>0$ and for $|R-R'|\to\infty$ with the ratio $|\tau-\tau'|/|\mathbf{r}-\mathbf{r}'|$ fixed and finite, the dominant portion of the β integral of (2.36) comes from near $\beta=\omega_1$, and, by a partial integration, we find[12]

$$H(x,x')\sim\frac{1}{8\pi E}\left[\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}-\left(\frac{m_1}{2\pi|R-R'|}\right)^{1/2} \times\frac{e^{[\omega_1(\tau-\tau')-m_1|R-R'|]}}{[\omega_1|R-R'|-m_1(\tau-\tau')]}\right]. \quad (2.40)$$

For $\tau-\tau'<0$, we can write down the analogous expression directly by making use of the fact that $H(x,x')$ is invariant under the transformation

$$\tau\to-\tau, \quad m_1\leftrightarrow m_2. \quad (2.41)$$

This symmetry applies because the propagators of both particles were given the same $i\epsilon$ treatment. The asymptotic behavior of the wave function, determined by (2.40), is pictured in the (r,τ) plane shown as Fig. 1. In a cone of finite aperture (which fills the entire space as the kinetic energy goes to zero), we see that $\varphi(x)$ is asymptotically like a Schrödinger scattering wave function

$$\varphi(x)\sim e^{i\mathbf{k}\cdot\mathbf{r}}+f(e^{ikr}/r), \quad (2.42)$$

with the scattering amplitude

$$f(\mathbf{k}'\leftarrow\mathbf{k})=\frac{1}{8\pi E}\int d^4x'\, e^{-i\mathbf{k}'\cdot\mathbf{r}'}V(x')\varphi(x'). \quad (2.43)$$

This is identically the same quantity as the original scattering amplitude (2.21), but expressed in terms of the rotated functions.

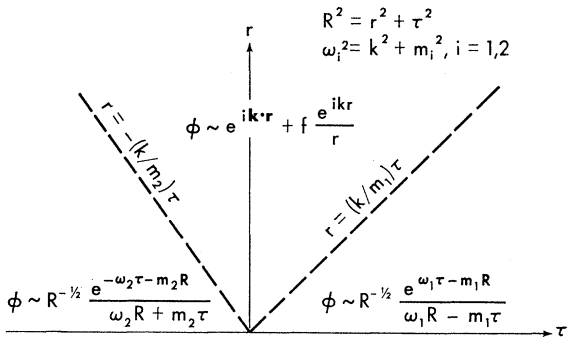

FIG. 1. Behavior of the Wick-rotated Bethe-Salpeter wave function $\varphi(r,\tau)$ in the energy region $m_1+m_2\leqq E\leqq m_1+m_2+M$, as $R\to\infty$ with r/τ constant.

The wave function $\varphi(x)$ is seen to grow exponentially for large τ-like distances. However, $\varphi(x)$ occurs multiplied by $V(x)$ in the integrals (2.32) and (2.43), and this product is well behaved at large distances in all directions. Therefore, although this feature is, perhaps, at first disconcerting, it can be abided. Some understanding of this unseemly behavior of $\varphi(x)$ may be gained from the following rough analogy. Think first of a one-dimensional (radial) Schrödinger equation. One of the two possible solutions of this second-order equation about the origin is discarded, and the other may be fixed by a choice of normalization. Thus, as one integrates out from $r=0$, the solution is completely determined. Approaching $r=\infty$, we have (a) for bound states, the requirement $\psi\to 0$ which can be satisfied only for certain values of some parameter (eigenvalues); (b) for scattering states, the phase shift is uniquely found.

Now, for comparison, think of the BS equation as a one-dimensional problem (in the τ variable) but of fourth order. Of the four independent solutions at the origin, two solutions are discarded as being too singular. The other two will occur in a linear combination, so that, in addition to a normalization factor, we are left with one free parameter as we integrate out to infinity.

Then, for $R\to\infty$:

(a) For bound states we have the requirement $\varphi\to 0$. Of the four possible solutions of the equation in this region, two are exponentially increasing, two decreasing. We thus have two conditions to be filled with the one free parameter, and this will work only for particular values of some other parameter in the equation (eigenvalues).

(b) For scattering states, the four possible solutions of the differential equation (2.38) for $\tau\to\infty$ have the following characters:

$$\phi_1\sim e^{(\omega_1+m_1)\tau}, \quad \phi_2\sim e^{(\omega_1-m_1)\tau}, \quad \phi_3\sim e^{-(\omega_2-m_2)\tau}, \quad \phi_4\sim e^{-(\omega_2+m_2)\tau}. \quad (2.44)$$

[12] Our conclusions concerning the asymptotic behavior of $H(x,x')$ away from the space axis are not in accord with those of A. R. Swift and B. W. Lee, J. Math. Phys. **5**, 908 (1964). In particular, we do not confirm their asymptotic evaluation of the integral I_2 in their Eq. (B7).

For $\tau \to +\infty$, our asymptotic formula (2.40) tells us that the solution must be dominated by ϕ_2. Thus our boundary condition may be stated: The *strongly* divergent solution ϕ_1 must be absent. This is one condition which fixes the one free parameter we had in φ starting at the origin. The solution is now well determined, and we can read off the phase shift.

To complete this admittedly nonrigorous discussion one should ask about $\tau \to -\infty$. The *strongly* divergent solution is now ϕ_4. But the symmetry under (2.41), which applies to $\varphi(x)$ as well as $H(x,x')$, guarantees that ϕ_4 is absent when ϕ_1 is.[13]

III. METHOD OF SOLUTION

It now remains to demonstrate that one can, as a practical matter, numerically solve the equation written down and described in the preceding section. It is clear that the ordinary angular momentum is a constant of the motion. We may thus separate out two angular variables and have left a two-dimensional $(|\mathbf{r}|,\tau)$ fourth-order elliptic equation. There are a number of standard numerical approaches to choose among: use mesh-point integration or variational principles; study the differential equation or the integral equation; use the coordinate representation or its Fourier transform. A few of these different routes were tried without success. Then, a variational method based on the integral equation was found to solve the problem very successfully.

