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# Sum Rules for Inelastic Electron Scattering*

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Sum rules are constructed for the analysis of inelastic electron scattering at high energy (~150 Mev) from light nuclei. Effects taken into account are: nucleon charge, recoil, and magnetic-moment currents; exchange currents; finite nucleon size; nuclear center-of-mass motion; and the kinematical factors describing the correct relation between initial and final electron energies, the scattering angle, and the nuclear excitation energy. It appears that a sensitive test of the role of exchange currents in the nuclear ground state is provided by a sum rule for the energy-weighted cross section for fixed-momentum transfer:

$$\sigma_E=\int_{|\mathbf{q}|=\text{const}} \epsilon\sigma(\epsilon,q)d\epsilon.$$

## I. INTRODUCTION

THIS paper describes the construction of sum rules for the analysis of inelastic scattering of high-energy electrons from light nuclei. Sum rules, rather than detailed inelastic spectrum calculations, are valuable because they do not require knowledge of the much too complicated final-state wave functions for excited nuclear systems. The purpose of such sum rules is to provide information about the nuclear Hamiltonian and the structure of the ground state beyond the static charge distributions, as determined by elastic scattering. In particular, one may inquire into the role of charge exchange forces which give rise to exchange currents, of finite size of the elementary nucleons, and of internucleon correlations in the ground state.

The original dipole sum rule of Thomas, Reiche, and Kuhn[1] for the absorption of light by atoms was extended by Heisenberg[2] in 1931, to include effects of retardation and indistinguishability of the atomic electrons in atomic x-ray studies. His result, expressed as a ratio of the total cross section for absorption of x-rays of momentum $\hbar\mathbf{q}$ by an atom of atomic number $Z$ to the single-particle cross section, is $Z+Z(Z-1)f_2$, where

$$f_2=\int|\psi_0(\mathbf{r}_1\cdots\mathbf{r}_Z)|^2e^{i\mathbf{q}\cdot(\mathbf{r}_1-\mathbf{r}_2)}d\tau_1\cdots d\tau_Z \qquad (1)$$

is the form factor for two-body correlations and corresponds to the square of the elastic scattering (i.e., one-body) form factor in the approximation of single-particle wave functions (not antisymmetrized). Feenberg and Siegert[3] first pointed out that exchange currents arising from charge exchange forces, which may be operating to bind the nuclear ground state, will modify the sum rule, Eq. (1), for the nuclear photoeffect. Starting with the work of Levinger and Bethe[4] in 1950, these modifications have been studied extensively in recent years with the aim of tying down the role of exchange forces by their contribution to the observed photoeffect in light nuclei.

In extending sum rules to apply to the case of electron-nuclear scattering, it is again of interest to see what can be learned about exchange forces and two-body correlations. The electron and photonuclear interactions differ in two principal features: (1) the Möller potentials for the electron interaction contain important scalar and longitudinal components in addition to the transverse parts operating in the photo process; (2) the possibility of precise measurements with monoenergetic electron beams means not only a more accurate comparison of calculations with experiments, but also a greater flexibility in the type of sum rules which can be constructed and studied. An approximate ratio of the form (1) can also be written for total electron cross sections taking into account only the Coulomb interaction with each nuclear particle. The four main corrections to this result which will be analyzed in this paper are (a) contributions from recoil and spin currents of the nucleons, (b) contributions from exchange currents arising in connection with charge exchange forces operating in nuclei, (c) kinematical corrections which arise from use of the energy and momentum conservation laws to relate the momentum transferred by the electron to the energy of the final nuclear states, and (d) corrections resulting from the finite extension of the nucleon charge and current distributions.

