Solution of a Bethe–Salpeter Equation*

CHARLES SCHWARTZ

Department of Physics, University of California, Berkeley, California

(Received 24 September 1964)

The Bethe-Salpeter equation arising from a ϕ^3 theory is solved numerically for several energies (bound states only) and symmetry states. The method used is a variational calculation in the four-dimensional Euclidean space arrived at by the transformation due to Wick. A high degree of accuracy is achieved by using only a very small amount of common computing machine capabilities.

WHILE there is no general theory for the interactions of the elementary particles, theorists wish to have some equations to play with, in attempts to describe experimental phenomena, even though the truth contained in these equations is admittedly well circumscribed. Thus it is that relativistic two-body equations, similar to that written down by Bethe and Salpeter[1] for quantum electrodynamics, have recently become quite popular objects of study. The main drawback seems to be that the equation can not be solved generally either by a perturbation expansion or by reduction to a single-variable differential or integral equation; and it is therefore considered by many people to be intractable. (Explicit solutions have been found for the special cases: exchanged-particle mass equals zero[2]; and total energy equals zero, for a few particular potentials.[3]) It was our thought that what is intrinsically a problem in two dimensions should be easily solvable with the help of an electronic computer; and in this paper we report the verification of this guess for a particular simple Bethe-Salpeter equation. All the important analytical work, done a decade ago, may be learned from the very clear paper by Wick[4]; we have simply added the numerical details.

I. THE BETHE-SALPETER EQUATION

We consider the problem of two scalar particles (masses m_1 and m_2) which interact via the exchange of a third scalar particle (mass κ). For a bound state of total energy E, the center-of-mass coordinate can be removed, and one has then to deal with a single four-dimensional independent variable, the internal coordinate $\mathbf{x}$. Wick then shows that one can transform to an imaginary relative-time variable, so that one is left with the following eigenvalue problem in a four-dimensional Euclidean space. We have taken $m_1 = m_2$ as the unit of length and use the parameter

$$\eta = E/(m_1 + m_2); \quad 0 \le \eta < 1. \tag{1}$$

$$L\psi = \lambda V\psi \tag{2}$$

[1] E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951); also J. Schwinger, Proc. Natl. Acad. Sci. U. S. **37**, 455 (1951).
[2] R. E. Cutkosky, Phys. Rev. **96**, 1135 (1954).
[3] A. Bastai, L. Bertocchi, G. Furlan, and M. Tonin, Nuovo Cimento **30**, 1532 (1963).
[4] G. C. Wick, Phys. Rev. **96**, 1124 (1954).

where

$$L = (-\Box + 1 - \eta^2)^2 - 4\eta^2(\partial^2/\partial x_4^2), \tag{3}$$

$$\Box = \sum_{i=1}^{4} \frac{\partial^2}{\partial x_i^2}; \tag{4}$$

and the interaction is

$$V = \frac{1}{\pi^2} \int d^4q \frac{e^{-i\mathbf{q}\cdot\mathbf{x}}}{q^2 + \kappa^2} = \frac{4\kappa}{R} K_1(\kappa R), \tag{5}$$

where

$$R = \left(\sum_{i=1}^{4} x_i^2\right)^{1/2}. \tag{6}$$

K_1 is the modified Bessel function with the properties

$$K_1(z) \underset{z \to 0}{\longrightarrow} \frac{1}{z} + \tfrac{1}{2}z \ln z + \cdots \tag{7}$$

$$\underset{z \to \infty}{\longrightarrow} (\pi/2z)^{1/2} e^{-z}. \tag{8}$$

With respect to the volume integral

$$\int d^4x$$

L is self-adjoint, and since both L and V are positive operators the Rayleigh quotient

$$[\lambda] = \int d^4x \ \psi^* L\psi \Big/ \int d^4x \ \psi^* V\psi \tag{9}$$

gives stationary approximations to the true value of λ which are also upper bounds. [We consider η as given and λ to be found; but once the calculation of $\lambda(\eta)$ has been completed we can of course recover $\eta(\lambda)$ without loss of accuracy.]