This generalized Schwinger variational principle for the scattering amplitude is derived in the next section and reads as follows:

$$[8\pi E f(\mathbf{k}' \leftarrow \mathbf{k})]$$
$$= \int d^4x\, e^{-i\mathbf{k}'\cdot\mathbf{r}} V(x)\phi_{\mathbf{k}}(x) + \int d^4x\, \phi_{\mathbf{k}'}(-x) V(x) e^{i\mathbf{k}\cdot\mathbf{r}}$$
$$- \int d^4x\, \phi_{\mathbf{k}'}(-x) V(x)\phi_{\mathbf{k}}(x) + \int\int d^4x d^4x'$$
$$\times \phi_{\mathbf{k}'}(-x) V(x) H(x,x') V(x')\phi_{\mathbf{k}}(x'). \quad (3.1)$$

The last integral in (3.1) looks forbidding, but the following simple device puts the entire calculation in a manageable form. Instead of the function $\phi(x)$ as the quantity to be varied, we take

$$U(x) = V(x)\phi(x). \quad (3.2)$$

Then, defining the Fourier transforms

$$\tilde{U}(p) = \int d^4x\, e^{-ip\cdot x} U(x) \quad (3.3)$$

and

$$H(x,x') = \int \frac{d^4p}{(2\pi)^4} H(p) e^{ip\cdot(x-x')}, \quad (3.4)$$

we find that the variational principle reads

$$[8\pi E f(\mathbf{k}' \leftarrow \mathbf{k})]$$
$$= \tilde{U}_{\mathbf{k}}(\mathbf{k}') + \tilde{U}_{\mathbf{k}'}(\mathbf{k}) - \int d^4x \frac{U_{\mathbf{k}'}(-x) U_{\mathbf{k}}(x)}{V(x)}$$
$$+ \int \frac{d^4p}{(2\pi)^4} \tilde{U}_{\mathbf{k}'}(p) H(p) \tilde{U}_{\mathbf{k}}(p). \quad (3.5)$$

The virtue of this form is that it leaves only two single integrals to be evaluated [no double integral as in (3.1)]; one of these is in x space, the other is in p space. A second advantage is that, while $\phi(x)$ has the unpleasant behavior at large τ, the function $U(x)$ is well behaved.

We now simplify to the equal-mass case:

$$m_1 = m_2 = m = 1, \quad \omega_1 = \omega_2 = \omega = (1+k^2)^{1/2} = \tfrac{1}{2}E. \quad (3.6)$$

For the Fourier transform of the Green's function we can use

$$H(p) = [(p^2 - k^2 - i\epsilon)^2 + 4\omega^2 p_4^2]^{-1} \quad (3.7)$$

but a few words must be said in explanation.[13a] Firstly, the formula (3.4) is not well defined by itself because of the exponential growth for $|\tau| \to \infty$. We need to evaluate the integral

$$\int d^4x \int d^4x' U_{\mathbf{k}'}(-x) H(x,x') U_{\mathbf{k}}(x') \quad (3.8)$$

in (3.1), and show that it can be expressed in the form

$$\int \frac{d^4p}{(2\pi)^4} \tilde{U}_{\mathbf{k}'}(p) H(p) \tilde{U}_{\mathbf{k}}(p) \quad (3.9)$$

of (3.5). We do this in three steps: First, un-Wick-rotate the two time variables. Second, Fourier-transform U and H. Third, attempt to re-Wick-rotate (counterclockwise) the fourth component of momentum. The functions U are taken to be sufficiently well behaved so that these steps are all trouble free, and the only point to watch is the original problem of the location of the poles in the momentum Green's function. What we are left with is a specific contour of integration for the p_4 variable which is illustrated in Fig. 2. What prevents us from simply integrating along the real p_4 axis are the

[13] The scattering states are even under this symmetry because the incident plane wave is even; but bound states of the BS equation may be even or odd. In nonrelativistic problems, the states are all even under this transformation.

[13a] *Footnote added in proof.* This part of the discussion was incorrectly given in the unpublished version of this paper, and some of the numerical results given there are consequently wrong (although only by an inconsequential amount). We thank Klaus Rothe for bringing this matter to our attention.

poles at

$$q_2=i[\omega-(\mathbf{p}^2+m^2)^{1/2}+i\epsilon]$$

and

$$q_3=i[-\omega+(\mathbf{p}^2+m^2)^{1/2}-i\epsilon].$$

These poles make their maximum intrusion for $\mathbf{p}^2=0$, and so we can specify that the integral (3.9) be evaluated along the fixed contour $C_{\max}$ which goes around the points $p_4=\pm[i(\omega-m)-\epsilon]$. Then the following device allows us to follow this prescription very easily: the integral (3.9) may be seen as an analytic function of k^2 (provided the functions U are taken to be analytic in k^2, which is easily done) and in particular may be continued into the region

$$-m^2<k^2\leq 0.$$

In particular for real negative k^2 (just like the bound state situation) the troublesome poles of Fig. 2 never cross the real axis, and so we can now move the contour $C_{\max}$ to coincide with the real axis. This will allow us to introduce polar coordinates for doing the $|\mathbf{p}|$, p_4 integral. We can summarize this discussion about the meaning of the last integral in Eq. (3.5) by the following:

Rule: Proceed as if there were no poles to worry about (i.e., assume k^2 negative); then analytically continue the result to the correct value of k^2. (3.10)

Next we shall separate the (three-dimensional) angular momentum for our problem just as one does for the Schrödinger problem; and by separating real and imaginary parts, convert the variational principle for $e^{i\delta}\sin\delta$ into one for $\tan\delta$. In Sec. IV can be found the unitarity theorem showing that the phase shift δ is real. The imaginary part of H [see (2.36)] can be Fourier transformed without difficulty to yield:

$$H_i=(\pi^2/2\omega)\delta(p^2-k^2)\delta(p_4). \tag{3.11}$$

The partial-wave expansion proceeds as with the Schrödinger problem:

$$f(\mathbf{k}'\leftarrow\mathbf{k})=k^{-1}\textstyle\sum_l P_l(\hat{k}'\cdot\hat{k})(2l+1)e^{i\delta_l}\sin\delta_l, \tag{3.12}$$

$$U_{\mathbf{k}}(x)=\textstyle\sum_l P_l(\hat{k}\cdot\hat{r})i^l(2l+1)u_l(r,\tau). \tag{3.13}$$

For the equal-mass case $u_l(r,\tau)$ is a real function, even in τ; and, using the symbol $\Re$ for "the real part of," we finally arrive at

$$\left[\frac{16\pi\omega\tan\delta_l}{k}\right]=2\tilde{u}_l(k,0)-\int d^4x\, u_l^2(r,\tau)/V(r,\tau)$$
$$+\Re\int\frac{d^4p}{(2\pi)^4}\tilde{u}_l^2(|\mathbf{p}|,p_4)H(|\mathbf{p}|,p_4), \tag{3.14}$$

where

$$\tilde{u}_l(|\mathbf{p}|,p_4)=4\pi\int_{-\infty}^{\infty}d\tau\, e^{-ip_4\tau}$$
$$\times\int_0^{\infty}r^2dr\, j_l(|\mathbf{p}|r)u_l(r,\tau) \tag{3.15}$$

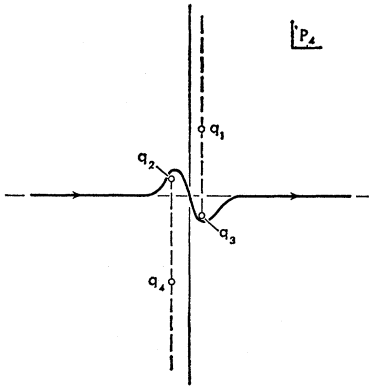

FIG. 2. The q_i, $i=1, 2, 3, 4$ indicate positions of the poles of the momentum Green's function in the p_4 plane. The p_4 plane is obtained from the original p_0 plane by a Wick rotation $p_0\rightarrow ip_4$. The heavy line is the contour of integration in the momentum integrals of (3.5) and (3.9).