In this work, we limit our considerations to light nuclei ($Z\leq 8$) so that the interaction with the electron may be accurately treated in the first Born approximation. A correct treatment of the nuclear center-of-mass motion is thus important, and this will be given, using the method of Gartenhaus and Schwartz.[5] This correction is of interest in relating the two-body correlation function, such as given in Eq. (1), with the elastic

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[2] W. Heisenberg, Physik. Z. **32**, 737 (1931).
[3] E. Feenberg, Phys. Rev. **49**, 328 (1936); A. J. F. Siegert, Phys. Rev. **52**, 787 (1937).
[4] J. S. Levinger and H. A. Bethe, Phys. Rev. **78**, 115 (1950).
[5] S. Gartenhaus and C. L. Schwartz, Phys. Rev. **108**, 482 (1957).

scattering form factor

$$f_{\mathrm{el}}=\int|\psi_0(\mathbf{r}_1\cdots\mathbf{r}_A)|^2 \times\exp[i\mathbf{q}\cdot(\mathbf{r}_1-A^{-1}\sum_i \mathbf{r}_i)]d\tau_1\cdots d\tau_A, \quad (2)$$

when shell-model wave functions are used.

Typical experimental conditions of interest here correspond to momentum transfers $|\mathbf{q}|\lesssim 200$ Mev/c, giving nucleon recoil velocities $v/c\lesssim 1/5$. We put 200 Mev/c as an upper limit on our considerations because our extensive ignorance of the relativistic many-body problem restricts us to a nonrelativistic one-time description of the nuclear system as a collection of "Pauli" particles with static interaction potentials. This means that we include corrections to the interaction currents which are related by the continuity equation,

$$\operatorname{div}\mathbf{j}+(i/\hbar)[H,\rho]=0, \quad (3)$$

to the appearance of charge exchange potentials in the nuclear Hamiltonian; but we ignore those meson resonance effects which are known to be of major importance for the nuclear photoprocesses at energies $\gtrsim 100$ Mev.

Because we limit our considerations to $|\mathbf{q}|\lesssim 200$ Mev/c, charge and single-particle magnetic moment scattering are the dominant contributions to the total cross sections for incident electrons. The resulting sum rule for fixed angle of electron scattering is insensitive to exchange current contributions, and the two-body correlation function $f_2$ in Eq. (1) is pretty well tied down by the two requirements that $\psi_0$ give the observed charge density, and that the exclusion principle be considered. Quantitative results are presented in Sec. III and the above given conclusion is the same whether we consider the low-resolution experiment which accepts electrons of all energies at a given scattering angle or the high-resolution experiment which selects both angle and final energy so as to keep the momentum transfer constant.

However, it appears that a sensitive test of the role of exchange currents in the nuclear ground state is provided by a sum rule for the energy-weighted cross section

$$\sigma_E=\int_{|\mathbf{q}|=\mathrm{const}}\epsilon\sigma(\epsilon,q)d\epsilon, \quad (3')$$

where $\epsilon$ is the energy of each final nuclear state (relative to its center-of-mass) and the momentum transfer $|\mathbf{q}|$ is held constant. In Sec. IV it is shown that the exchange currents contribute as much as 40% of $\sigma_E$ for Rosenfeld-type forces.

The relative importance of the contribution of the exchange currents to $\sigma_E$ may be understood as follows. Notice that $\sigma_E$ weights against the elastic in favor of the inelastic scattering, and recall from Eq. (3) that the exchange currents contribute, in Born approximation, through what may be described as an oscillating component of the charge distribution of the nucleus, which exchanges energy with the scattering electron.

## II. FORMAL DEVELOPMENT

In this section we construct a sum rule for the total cross section, $\int dk_f(d^2\sigma/d\Omega dk_f)$, for an electron to be scattered into a given solid angle $d\Omega$ by a nucleus of charge $Ze$. The electron is treated in Born approximation and the relevant matrix element is

$$(\mathrm{M.E.})_{f0}=\int d^4x\langle f|\Gamma_\mu(x)|0\rangle j_\mu(q)e^{iq\cdot x}/q^2, \quad (4a)$$

where $j_\mu(q)=\langle\bar{u}(k_f)|\gamma_\mu|u(k_0)\rangle$ is the electron current corresponding to a four-momentum transfer $q^2=(\Delta E)^2-\mathbf{q}\cdot\mathbf{q}$; $\bar{u}=u^\dagger\beta$, $\gamma_\mu=(\beta,\beta\boldsymbol{\alpha})$, and $\langle f|\Gamma_\mu(x)|0\rangle$ expresses the nuclear transition current densities at $x$ which we take of the following form:

$$\langle f|\Gamma_\mu(x)|0\rangle=\int d^4x_1\cdots d^4x_A\,\bar{\psi}_f(x_1,\cdots x_A) \times\sum_\alpha\sum_{i=1}^{A}\rho_i^\alpha(x-x_i)\Gamma_\mu^\alpha(i)\psi_0(x_1,\cdots x_A). \quad (4b)$$