Equation (2) is invariant under rotations in the three-dimensional subspace (but not in the complete four-space except for $\eta = 0$). Therefore we can make the usual separation of the orbital angular momentum.

$$\psi = Y_{lm}(\vartheta, \varphi)\chi_l(R, \theta), \tag{10}$$

CHARLES SCHWARTZ

where we have taken the angles to be defined by

$$x_4 = R \cos\theta,$$
$$x_3 = R \sin\theta \cos\vartheta,$$
$$x_2 = R \sin\theta \sin\vartheta \cos\varphi, \tag{11}$$
$$x_1 = R \sin\theta \sin\vartheta \sin\varphi,$$

and the Y_{lm} are conventionally normalized as

$$\int_0^{2\pi} \int_0^{\pi} Y_{lm}^*(\vartheta,\varphi) Y_{l'm'}(\vartheta,\varphi) \sin\vartheta \, d\vartheta \, d\varphi = \delta_{ll'}\delta_{mm'}. \tag{12}$$

$$|m| \le l = 0, 1, 2, \cdots.$$

It will be convenient to introduce the four-dimensional spherical harmonics

$$|nlm\rangle = Y_{lm}(\vartheta,\varphi)[(2^{2l+1}(n+1)(n-l)\,!\,l\,!^2)/$$
$$(\pi(n+l+1)\,!)]^{1/2} \sin^l\theta \, C_{n-l}^{l+1}(\cos\theta),$$
$$l \le n = 0, 1, 2, \cdots \tag{13}$$

involving the Gegenbauer[5] polynomials C_n^ν. These are orthonormal with the volume of integration for the third angle as

$$\int_0^\pi \sin^2\theta \, d\theta, \tag{14}$$

and are eigenfunctions of the four-dimensional Laplacian.

$$\Box \, |nlm\rangle = \left(\frac{\partial^2}{\partial R^2} + \frac{3}{R}\frac{\partial}{\partial R} - \frac{n(n+2)}{R^2} \right) |nlm\rangle. \tag{15}$$

We should note that our equation is also invariant under the "time reflection" $x_4 \to -x_4$, and under this transformation our four-dimensional harmonics transform as

$$|nlm\rangle \to (-1)^{n-l}|nlm\rangle. \tag{16}$$

The quantum numbers of a state will thus be given as $l^\pm$, with the upper or lower sign according as $n-l$ is always even or odd. (The eigenvalues are $(2l+1)$-fold degenerate in the m index, which we shall drop; and instead of "principal quantum numbers" for each symmetry state we shall simply say "lowest" or "first excited," etc.)

The mixing of the n index is of course due to the term $\partial^2/\partial x_4^2$, and we resolve this in our polar coordinates as

$$(\partial/\partial x_4) = \cos\theta(\partial/\partial R) - (1/R)\sin\theta(\partial/\partial\theta). \tag{17}$$

Then we calculate the following formulas:

$$\cos\theta|nlm\rangle = A_n^{(l)}|(n-1)lm\rangle + A_{n+1}^{(l)}|(n+1)lm\rangle \tag{18a}$$

and

$$\sin\theta(\partial/\partial\theta)|nlm\rangle = -(n+2)A_n^{(l)}|(n-1)lm\rangle$$
$$+ nA_{n+1}^{(l)}|(n+1)lm\rangle, \tag{18b}$$

with

$$A_n^{(l)} = ((n-l)(n+l+1)/4n(n+1))^{1/2}. \tag{19}$$

$$A_0^{(0)} = 0.$$

Thus we arrive at the structure for our trial function

$$\psi_{lm} = \sum_n |nlm\rangle f_{nl}(R), \tag{20}$$

where

$$n = l, l+2, l+4, \cdots \tag{21}$$

for the l^+ states; or

$$n = l+1, l+3, l+5, \cdots \tag{22}$$

for the l^- states.

Now we must choose the radial functions $f_{nl}(R)$. First, examining near the origin, one easily sees (following Wick) the required behavior

$$f_{nl}(R) \sim R^n \quad (R \to 0). \tag{23}$$

Noting that L and V are even functions of R one might guess that the next term after R^n in the small R expansion of f_{nl} would be R^{n+2}. However because of the $1/R^2$ singularity of the potential V one finds the next term to be $\sim R^{n+2} \ln R$. It would be inconvenient to put logarithmic functions into the basis; and so we shall compensate by allowing the intermediate powers of $R - R^n$, R^{n+1}, R^{n+2}, R^{n+3}, $\cdots$ —in order to gain more flexibility in this region. (An analogous situation, where known logarithm terms are represented by powers, may be found in the study of the helium atom.[6])

Now look at $R \to \infty$ (following Wick again):

$$\psi \sim e^{-g(\theta)R}, \tag{24}$$

where

$$g = [1 + \eta^2 \cos 2\theta \pm 2\eta \cos\theta(1 - \eta^2 \sin^2\theta)^{1/2}]^{1/2}. \tag{25}$$

This may be simplified in particular directions: Along the time-like direction

$$\psi \sim e^{-[(1-\eta)R \cos\theta]}; \tag{26}$$

while along the space-like direction

$$\psi \sim \exp[-(1-\eta^2)^{1/2}R \sin\theta], \tag{27}$$

which is familiar from the Schrödinger equation.