or, conversely,

$$u_l(r,\tau)=\frac{4\pi}{(2\pi)^4}\int_{-\infty}^{\infty}dp_4e^{ip_4\tau}$$
$$\times\int_0^{\infty}|\mathbf{p}|^2d|\mathbf{p}|\, j_l(|\mathbf{p}|r)\tilde{u}_l(|\mathbf{p}|,p_4). \tag{3.16}$$

The basis functions for the calculations were chosen to be

$$\tilde{u}_l=\Sigma_{n,m}a_{nm}\sin^l\vartheta\cos^{n-l}\vartheta\frac{2\pi k^l(p/\alpha)^n}{(1+p^2/\alpha^2)^{n+m}}, \tag{3.17}$$

where

$$p=(|\mathbf{p}|^2+p_4^2)^{1/2},\quad \cos\vartheta=p_4/p, \tag{3.18}$$

and

$$n=l,\ l+2,\ l+4,\ \cdots,\quad m=1,\ 2,\ 3,\ \cdots. \tag{3.19}$$

The detailed procedures for evaluating the integrals of (3.14) with these basis functions are described in Appendix B.

The variational principle (3.14) can also be used to find bound states by dropping the first term, and converting it to a stationary quotient for the potential-strength parameter λ (at a given energy E). Calculations of some bound states for this equation, using the Rayleigh-Ritz method on the differential equation, have already been published.[2] In that approach, it was difficult to get accurate answers for weakly bound states (k^2 very small and negative), because the correct

TABLE I. Eigenvalue in coupling constant λ for first bound s state for $m_1=m_2=m=M$, $E/2m=0.9$.

Size of matrix	Old way at $\alpha=0.3$ (differential eq.)	New way at $\alpha=1.3$ (integral eq.)
1	2.90247	1.740278
2	2.18762	1.671994
4	1.84898	1.665830
6	1.76092	1.665305
9	1.71556	1.665217
12	1.69850	1.665190
16	1.68794	1.665180
20	1.68179	1.665174
25	1.67729	
extrapolated	1.667±0.008	1.66517±0.00001

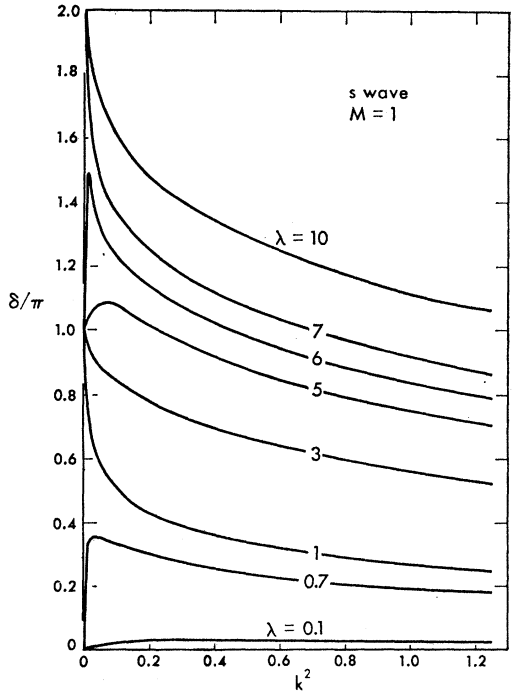

FIG. 3. S-wave phase shifts versus k^2 for various coupling strengths λ and for exchanged-particle mass $M=1$. Zero-energy bound states occur for $\lambda=0.76222$ and $\lambda=6.01$.

wave function has a very asymmetric (in r,τ) asymptotic behavior. In Table I are shown comparative results of that old and this new method for $k^2=-0.19$ ($E/2m=0.9$); and it is clear that the new method converges very much more rapidly than the old. Credit for this improvement is due to the presence of the Green's function in the new calculation; for it is the Green's function which determines the asymptotic behavior that was causing the difficulty. We achieved similarly excellent results right up to zero binding energy ($k^2\to 0_-$); the critical values of λ needed to produce a bound state at $E=2m$—for several values of the exchanged particle mass M and for several partial waves—are given in Table II.

TABLE II. Computed values of $\pi\lambda/M$ which give a first bound state at $E=2m$. One nonrelativistic value ($M\to 0$) for $l=0$ is known, from solution of the Schrödinger equation, to be 1.67981.

M	$l=0$	$l=1$	$l=2$	$l=3$	$l=4$
10	6.585	143.3	729		
5	4.455	75.72	369		
2	2.970	36.12	158.2		
1	2.395	23.18	90.67	250	567
0.5	2.071	16.64	59.3		
0.2	1.853	12.5	39.7		
0.1	1.771	11.1	33.6		
0.05	1.727				
0.02	1.700				
0.01	1.690				
0.0035	1.684				
0.001	1.681				

The variables in the stationary calculation with the basis (3.17) are the linear amplitudes a_{nm} and the scale parameter α. The sequence of approximations was defined by taking all terms of (3.17) with $m+n-l$ less than 1, 2, 3, etc.: This gave the sequence of basis dimensions 1, 2, 4, 6, 9, 12, 16, 20, 25, 30. We collected much data at sizes up to 20 and 30 in order to be thoroughly convinced of the true convergence of our numerical results; but it also became clear that one could frequently get fairly useful approximations

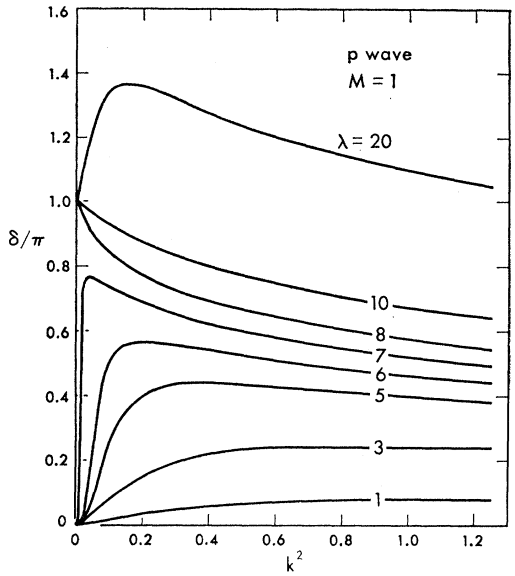

FIG. 4. P-wave phase shifts versus k^2 for various coupling strengths λ and for exchanged-particle mass $M=1$. A zero-energy bound state occurs at $\lambda=7.377$.

(within 10%) by using just one or two terms. In several ways our computer programs were "over-engineered," since we did not know beforehand just how well our method would work. It may now be said that one could probably do reasonable computations of the BS equation using only a desk calculator.

The variation of the scale parameter α is discussed in another paper,[14] where we also present curves of $\omega\tan\delta/k$ as a function of α for a series of basis sizes. In Table III are listed results at three values of α for $l=0$, $M=1$, $k^2=0.4$; and a very rapid convergence is again apparent. It is probably true that this variational method gives a rigorous upper (or lower) bound for the true phase shift, but we draw our confidence in the reliability of the numerical results chiefly from the apparent good convergence, as represented by the data in Table III. For the following collection of numerical results α was simply kept fixed at $1.5M$.