In Eq. (4b) the index $\alpha$ labels the several types of current interactions, such as charge, magnetic moment, and exchange, with $\rho^\alpha(x-x_i)$ the invariant density function for the electromagnetic interactions of the $i$th-bound nucleon. As a consequence of the assumed one-particle coordinate dependence of the densities $\rho^\alpha$, the matrix element Eq. (4a) becomes

$$\sum_\alpha\sum_i\frac{j_\mu(q)}{q^2}f_i^\alpha(q)\langle f|\Gamma_\mu^\alpha(i)e^{i\mathbf{q}\cdot x_i}|0\rangle, \quad (5)$$

with

$$f_i^\alpha(q)=\int d^4x\,\rho_i^\alpha(x)e^{iq\cdot x}$$

the form factor corresponding to the four-momentum transfer $q^2$.[6]

Reducing to a one-time description of the nuclear particles, we obtain finally,

$$(\mathrm{M.E.})_{f0}=\delta(\mathrm{Energy})\sum_{\alpha,i}\int(d\tau)\varphi_f^*(\mathbf{r}_1,\cdots\mathbf{r}_A) \times e^{i\mathbf{q}\cdot\mathbf{r}_i}\Gamma_\mu^\alpha(i)\varphi_0(\mathbf{r}_1,\cdots\mathbf{r}_A)\frac{j_\mu(q)f_i^\alpha(q)}{q^2}, \quad (6)$$

[6] The general current operator deduced from a field-theoretic analysis of the many-body problem in quantum field theory has terms whose coordinate dependence cannot be expressed simply by the invariant $(x-x_i)^2$. Crossed diagram corrections to the ladder approximation to the Bethe-Salpeter equation for the two-body problem give rise to such terms. They may depend on the four-momentum transfer to various nucleons which $\neq q^\mu$ for inelastic processes. This uncertainty as to general form-factor arguments is unimportant in the energy regions to which we limit out discussions. This is because $(f_i^\alpha)^2$ lies within 20% of unity and the peak of the inelastic cross section corresponds closely to a direct ejection of a single target nucleon.

where $\varphi(\mathbf{r}_1\cdots\mathbf{r}_A)$ represents a space wave function for $A$ particles multiplied by $A$ two-component Pauli spinors, appropriately antisymmetrized. Kinematical errors introduced by a nonrelativistic description of the nucleon motion are $\sim q^2/M^2<5\%$ for the energy range considered here. There are also dynamical corrections of this same order to the single-nucleon current operator as is evident when the scattering of electrons by Dirac protons with added anomalous moment, as calculated by Rosenbluth, is compared with the analogous result for a Pauli proton. The form factors in Eq. (6) will be taken to be the same as those measured by Hofstadter[7] and collaborators for free nucleons. The best argument for this assumption is the observed absence of magnetic moment quenching in nuclei.[8]

There is no real meson production to states which can be reached at these energies, and in order to specify the current operator $\Gamma_\mu{}^\alpha(i)$ one assumes that all manifestations of the meson cloud not contained in the free nucleon's anomalous moment and form factors, can be summarized in static binding potentials. The nuclear Hamiltonian is thus written as

$$H=\sum_i\frac{p_i^2}{2M}+\sum_{i<j}V_0(i,j)+\sum_{i<j}\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_jV_\tau(i,j). \qquad (7)$$

The presence of the charge exchange interaction $V_\tau$ gives rise to the so-called "gauge-currents," which have been discussed extensively by Sachs[9] in terms of line integrals. Their presence is required to maintain the continuity equation. There is no unique treatment of these path-dependent line currents since only the divergence of the current is fixed by the continuity equation. For simplicity, we shall use straight-line integrals in discussing these terms. The uncertainty due to these terms is quite small because the over-all contribution of the gauge currents is small. However, we shall see that the charge exchange forces give rise to important and uniquely specified contributions to the energy-weighted cross sections.