We shall, for simplicity, try to use just a single, direction-independent exponential

$$f \sim e^{-\alpha R} \quad (R \to \infty). \tag{28}$$

If the parameter α is taken between $(1-\eta)$ and $(1-\eta^2)^{1/2}$, we may expect fairly good convergence (see Ref. 6) provided η is not too close to one. (Thus we expect the most difficulty in the limit of weak binding.)

[5] See W. Magnus and F. Oberhettinger, *Functions of Mathematical Physics* (Chelsea Publishing Company, New York, 1954), p. 76.

[6] See C. Schwartz, in *Methods in Computational Physics* (Academic Press Inc., New York, 1963), Vol. II.

TABLE I. Lowest $l=0^+$ eigenvalue (λ).

K	Size of matrix	$\eta=0.0$, $\alpha=1.0$	$\eta=0.9$, $\alpha=0.3$
0	1	3.52070	2.90247
1	2	3.50070	2.18762
2	4	3.45316	1.84898
3	6	3.43165	1.76092
4	9	3.42375	1.71556
5	12	3.42083	1.69850
6	16	3.41966	1.68794
7	20	3.41915	1.68179
8	25	3.41888	1.67729
Extrapolation		3.4185 ± 0.0002	1.667 ± 0.008

Our final structure for the trial function is now

$$\psi_{lm}=\sum_{n\geq l}\sum_{k\geq n}C_{n,k}\,|nlm\rangle R^{k}e^{-\alpha R}. \qquad (29)$$

With this basis the matrix elements of L are simply computed, and for those of V we need only the integrals

$$J_{n}=b\int_{0}^{\infty}dt\,e^{-at}t^{n+1}K_{1}(bt). \qquad (30)$$

These may be tabulated, along with

$$I_{n}=\int_{0}^{\infty}dt\,e^{-at}t^{n}K_{0}(bt), \qquad (31)$$

by means of the recursion formulas

$$I_{n}=(1/a)[nI_{n-1}-J_{n-1}], \qquad (32)$$

$$J_{n}=(a/a^{2}-b^{2})[nJ_{n-1}-(n+1)(b^{2}/a)I_{n}], \qquad (33)$$

starting from

$$I_{0}=(\ln\{[a+(a^{2}-b^{2})^{1/2}]/b\}/(a^{2}-b^{2})^{1/2}), \qquad (34)$$

$$J_{0}=-b(\partial/\partial b)I_{0}, \qquad (35)$$

TABLE II. Computed values of λ $(m_1=m_2=\kappa)$.

State $\backslash\eta$	0.0	0.4	0.6	0.8	0.9
Lowest 0^+	3.419	3.115	2.718	2.10	1.67
(1st ex.) 0^+	16.92	15.7	14.3	12.0	10.1
Lowest 0^-	16.38	15.85	15.18	14.2	13.6
Lowest 1^+	16.38	15.23	13.74	11.4	9.80

(or the appropriately modified formulas for the cases $a\leq b$). The formulas (32), (33) involve numerical cancellations which get worse as the order n increases. This may be overcome simply by computing in extra precision; or one can compute I_n and J_n at the maximum value of n needed first (by means of well-convergent infinite series) and then safely iterate to the lower values of n.

II. RESULTS AND CONCLUSIONS

In our calculations the approximations were so ordered that one fixed the maximum power of R in (29), i.e., keep all $k\leq K$ and all $n\leq K$. Table I shows the convergence of the results at the two extreme energy values which we investigated (with $\kappa=m_1=m_2$). Table II summarizes all our numerical results.

It is clear from the contents of Table I that we have achieved a very efficient solution of the equation, since the results converge quite rapidly; each λ value was obtained at a cost of 2 to 4 sec computation time on an IBM 7090 computer. We could obviously have continued the calculations to get much more accurate results at still very little cost, or alternatively we see that this type of problem could be readily handled by computers much smaller and slower. We conclude that the Bethe-Salpeter equation—after being given the Wick treatment—may, in fact, as he predicted, be accurately solved by straightforward numerical means.

One can look forward to an enlargment of the scope of this work. A few coupled equations, as one would expect for particles with spin, should not offer new difficulties. For the treatment of more singular (repulsive) potentials one would use trial functions suitably modified at $R\to0$ (see Ref. 3), but no serious problems are expected. The major task remaining here is to see if we can carry out a similar analytical-numerical resolution of the Bethe-Salpeter equation for scattering states $(\eta\geq1)$.

The author is grateful to C. Zemach for essential encouragements.

Note added in proof. S. H. Vosko [J. Math. Phys. **1**, 505 (1960)] reports a numerical calculation of the Bethe-Salpeter equation for the case of very weak binding (η close to one), where the method described in the present paper would not work well.