In Fig. 3 are shown curves of s-wave phase shifts versus energy for $M=1$ at several coupling strengths. The familiar jumps by π at $k=0$ may be seen whenever

[14] C. Schwartz, following paper, Phys. Rev. 141, 1468 (1965).

a new bound state appears; otherwise the behavior is very smooth and simple. In Fig. 4 are similar data for p waves.

Partial-wave contributions to the total cross section,

$$Q_l=(2l+1)(\sin^2\delta_l/k^2),$$

are shown in Fig. 5 for $M=1$, $\lambda=5$. At this value of λ there is one very deeply bound s state.[15] The s-wave cross section is very large at $k=0$ because of a "virtual state": The next bound state occurs at $\lambda=6.01$, which is quite nearby. Q_0 then bounces through zero at $k^2=0.2$; this happens because the phase shift (see Fig. 3) is passing through π on its return toward zero as $k\to\infty$.

The p-wave cross section in Fig. 5 shows a large peak at low energy. However, in standard parlance, this is not really a resonance, since δ (see Fig. 4) does not pass through $\frac{1}{2}\pi$ at this value of λ. The phase shift merely stays close to about $\frac{4}{5}$ of $\frac{1}{2}\pi$ over a wide range of energy ($0.2<k^2<1$), and so Q_1 is close to its maximum value of $3/k^2$. Then as k approaches zero, Q_1 must vanish as

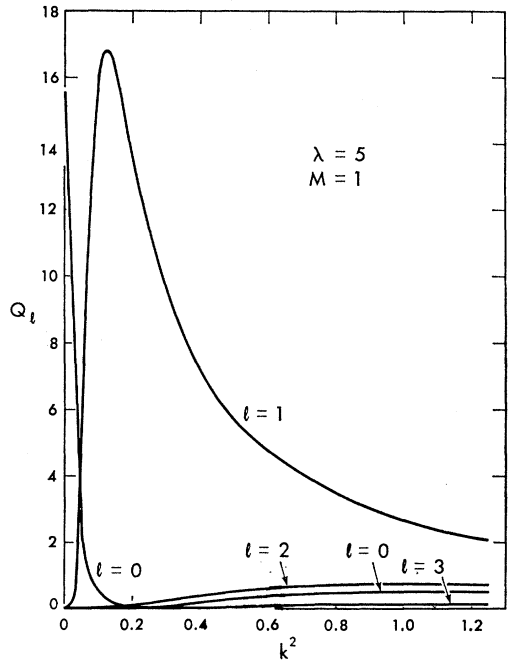

FIG. 5. Partial-wave cross sections $Q_l=(2l+1)\sin^2\delta_l/k^2$ versus k^2 for $\lambda=5$, $M=1$. The Q_l for the higher l are well represented by their Born approximations.

k^4, and these two forces working against each other produce the large peak seen in Fig. 5.

Contributions from $l=2$ and $l=3$ are also shown in Fig. 5, and these are seen to be rather small. In Table IV are given values of Q_2, Q_3, and Q_4 calculated at $\lambda=5$, along with a comparison to their Born approximation values. We see that, as l increases, the contribution to the scattering becomes smaller and the Born approximation becomes more accurate. A semiquantitative understanding of these two observations may be had by recalling that the expansion parameter of the Born series is λ/λ^*, where λ^* is the value of the coupling at which a bound state appears. Looking across the $M=1$ row of Table II, we see that, in going from $l=2$ to $l=3$, the ratio of λ^* values increases by 2.8, and from $l=3$ to $l=4$ by 2.3. The accuracy of the Born approximation (judged by R_l-1 from Table IV) is seen to increase by roughly these factors, going from $l=2$ to $l=3$ to $l=4$; and the magnitudes of the cross sections decrease very roughly as the square of these ratios. We therefore conclude that the entire elastic-scattering problem for the BS equation may be very well solved

TABLE III. Computed values of $(\omega/k)\tan\delta_0$ for $M=m=1$, $k^2=0.4$, $\lambda=1$ at several values of α and several matrix sizes. Convergence is seen to be rapid enough that accuracy for ordinary purposes is achieved with fairly small matrices.

Matrix size	$\alpha=1.2$	$\alpha=1.6$	$\alpha=2.0$
1	2.46995	3.23430	3.02017
2	3.40599	3.26699	3.30337
4	3.49375	3.54227	3.44639
6	3.55633	3.55208	3.53635
9	3.56117	3.56121	3.55757
12	3.56345	3.56342	3.56100
16	3.56368	3.56372	3.56289
20	3.56386	3.56388	3.56358
Extrapolated value	3.5639±1	3.5640±1	3.5640±2

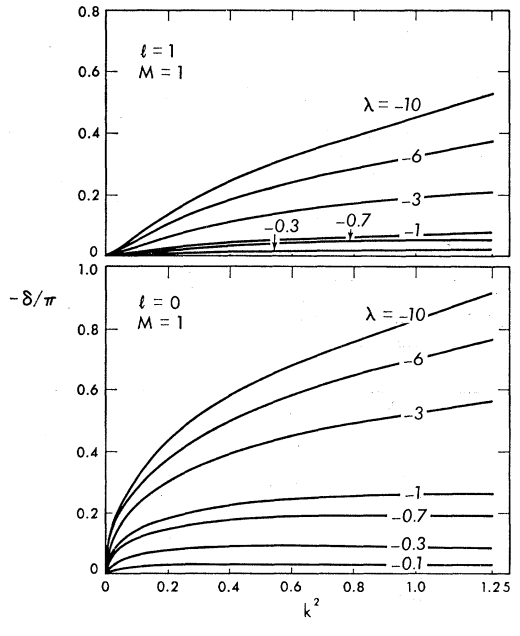

FIG. 6. S- and P-wave phase shifts versus k^2 for various *negative* coupling strengths and for exchanged-particle mass $M=1$.

[15] In fact, at $\lambda=3.4$ the total energy E of that bound s state reaches zero, so at still stronger coupling strengths we really cannot say what happens to that bound state without going outside the given BS equation. In the present section, we merely wish to show that ordinary results for the BS equation look just like the familiar results for a Schrödinger equation; and therefore we shall ignore such extraneous problems.

TABLE IV. Computed values of the partial cross sections, $Q_l=(2l+1)\sin^2\delta_l/k^2$, for $\lambda=5$, $M=1$. R_l is the ratio of Q_l to its Born approximation.

k^2	Q_2	R_2	Q_3	R_3	Q_4	R_4
0	0	(1.120)	0	(1.017)	0	(1.0038)
0.1	0.01393	1.242	0.0^48413	1.036	0.0^65851	1.0067
0.2	0.09753	1.358	0.0^21669	1.061	0.0^43448	1.013
0.4	0.3961	1.451	0.01708	1.105	0.0^38888	1.028
0.6	0.6300	1.400	0.0461	1.127	0.0^23381	1.041
0.8	0.731	1.298	0.0778	1.129	0.0^2870	1.047
1.0	0.744	1.194	0.1051	1.118	0.01470	1.052
1.25	0.705	1.088	0.1292	1.091	0.0223	1.045

by doing these numerical calculations for the lower l values and then adding the Born approximation values for the higher partial waves.