With these assumptions, Eqs. (6) and (7) combine to give as the scattering matrix element[10]

$$(\text{M.E.})_{f0}=\delta(\text{Energy})\mathfrak{M}_{f0};$$

$$\mathfrak{M}_{f0}=\left(\frac{e}{q^2}\right)\sum_{i=1}^{A}\int d\tau_1\cdots d\tau_A\ \varphi_f^*(\mathbf{r}_1,\cdots\mathbf{r}_A)f(q)\Big[\langle u(k_f)|u(k_0)\rangle e_ie^{i\mathbf{q}\cdot\mathbf{r}_i}-\langle u(k_f)|\boldsymbol{\alpha}|u(k_0)\rangle$$
$$\cdot\Big\{(\mathbf{p}_ie^{i\mathbf{q}\cdot\mathbf{r}_i}+e^{i\mathbf{q}\cdot\mathbf{r}_i}\mathbf{p}_i)e_i/2M+i\boldsymbol{\sigma}_i\times\mathbf{q}e^{i\mathbf{q}\cdot\mathbf{r}_i}\mu_i/2M+2e\sum_{j(\neq i)}(\boldsymbol{\tau}_j\times\boldsymbol{\tau}_i)_3\int_{\mathbf{r}_j}^{\mathbf{r}_i}d\mathbf{s}\ V_\tau(r_{ij})\exp(i\mathbf{q}\cdot\mathbf{s})\Big\}\Big]$$
$$\times\varphi_0(\mathbf{r}_1,\cdots\mathbf{r}_A). \qquad (8)$$

The first thing which we do here is to extract the statement of momentum conservation from Eq. (8). This we shall do according to the method of Gartenhaus and Schwartz[5] in order to keep the interaction terms symmetrical in all the nuclear particles. The result of the Gartenhaus-Schwartz transformation is that the coordinates $\mathbf{r}_i$ in Eq. (8) are replaced by $\boldsymbol{\varrho}_i=\mathbf{r}_i-\mathbf{R}$, with $\mathbf{R}=(1/A)\sum_i\mathbf{r}_i$, and momentum operators $\mathbf{p}_i$ by $\boldsymbol{\pi}_i=\mathbf{p}_i-(1/A)\mathbf{P}$, with $\mathbf{P}=\sum_i\mathbf{p}_i$. The added terms generally contribute $1/A$ corrections and will be kept only as they appear in the leading terms of all subsequent calculations.

We next construct measurable cross sections from this matrix element which do not require any statements about the excited nuclear states, $\varphi_f$. The simplest problem from the experimental point of view is the low-resolution total cross section (elastic plus inelastic) for electrons of all energies emerging into a given solid angle $d\Omega$. To simplify the development, consider first just the first term in Eq. (8) corresponding to the Coulomb interaction. The cross section is

$$\sigma_cd\Omega_f=2\pi d\Omega_f\Big\langle\sum_f\int k_f^2dk_f\ \delta(\text{Energy})|\mathfrak{M}_{f0}|^2\Big\rangle, \qquad (9)$$

where $\sum_f$ denotes the sum over final nuclear states and $\langle\ \rangle$ indicates sum over final and average over initial electron spins. In the closure approximation this reduces to

$$\sigma_cd\Omega_f=\frac{e^2\cos^2(\theta/2)}{4k_0^2\sin^4(\theta/2)}\frac{d\Omega_f}{1+(2k_0/AM)\sin^2(\theta/2)}f^2(q)$$
$$\times\frac{1}{4}\frac{\text{Sp}}{1+\cos\theta}\int(d\tau)\varphi_0^*\sum_i\sum_j(1+\boldsymbol{\alpha}\cdot\hat{k}_0)e_i$$
$$\times\exp[-i\mathbf{q}\cdot\boldsymbol{\varrho}_i](1+\boldsymbol{\alpha}\cdot\hat{k}_f)e_j\exp[i\mathbf{q}\cdot\boldsymbol{\varrho}_j]\varphi_0$$
$$\equiv\sigma_0f^2(q)\mathfrak{F}d\Omega_f, \qquad (10)$$