We also looked briefly at some scattering by repulsive potentials (λ negative). Some resulting curves of δ versus k^2 are shown in Fig. 6, and it is seen that they have the general appearance of what one would expect from a repulsive potential in a Schrödinger equation.

The scattering data discussed so far has emphasized $M/m=1$ as a typical relativistic situation. For $M\to 0$, one may expect an approach toward the nonrelativistic results (since the restriction to elastic scattering implies small k if M is small), and this is borne out by the data of Table II. We have done some scattering calculations for M larger than m; and the general nature of the results, as compared to those for $M=m$, may be described in terms of the following scaling of the variables λ and k. For $k^2\ll m^2$, the data of Table II indicates how λ varies with M for a fixed value of the phase shift. For $k^2\gg m^2$, one sees from the differential equation that results depend only on the two ratios k/M and λ/M^2. Thus, for fixed λ, we can use the Born approximation for very large M, nonrelativistic results for very small M, and our numerical procedures for intermediate values.

IV. MISCELLANEOUS TOPICS

A. The Nonrelativistic Limit

We shall first observe that the BS equation reduces to the Schrödinger equation, if the integral terms in the Green's function (2.36) are dropped, and we shall then consider under what circumstances this is permissible. (This first result may be seen as easily from the unrotated equations, but the subsequent studies are simpler in the Euclidean metric space.)

When the integral terms in (2.36) are dropped, $H(x,x')$, and therefore also $\varphi(x)$, become independent of the time variable. The integral over τ' in (2.32) may then be done:

$$\int V(x')d\tau'=\int d\tau'\frac{\lambda}{\pi^2}\int d^4p\frac{e^{ip\cdot x'}}{p^2+M^2}$$
$$=\frac{2\lambda}{\pi}\int d\mathbf{p}\frac{e^{i\mathbf{p}\cdot\mathbf{r}'}}{\mathbf{p}^2+M^2}=4\pi\lambda\frac{e^{-Mr'}}{r'}, \quad (4.1)$$

The equation reduces to

$$\varphi(\mathbf{r})=e^{i\mathbf{k}\cdot\mathbf{r}}+\frac{2\lambda}{E}\int d\mathbf{r}'\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\frac{e^{-Mr'}}{r'}\varphi(\mathbf{r}'), \quad (4.2)$$

which should be compared with (1.1). If the E in (4.2) is replaced by m_1+m_2, then this reduced BS equation is identical to the Schrödinger equation with the Yukawa potential (2.26) already inferred by comparison of the Born approximations.

To decide when the above approximation to $H(x,x')$ is valid, we distinguish two regions of the variables R, R':

$$|R-R'|\times\min(m_1,m_2)\gg 1, \quad (4.3)$$

$$|R-R'|\times\min(m_1,m_2)\gtrsim 1. \quad (4.4)$$

Since Q in (2.36) always exceeds m_1 or m_2, (4.3) is the region in which the asymptotic evaluation (2.40) is valid. In this region, the integral terms are ignorable relative to the nonrelativistic term provided that

$$\omega_{1,2}|\tau-\tau'|\ll m_{1,2}|R-R'|$$

or, equivalently,

$$|\mathbf{r}-\mathbf{r}'|\gg(k/m_{1,2})|\tau-\tau'|. \quad (4.5)$$

Since the interaction treats $|\mathbf{r}-\mathbf{r}'|$ and $|\tau-\tau'|$ symmetrically, this means that for relative time effects to be small we must have

$$k\ll m_{1,2}. \quad (4.6)$$

In order that the region (4.4), which is not covered by the above argument, also be ignorable, it is sufficient that it be a domain small in extent compared to the region of variation of R, R' which is relevant to the dynamics. This latter region is

$$RM\lesssim 1, \quad R'M\lesssim 1,$$

since M^{-1} is the range of the interaction. Hence we need, in addition to (4.6), the requirement

$$M\ll m_{1,2}. \quad (4.7)$$

In conclusion, we can state that the BS equation reduces to the Schrödinger equation when both the de Broglie wavelength associated with the relative momentum and the range of the interaction are large compared to the Compton wavelengths of the interacting particles.

This conclusion may be demonstrated numerically with our calculations of the coupling constant values needed to produce a bound state at zero binding ($E=m_1+m_2$). This gives us one set of conditions (4.6) directly via $k=0$, and we can study the limit $M\to 0$ (4.7). (The other limit, $M=0$, $E\to m_1+m_2$, may be found from the solutions of Cutkosky.[2]) At zero binding, the s-wave part of (4.2) is equivalent to the

differential equation

$$\left(\frac{d^2}{dz^2}+\lambda_0\frac{e^{-z}}{z}\right)u(z)=0, \quad \lambda_0=\frac{2\pi\lambda}{M(m_1+m_2)}, \quad (4.8)$$

for which the lowest eigenvalue is known to be $\lambda_0=1.679809$. Several numerically computed values of λ are given in Table II (for $m_1=m_2=1$), and it is seen that the proper limit is approached.

B. Elastic Unitarity and Other Properties

The main objectives of this subsection are to show that, subject to the threshold condition (2.31), the BS scattering amplitude satisfies the unitarity condition for elastic scattering and to prepare for the derivations of the variational principles below. Such discussions in nonrelativistic theory may be founded, equally well, on the differential or the integral equation. The use of the differential BS equation, however, appears to involve some delicacy in the process of integrating by parts, while the integral equation, having a structure similar to its nonrelativistic counterpart, is easier to deal with. We begin by listing some properties of the BS functions. The wave functions will be given subscripts to specify the incident momenta. We repeat the basic equations:

(A) $$\varphi_{\mathbf{k}}(x)=e^{i\mathbf{k}\cdot\mathbf{r}}+\int H(x,x')V(x')\varphi_{\mathbf{k}}(x')d^4x',$$

(B) $$f(\mathbf{k}'\leftarrow\mathbf{k})=(8\pi E)^{-1}\int e^{-i\mathbf{k}'\cdot\mathbf{r}}V(x)\varphi_{\mathbf{k}}(x)d^4x.$$

We also note

(C) $$\varphi_{\mathbf{k}}(\mathbf{r},\tau)=\varphi_{-\mathbf{k}}(-\mathbf{r},\tau),$$

(D) $$H(x,x')=H(-x',-x).$$

From (2.36),

(E) $$\frac{1}{2i}[H(x,x')-H^*(x,x')]=\frac{1}{8\pi E}\frac{\sin k|\mathbf{r}-\mathbf{r}'|}{|\mathbf{r}-\mathbf{r}'|}=\frac{k}{8\pi E}\int\frac{d\Omega_{k''}}{4\pi}e^{i\mathbf{k}''\cdot\mathbf{r}}e^{-i\mathbf{k}''\cdot\mathbf{r}'}.$$

(F) $V(x)$ is real and invariant under $\mathbf{r}\rightarrow-\mathbf{r}$ or $x\rightarrow-x$.

(G) $H(x,x')$, and hence $\varphi(x)$, is invariant under change of sign of the time variables, provided m_1 and m_2 are also interchanged. Then $\varphi(r,\tau)$ is an even function of τ if $m_1=m_2$.

Because of (D), (F), we have

(H) $$\varphi_{\mathbf{k}'}(-x)=e^{-i\mathbf{k}'\cdot\mathbf{r}}+\int d^4x'\varphi_{\mathbf{k}'}(-x')V(x')H(x',x),$$

and hence, as an alternative to (B),

(I) $$f(\mathbf{k}'\leftarrow\mathbf{k})=(8\pi E)^{-1}\int\varphi_{\mathbf{k}'}(-x)V(x)e^{i\mathbf{k}\cdot\mathbf{r}}d^4x.$$

By setting $\mathbf{r}\rightarrow-\mathbf{r}$ in (B) and using (C), we obtain the parity property

(J) $$f(\mathbf{k}'\leftarrow\mathbf{k})=f(-\mathbf{k}'\leftarrow-\mathbf{k}).$$

Setting $x\rightarrow-x$ in (B) and comparing with (I), we get

(K) $$f(\mathbf{k}'\leftarrow\mathbf{k})=f(\mathbf{k}\leftarrow\mathbf{k}').$$

For brevity, we shall write $\tilde{\varphi}_{\mathbf{k}}(x)$ for $\varphi_{\mathbf{k}}(-x)$ and

$$\langle\varphi|\psi\rangle\equiv\int d^4x\,\varphi^*(x)\psi(x).$$

To obtain the unitarity condition, first consider X_1 and X_2 (we use now a more compact notation):

$$X_1=\langle\tilde{\varphi}_{-\mathbf{k}'}|VHV\varphi_{\mathbf{k}}\rangle=\langle\tilde{\varphi}_{-\mathbf{k}'}|V(-e^{i\mathbf{k}\cdot\mathbf{r}}+\varphi_{\mathbf{k}})\rangle$$
$$=-8\pi Ef^*(\mathbf{k}'\leftarrow\mathbf{k})+\langle\tilde{\varphi}_{-\mathbf{k}'}|V\varphi_{\mathbf{k}}\rangle, \quad (4.9)$$

$$X_2=\langle\tilde{\varphi}_{-\mathbf{k}'}|VH^*V\varphi_{\mathbf{k}}\rangle=\langle(-e^{i\mathbf{k}'\cdot\mathbf{r}}+\tilde{\varphi}_{-\mathbf{k}'})|V\varphi_{\mathbf{k}}\rangle$$
$$=-8\pi Ef(\mathbf{k}'\leftarrow\mathbf{k})+\langle\tilde{\varphi}_{-\mathbf{k}'}|V\varphi_{\mathbf{k}}\rangle. \quad (4.10)$$

Invoking (E), we have

$$\mathrm{Im}f(\mathbf{k}'\leftarrow\mathbf{k})=(16\pi iE)^{-1}(X_1-X_2)$$
$$=k(8\pi E)^{-2}\int\frac{d\Omega_{k''}}{4\pi}\langle\varphi_{-\mathbf{k}'}|Ve^{i\mathbf{k}''\cdot\mathbf{r}}\rangle\times\langle e^{-i\mathbf{k}''\cdot\mathbf{r}}|V\varphi_{\mathbf{k}}\rangle. \quad (4.11)$$

The desired condition then follows:

(L) $$\mathrm{Im}f(\mathbf{k}'\leftarrow\mathbf{k})=(k/4\pi)\int d\Omega_{k''}f^*(\mathbf{k}'\leftarrow\mathbf{k}'')f(\mathbf{k}''\leftarrow\mathbf{k}).$$

(E), and hence (L), are only valid if the Wick rotation is valid, that is, for $E<m_1+m_2+M$.

On the other hand, elastic unitarity below

$$E=m_1+m_2+M$$

may not be physically correct even though our BS equation obeys it. This possibility arises if there is a bound state of particles 1 and 2. Such a state is predicted by the BS equation itself, if λ is large enough. In this case, the ladder approximation may be insufficient even for low-energy scattering where it is expected to work best.

C. Variational Principles

The BS analog of the Schwinger variational principle[16] can be taken over from nonrelativistic quantum

[16] J. Schwinger, Harvard University, 1947 (unpublished); and J. M. Blatt and J. D. Jackson, Phys. Rev. **76**, 18 (1949).

mechanics directly. It is

$$[8\pi Ef(\mathbf{k}'\leftarrow\mathbf{k})]$$
$$=\int dx\, e^{-i\mathbf{k}'\cdot\mathbf{r}}V(x)\varphi_{\mathbf{k}}(x)+\int dx\,\varphi_{\mathbf{k}'}(-x)V(x)e^{i\mathbf{k}\cdot\mathbf{r}}$$
$$-\int dx\,\varphi_{\mathbf{k}'}(-x)V(x)\varphi_{\mathbf{k}}(x)+\int dxdx'\,\varphi_{\mathbf{k}'}(-x)V(x)$$
$$\times H(x,x')V(x')\varphi_{\mathbf{k}}(x'). \quad (4.12)$$

It is easy to verify that if $\varphi_{\mathbf{k}}$, $\varphi_{\mathbf{k}'}$ are taken as trial functions, then the right-hand side of (4.12) is an extremum when (A) and (H) above are satisfied and the extremum value is $8\pi Ef(\mathbf{k}'\leftarrow\mathbf{k})$ as indicated.

The usual derivations of the Kohn method for scattering with the Schrödinger equation involve some partial integrations, and these may be difficult to study in our generalized problem. We now give a derivation (following Kato[17]) which proceeds from the Schwinger result, (4.12). Let $J_s(\varphi_{\mathbf{k}'},\varphi_{\mathbf{k}})$ represent the right-hand side of (4.12) in terms of trial functions $\varphi_{\mathbf{k}}(x)$, $\varphi_{\mathbf{k}'}(x)$. Define $\chi_{\mathbf{k}}(x)$ by

$$\varphi_{\mathbf{k}}(x)=e^{i\mathbf{k}\cdot\mathbf{r}}+\int H(x,x')V(x')\chi_{\mathbf{k}}(x')d^4x', \quad (4.13)$$

so that

$$V(x)\chi_{\mathbf{k}}(x)=\mathfrak{D}\varphi_{\mathbf{k}}(x), \quad (4.14)$$

with $\mathfrak{D}$ the differential operator of the BS equation. Now write ($\mathfrak{D}^{(-)}$ is formed from $\mathfrak{D}$ by $x\rightarrow -x$)

$$[8\pi Ef(\mathbf{k}'\leftarrow\mathbf{k})]$$
$$=J_s(\chi_{\mathbf{k}'},\chi_{\mathbf{k}})+\int\{[\mathfrak{D}^{(-)}-V(x)]\varphi_{\mathbf{k}'}(-x)\}V^{-1}(x)$$
$$\times[\mathfrak{D}-V(x)]\varphi_{\mathbf{k}}(x)d^4x. \quad (4.15)$$