with

$$\sigma_0=\frac{e^4}{4k_0^2}\frac{\cos^2(\theta/2)}{\sin^4(\theta/2)}\left[1+\frac{2k_0}{AM}\sin^2(\theta/2)\right]^{-1},$$

$$\mathfrak{F}=Z+Z(Z-1)\int(d\tau)\varphi_0^*\exp[i\mathbf{q}\cdot(\boldsymbol{\varrho}_i-\boldsymbol{\varrho}_j)]\varphi_0.$$

[7] Hofstadter, Bumiller, and Yearian, Revs. Modern Phys. (to be published).
[8] R. J. Blin-Stoyle, Revs. Modern Phys. 28, 75 (1956).
[9] R. G. Sachs, Phys. Rev. 74, 433 (1948).
[10] A common form factor $f$ is consistent with present data: $\rho_i(x)=\frac{1}{2}e(1+\tau_{3i})\rho(x)=e_i\rho(x)$, $\mu_i(x)\equiv\mu[\frac{1}{2}(1+\tau_{3i})+\kappa\tau_{3i}]\rho(x)=\mu_i\rho(x)$.

This result is the usual sum rule, Eq. (1), due to Heisenberg. (Note that the center-of-mass corrections drop out.)

The matrix elements in Eq. (9) contain several factors which vary with the final-state energy and, in the above closure calculation, these have been treated as constants. These factors are (a) the momentum transfer $\mathbf{q}$ which is related to the nuclear excitation, $\epsilon$, by the energy conservation relation

$$k_0=k_f+|\mathbf{q}|^2/2AM+\epsilon,\quad \mathbf{q}=\mathbf{k}_f-\mathbf{k}_0; \tag{11}$$

(b) the volume in phase space available to the scattered electron

$$k_f^2\frac{dk_f}{d(k_f+\epsilon-k_0+|\mathbf{q}|^2/2AM)};$$

and (c) the nucleon form factor

$$f(q)\cong 1+\tfrac{1}{6}q^2\langle r^2\rangle.$$

Equation (10) is usually applied with factors (a), (b), and (c) evaluated for zero nuclear excitation ($\epsilon=0$) and it is one of our first tasks to evaluate $\epsilon$ corrections to this sum rule. We treat these corrections by expanding those factors which depend on the final nuclear excitation energy in a power series about $\epsilon=0$, keeping terms through second order in $\epsilon/k_0$. To estimate the accuracy of this expansion, we note that the most probable value of $\epsilon$ is approximately that for single particle ejection from the nucleus; *viz.*,

$$\epsilon_{\text{m.p.}}+|\mathbf{q}|^2/2AM\cong|\mathbf{q}|^2/2M,$$

or

$$\epsilon_{\text{m.p.}}\cong(|\mathbf{q}|^2/2M)(1-1/A)\approx 20\text{ Mev}$$

for $k_0=150$ Mev, and $\theta=\pi/2$.

We remark here that an accurate $\epsilon/k_0$ expansion limits the electron energy from below while ignorance of mesonic and relativistic effects limits it from above. Physically, the region of electron energies is bounded by the requirement that a complete set of nuclear states be energetically available to the target system. This will not be the case for too low an incident electron energy. For too high an electron energy Eq. (7) will be an inadequate approximation to the nuclear Hamiltonian since dynamical effects of meson production by the electron current, which are here neglected, become appreciable.

The following relations,

$$\langle\varphi_f|\epsilon^n\Omega|\varphi_0\rangle=\langle\varphi_f|\epsilon^{n-1}[H,\Omega]|\varphi_0\rangle, \tag{12}$$

can then be used to remove $\epsilon$ from the matrix element and permit thereby the closure sum to be effected.

The Hamiltonian commutators which are introduced in this way carry interesting information about the nuclear forces since one may be taking derivatives of potentials, or picking out the $\boldsymbol{\sigma}$- or $\boldsymbol{\tau}$-dependent parts of the interaction. In the present approach to a low-resolution sum rule these terms show up only as small corrections to the total cross section. In Sec. IV we shall design a sum rule, Eq. (3′), for $\sigma_E$ with the aim of emphasizing these commutator terms.