One sees that the right-hand side of (4.15) is stationary about the true value of φ (and thus also of χ), since each term of (4.5) is separately stationary and the stationary value is clearly $8\pi Ef(\mathbf{k}'\leftarrow\mathbf{k})$ as indicated. Now after some algebraic manipulations, not involving partial integrations, we get

$$[8\pi Ef(\mathbf{k}'\leftarrow\mathbf{k})]=\int e^{-i\mathbf{k}'\cdot\mathbf{r}}V(x)\chi_{\mathbf{k}}(x)d^4x$$
$$-\int\varphi_{\mathbf{k}'}(-x)[\mathfrak{D}-V(x)]\varphi_{\mathbf{k}}(x)d^4x. \quad (4.16)$$

This is the generalized Kohn variational principle. The first term

$$\int e^{-i\mathbf{k}'\cdot\mathbf{r}}V(x)\chi_{\mathbf{k}}(x)d^4x \quad (4.17)$$

is identified from (4.13) as the trial scattering amplitude describing the asymptotic behavior of the trial function $\varphi_{\mathbf{k}}(x)$.

Generally, one would imagine that the Kohn form (4.16) would be easier to apply in practice than the Schwinger form (4.12), because the former involves the local operator $\mathfrak{D}-V$ while for the latter one needs to evaluate the double integral

$$\int dx\int dx'\,\varphi_{\mathbf{k}'}(-x)V(x)H(x,x')V(x')\varphi_{\mathbf{k}}(x'), \quad (4.18)$$

and it looks as if this would be much harder to do. However, there is an important constraint in the Kohn procedure: The trial function must have the correct asymptotic *form*, as shown by (4.13). In ordinary Schrödinger problems, one can easily write for a trial function

$$\varphi=(\sin(kr+\delta))/kr+\text{short-range terms} \quad (4.19)$$

and then proceed to use (4.16). However, for our Bethe-Salpeter problem, the exact asymptotic form of φ is a complicated thing [see Eq. (2.40)], and we have not been able to see how the Kohn method can be put to practical use here.

V. SUMMARY

We have shown, by rather ordinary analytical and numerical techniques, that a Bethe-Salpeter equation can be understood and solved almost as expeditiously as its nonrelativistic counterpart, the Schrödinger equation. The results are not yet directly applicable to physical problems because nothing has yet been said about spin, inelastic channels, nonlocal interactions, and so forth. We expect to treat some of these questions shortly.

The present study has emphasized some "traditional" language—coordinate space, wave function, variational principles—rather than the contemporary language of S-matrix theory. We note, however, that in the final working formula (3.5) the wave function has disappeared from view, and what remains (the U function) is really the T matrix in some slight disguise. One is then led to ask if our complete study of the Wick rotation is really necessary, since, at the final stage, it may be viewed merely as a trick for evaluating integrals in a variational principle for the original Lorentz problem. The reply is that a good understanding of the orders of magnitude of functions and integrals, which is facilitated by the rotation, is very helpful in the construction of analytical and numerical procedures.

To conclude, we refer again to the data presented and discussed in Sec. III. There it was seen that by direct application of the numerical program, and with the appropriate supplement of familiar Born approximation and nonrelativistic results, we could cover the entire range of values of the parameters l (orbital

[17] T. Kato, Phys. Rev. **80**, 475 (1950).

angular momentum), M (exchanged-particle mass), and λ (interaction strength). Thus within the restrictions of our model—the ladder approximation with $m_1=m_2$, and energies below the inelastic threshold—we have provided a well-rounded picture of scattering with the Bethe-Salpeter equation.

APPENDIX A: AN INTEGRAL FORMULA

Consider

$$F(x,x')=\int\frac{d^n p d^{n'}p' e^{i(px-p'x')}}{(p^2-p'^2+Q^2)^{1+\mu}}, \tag{A1}$$

where p, x are vectors in n-dimensional Euclidean space, p', x' are vectors in an n'-dimensional Euclidean space, and Q is a complex number in the fourth quadrant so that $\mathrm{Im}Q^2<0$. As a function of any component of p', the integrand is seen to have poles only in the second and fourth quadrants; hence the contours of integration over p' can be rotated counterclockwise to the imaginary axes. In order to keep the product $p'x'$ real, we simultaneously carry out an analytic continuation of $F(x,x')$, rotating the components of x' clockwise by an equal amount. This can be formalized by setting $p'=qe^{i\varphi}$, $x'=ye^{-i\varphi}$ with q, y real and letting φ vary from zero to $\frac{1}{2}\pi-\delta$. Making these changes in (A1) and setting $q\rightarrow -q$, we get

$$F(x,x')=i^{n'}\int\frac{d^n p d^{n'}q e^{i(px+qy)}}{(p^2+q^2+Q^2)^{1+\mu}}. \tag{A2}$$

Now transform to spherical coordinates in $n+n'$ dimensions: Set

$$P=(p^2+q^2)^{1/2},\quad R=(x^2+y^2)^{1/2},\quad n+n'=2\nu+2, \tag{A3}$$

and do the angular integration first[18]:

$$\begin{aligned}F(x,x')&=\frac{i^{n'}2(\pi)^{\nu+1/2}}{\Gamma(\nu+\frac{1}{2})}\int_0^\infty\frac{P^{2\nu+1}dP}{(P^2+Q^2)^{1+\mu}}\\&\quad\times\int_0^\pi d\vartheta\,(\sin\vartheta)^{2\nu}e^{iPR\cos\vartheta}\\&=i^{n'}(2\pi)^{\nu+1}\int_0^\infty\frac{P^{2\nu+1}dP}{(P^2+Q^2)^{1+\mu}}\frac{J_\nu(PR)}{(PR)^\nu}\\&=\frac{i^{n'}(2\pi)^{\nu+1}}{2^\mu\Gamma(1+\mu)}\left(\frac{Q}{R}\right)^{\nu-\mu}K_{\nu-\mu}(QR).\end{aligned} \tag{A4}$$

The analytic continuation of x' may now be retraced to its starting point; then R becomes equal to $(x^2-x'^2)^{1/2}$, where the root is to be taken so that R is always in the first quadrant. This gives us the result that QR, the argument of the Bessel function in our result (A4), is always in the right half plane, although sometimes only by an infinitesimal amount.