The measurement of $\sigma_E$ poses an additional experimental problem in that high-resolution studies are required to construct the weighted sum. In high-resolution work the experimental advantage of electro-disintegration (over photo-disintegration) comes out most strongly. The possibility of high-resolution experiments suggests a further constraint in the computation of $\sigma_E$ from measured inelastic spectra which will greatly simplify the theoretical analysis. This is to keep the momentum transfer $|\mathbf{q}|$ fixed, instead of the scattering angle, by adding cross sections for different scattering angles and final electron energies, as related by the conservation laws Eq. (11) for constant $|\mathbf{q}|$. As is evident from Eqs. (8) and (11), a constant $|\mathbf{q}|$ cross section is the natural one for study since the exponential factors need not be expanded about $\epsilon=0$ in its construction.

With this in mind, we proceed now to a calculation of the low-resolution cross-section sum rule.

### III. CALCULATION OF LOW-RESOLUTION CROSS SECTION

First, we carry out the expansion of the $\mathbf{q}$-dependent terms in the matrix element (8) about their $\epsilon=0$ value, reduce the $\epsilon$ dependence with relations (12) and carry out the closure sum. We again, for illustrative purposes, confine our remarks to the Coulomb matrix element. Their extension to the current interactions is straightforward and will be indicated at the end of this section.

Using the kinematical relations (11) and denoting by $\mathbf{q}_0$ the value of the momentum transfer $\mathbf{q}$ corresponding to $\epsilon=0$, we have

$$k_f^2/(q^2)^2=[4k_0\sin^2(\theta/2)]^{-2}, \tag{13}$$

$$\frac{dk_f}{d(k_f+\epsilon-k_0+|\mathbf{q}^2|/2AM)}=\frac{1+\epsilon/AM+O(\epsilon^2/A^2M^2)}{1+(2k_0/AM)\sin^2(\theta/2)},$$

$$\exp(i\mathbf{q}\cdot\boldsymbol{\varrho}_i)=\exp(-\epsilon\hat{k}_f\cdot\nabla_{\mathbf{q}0})\exp(i\mathbf{q}_0\cdot\boldsymbol{\varrho}_i)$$
$$\cong[1+i\epsilon\hat{k}_f\cdot\boldsymbol{\varrho}_i-\tfrac{1}{2}\epsilon^2(\hat{k}_f\cdot\boldsymbol{\varrho}_i)^2]\exp(i\mathbf{q}_0\cdot\boldsymbol{\varrho}_i),$$
$$f(q)\cong1+\tfrac{1}{6}q^2\langle r^2\rangle\approx1-\tfrac{1}{6}q_0^2\langle r^2\rangle(1-\epsilon/k_0). \tag{14}$$

Inserting these into (8) and (9) we find (for the Coulomb interaction)

$$\sigma_c d\Omega_f = \sigma_0 f^2(q_0) d\Omega_f \Big\{ \mathfrak{F} + \frac{1}{2}\left(\frac{1}{AM} + \frac{q_0^2\langle r^2\rangle}{3k_0}\right) \times \int (d\tau)\varphi_0^* \sum_{i,j} \{e_i \exp(i\mathbf{q}_0\cdot\boldsymbol{\varrho}_i), [H, e_j \exp(-i\mathbf{q}_0\cdot\boldsymbol{\varrho}_j)]\}\varphi_0$$
$$+\frac{i}{2}\int (d\tau)\varphi_0^* \sum_{i,j} [e_i \exp(i\mathbf{q}_0\cdot\boldsymbol{\varrho}_i), \{H, e_j \hat{k}_f\cdot\boldsymbol{\varrho}_j \exp(-i\mathbf{q}_0\cdot\boldsymbol{\varrho}_j)]\}\varphi_0$$
$$+\frac{1}{4}\int (d\tau)\varphi_0^* \sum_{i,j} (\hat{k}_f\cdot\nabla_{\mathbf{q}_0})^2[H, e_i \exp(i\mathbf{q}_0\cdot\boldsymbol{\varrho}_i)][H, e_j \exp(-i\mathbf{q}_0\cdot\boldsymbol{\varrho}_j)]\varphi_0$$
$$+2\int (d\tau)\varphi_0^* \sum_{i,j} [H, \hat{k}_f\cdot\nabla_{\mathbf{q}_0} e_i \exp(i\mathbf{q}_0\cdot\boldsymbol{\varrho}_i)][H, \hat{k}_f\cdot\nabla_{\mathbf{q}_0} e_j \exp(-i\mathbf{q}_0\cdot\boldsymbol{\varrho}_j)]\varphi_0 \Big\}, \quad (15)$$