The applications needed for the present paper are:

$n=3$, $n'=1$, $\nu=1$, $\mu=1$:

$$\int\frac{d^4p\,e^{ip(x-x')}}{(\mathbf{p}^2-p_0^2+\beta^2-k^2-i\epsilon)^2}=2\pi^2 iK_0((\beta^2-k^2)^{1/2}|x-x'|); \tag{A5}$$

$n=4$, $n'=1$, $\nu=\frac{3}{2}$, $\mu=1$: [note $K_{1/2}(z)=(\pi/2z)^{1/2}e^{-z}$]

$$\int\frac{d\beta d^4p\,e^{i\beta(t-t')}e^{ip(x-x')}}{(\beta^2+\mathbf{p}^2-p_0^2-k^2-i\epsilon)^2}=2i\pi^3\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}; \tag{A6}$$

$n=3$, $n'=1$, $\nu=1$, $\mu=0$:

$$\int\frac{d^4p\,e^{ip(x-x')}}{(\mathbf{p}^2-p_0^2+M^2-i\epsilon)}=i(2\pi)^2\frac{M}{|x-x'|}K_1(M|x-x'|). \tag{A7}$$

APPENDIX B: CALCULATIONAL TECHNIQUES

A formally simple choice of basis would be

$$\tilde{u}_l(|\mathbf{p}|,p_4)=\sum_n g_{nl}(p)\sin^l\vartheta\,C_{n-l}^{l+1}(\cos\vartheta), \tag{B1}$$

using the Gegenbauer polynomials. Then we would find

$$u_l(r,\tau)=\sum_n f_{nl}(R)\sin^l\theta\,C_{n-l}^{l+1}(\cos\theta), \tag{B2}$$

where

$$f_{nl}(R)=\frac{(-1)^{(n-l)/2}}{4\pi^2}\int_0^\infty p^3dp\frac{J_{n+1}(pR)}{pR}g_{nl}(p), \tag{B3}$$

$$R=(r^2+\tau^2)^{1/2},\quad \cos\theta=\tau/R.$$

Then with $g_{nl}(p)$ expanded as

$$\sum_m a_{nm}2\pi k^l\frac{(p/\alpha)^n}{(1+p^2/\alpha^2)^{n+m}}, \tag{B4}$$

we would find the corresponding terms in $f_{nl}(R)$ to be

$$\sum_m a_{nm}\frac{k^l}{4\pi}\frac{(-1)^{(n-l)/2}\alpha^4}{(n+m-1)!}\left(\frac{\alpha R}{2}\right)^{n+m-2}K_{|2-m|}(\alpha R). \tag{B5}$$

Since the potential V is a function of R only, we can then do the angular integral in the second term of (3.14) and get

$$-\frac{1}{\lambda}\sum_{nm}\sum_{n'm'}a_{nm}a_{n'm'}Y_{n'm'}^{nm}, \tag{B6}$$

[18] We take the definitions of the special functions and integrals involving them from W. Magnus and F. Oberhettinger, *Functions of Mathematical Physics* (Chelsea Publishing Company, New York, 1949).

where

$$Y_{n'm'}{}^{nm}=\delta_{nn'}\frac{\alpha^3(n+l+1)!B(m,m',2n+m+m')k^{2l}}{(n+1)(n-l)!(l!)^2M(n+m-1)!(n+m'-1)!2^{2n+2l+m+m'+1}} \tag{B7}$$

with the integrals

$$B(m,m',N)\equiv\int_0^\infty dt\frac{K_{m-2}(t)K_{m'-2}(t)t^N}{K_1\left(\frac{M}{\alpha}t\right)}. \tag{B8}$$

A numerical table of these integrals may be safely (i.e., without serious loss of accuracy due to subtraction of nearly equal terms) built by using the recursion formula

$$K_\nu(t)=\frac{(2\nu-2)}{t}K_{\nu-1}(t)+K_{\nu-2}(t),$$

after we have first got the values for m, $m'=1$ and 2 for all N needed. These three sets of integrals were obtained directly by numerical integration.

The trial functions actually used were not exactly in the form (B1), but can be readily written as linear combinations of these terms. The choice (3.17) was made in order to reduce the loss of accuracy encountered in combining the several parts of the integral involving the Green's function in (3.14). Consequently, some slight loss of accuracy was introduced into the V^{-1} terms described above, but this seemed a reasonable compromise. With the actual basis (3.17), the last term of (3.14), the H matrix, can be given as

$$\sum a_{nm}a_{n'm'}Z_{nm}{}^{n'm'},$$

where

$$Z_{nm}{}^{n'm'}=k^{2l}\sum_\sigma(-1)^\sigma\binom{l+1}{\sigma}$$
$$\times A(\tfrac{1}{2}(n+n')+1-l+\sigma,\ n+n'+m+m',\ \tfrac{1}{2}(n+n')+2), \tag{B9}$$

in terms of the integrals ($x=p/\alpha$)

$$A(L,M,N)=\mathfrak{R}\int_0^\infty\frac{dx\,x^{2N-1}}{(1+x^2)^M}\frac{1}{\pi}$$
$$\times\int_0^\pi\frac{d\vartheta\cos^{2L-2}\vartheta}{[(x^2-k^2/\alpha^2-i\epsilon)^2+(4\omega^2/\alpha^2)x^2\cos^2\vartheta]}. \tag{B10}$$

A safe recursion formula for going from high to low M values is

$$A(L,M,N)=A(L,M+1,N+1)+A(L,M+1,N); \tag{B11}$$

and, for k^2 not too negative, we can safely step up in L with the formula

$$(4\omega^2/\alpha^2)A(L,M,N)=-A(L-1,M-1,N)+(1+2k^2/\alpha^2)A(L-1,M,N)$$
$$-(k^4/\alpha^4)A(L-1,M,N-1)$$
$$+D(L-1)E(M,N-1),\quad\text{for}\quad L>1, \tag{B12}$$

where

$$D(L)=\frac{(L-3/2)!}{(L-1)!(-1/2)!}$$

and

$$E(M,N)=\frac{(N-1)!(M-N-1)!}{2(M-1)!}.$$

Thus, we must start by evaluating the integrals A with $L=1$ at the maximum M for the set of N values needed. Following the rule (3.10) we first do the ϑ integral for negative k^2, finding

$$\frac{1}{\pi}\int_0^\pi\frac{d\vartheta}{(x^2-k^2/\alpha^2-i\epsilon)^2+(4\omega^2/\alpha^2)\cos^2\vartheta}$$
$$=\frac{1}{S(x)(x^2-k^2/\alpha^2-i\epsilon)}, \tag{B13}$$

where

$$S(x)=+((x^2-k^2/\alpha^2)^2+4(\omega^2/\alpha^2)x^2)^{1/2}. \tag{B14}$$

We now analytically continue to the correct k^2 value under the x integral, and, taking the real part in the familiar way, obtain

$$A(1,M,N)=\text{P.v.}\int_0^\infty\frac{dx\,x^{2N-1}}{(1+x^2)^M}\frac{1}{S(x)(x^2-k^2/\alpha^2)}. \tag{B15}$$

These integrals were evaluated by numerical integration after the principal-value part was subtracted off and done analytically. The actual integration variable used was

$$x^{1/2},\quad\text{for}\quad 0\leqq x\leqq 1$$

and

$$x^{-1/2},\quad\text{for}\quad 1\leqq x\leqq\infty; \tag{B16}$$

this was done to accommodate the rather sharp behavior of $S(x)$ for small values of k^2/α^2.

The evaluation of these starting numerical integrals for the H and V^{-1} matrix elements took 10 to 15 sec (given fixed values for M, k, α); then we could do many l, λ values at a fraction of a second each. All the data in the figures and tables in this paper were collected in about 12 min of computer (IBM 7094) time.