through terms of order indicated in the above expansions. With the Hamiltonian (7) the commutators linear in $H$ are evaluated directly to be

$$\sum_{i,j} [e_i \exp(i\mathbf{q}_0\cdot\boldsymbol{\varrho}_i), \{H, e_j \exp(-i\mathbf{q}_0\cdot\boldsymbol{\varrho}_j)]\}$$
$$= Ze^2\frac{q_0^2}{M}\left(1-\frac{1}{A}\right) - \sum\sum_{i\neq j} e_i e_j \frac{q_0^2}{AM}\exp(i\mathbf{q}_0\cdot\boldsymbol{\varrho}_{ij})$$
$$- e^2\sum\sum_{i\neq j} V_\tau(ij)[1-\exp(i\mathbf{q}_0\cdot\boldsymbol{\varrho}_{ij})] \times(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j - \tau_{3i}\tau_{3j});$$
$$\frac{i}{2}\sum_{i,j} [e_i \exp(i\mathbf{q}_0\cdot\boldsymbol{\varrho}_i), \{H, \hat{k}_f\cdot\boldsymbol{\varrho}_j e_j \exp(-i\mathbf{q}_0\cdot\boldsymbol{\varrho}_j)]\}$$
$$= -Ze^2\frac{2k_0\sin^2(\theta/2)}{M}\left(1-\frac{1}{A}\right) \times\left(1-\frac{Z-1}{A}\exp(i\mathbf{q}_0\cdot\boldsymbol{\varrho}_{ij})\right)$$
$$+\frac{Z(Z-1)}{A}\frac{q_0^2}{2M}\hat{k}_f\cdot\nabla_{\mathbf{q}_0}\exp(i\mathbf{q}_0\cdot\boldsymbol{\varrho}_{ij})$$
$$-\sum\sum_{i\neq j}\frac{\hat{k}_f\cdot\nabla_{\mathbf{q}_0}}{2}\exp(i\mathbf{q}_0\cdot\boldsymbol{\varrho}_{ij})V_\tau(ij) \times(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j - \tau_{3i}\tau_{3j});\quad \boldsymbol{\varrho}_{ij}\equiv\boldsymbol{\varrho}_i-\boldsymbol{\varrho}_j = \mathbf{r}_i - \mathbf{r}_j. \quad (16)$$

We notice the appearance in the above of $1/A$ recoil terms expressing the interparticle correlation, for $i\neq j$, through the center-of-mass motion. In the calculation of the commutators involving $H$ to the second power the cross $i\neq j$ terms become very complicated. Hence, we shall keep for consideration only the one-particle terms $(i=j)$ in the second-order correction to the exponential (14). We shall show that these contribute negligibly.

With this simplification, the last two lines of Eq. (15) become

$$\frac{1}{2}\int (d\tau)\varphi_0^* \sum_i e_i \exp(i\mathbf{q}_0\cdot\boldsymbol{\varrho}_i) \times\{\hat{k}_f\cdot\boldsymbol{\varrho}_i, [H^2, \hat{k}_f\cdot\boldsymbol{\varrho}_i]\} e_i \exp(-i\mathbf{q}_0\cdot\boldsymbol{\varrho}_i)\varphi_0$$
$$=\frac{Z}{M}\int (d\tau)\varphi_0^* e_1^2 \exp(i\mathbf{q}_0\cdot\boldsymbol{\varrho}_i) H \exp(-i\mathbf{q}_0\cdot\boldsymbol{\varrho}_i) - \tfrac{2}{3}Ze^2\frac{\langle T\rangle}{M} - \frac{Ze^2k_0^2}{M^2}\sin^2(\theta/2),$$

where $\langle T\rangle$ represents the average ground-state kinetic energy per nucleon, and the first term is worked out in (16).

It is at any point near here that the theoretical preference for constant $|\mathbf{q}|$ cross section is most evident. Collecting these results the Coulomb contribution to the total cross section is

$$\sigma_c d\Omega_f = \sigma_0 f^2(q_0) d\Omega_f \Big\{ \mathfrak{F} - Z\left(1-\frac{1}{A}\right)\frac{2k_0\sin^2(\theta/2)}{M}\left[1-\frac{Z-1}{A}\langle\exp(i\mathbf{q}_0\cdot\boldsymbol{\varrho}_{ij})\rangle\right]$$
$$+\frac{1}{2}\left(\frac{1}{M}+\frac{q_0^2\langle r^2\rangle}{3k_0}\right)\frac{Zq_0^2}{M} - \frac{2}{3}\frac{Z\langle T\rangle}{M} - \frac{Zk_0^2}{M^2}\sin^2(\theta/2) - \frac{q_0^2\langle r^2\rangle}{6k_0}\frac{q_0^2}{AM}Z(Z-1)\langle\exp(i\mathbf{q}_0\cdot\boldsymbol{\varrho}_{ij})\rangle$$
$$-\frac{1}{2}\left(\frac{1}{AM}+\frac{q_0^2\langle r^2\rangle}{3k_0}\right)\sum_{i\neq j}\langle V_\tau(ij)[1-\exp(i\mathbf{q}_0\cdot\boldsymbol{\varrho}_{ij})](\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j-\tau_{3i}\tau_{3j})\rangle + \frac{Z(Z-1)}{A}\frac{q_0^2}{2M}\hat{k}_f\cdot\nabla_{\mathbf{q}_0}\langle\exp(i\mathbf{q}_0\cdot\boldsymbol{\varrho}_{ij})\rangle$$
$$-\tfrac{1}{2}\hat{k}_f\cdot\nabla_{\mathbf{q}_0}\sum_{i\neq j}\langle V_\tau(ij)\exp(i\mathbf{q}_0\cdot\boldsymbol{\varrho}_{ij})(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j-\tau_{3i}\tau_{3j})\rangle\Big\}, \quad (17)$$

where $\langle\,\rangle$ denotes ground-state expectation value.

Before proceeding to the current terms, let us establish the order of magnitude of the correction terms in the above formula by evaluating them for the particular case of electron scattering from deuterons. In this example $\mathfrak{T}=1$ and the various kinematic corrections in the above serve to increase the cross section by 2% when evaluated for 90° scattering of 140-Mev incident electrons. In the evaluation of this number, a Rosenfeld force[11] is assumed to operate between the nucleons of the deuteron, and a Hulthén wave function is used to describe the ground state of the deuteron. The correction decreases to $\sim 0\%$ if the binding forces are assumed to have no isotopic spin dependence. For the deuteron, we have also calculated the two-body correlation terms in the second-order commutators of $H$ which were neglected according to the discussion below Eq. (16), and found them to contribute $<1\%$.

Additional contributions to the sum rule for the total differential cross section come from the interaction of the electron with the currents of the nucleons. The contributions of the recoil and spin currents are calculated explicitly for the deuteron in the work of Jankus.[12] Our considerations here then provide a measure of the gauge current and kinematic corrections to Jankus' sum rule. In view of the smallness of the correction terms in (17), we confine our analysis of them here to the deuteron. We find, using the same methods as applied in the discussion of charge scattering, that the phase space, kinematical, and gauge current corrections are negligible and add up to $<1\%$ for the previous physical parameters of 140-Mev incident electrons scattered by deuterons through 90°. The electron-deuteron sum rule reads finally

$$\sigma_c d\Omega_f = \sigma_0 f^2(q_0) d\Omega_f \times\left(1+\frac{q_0^2}{4M^2}+\{2\sec^2(\theta/2)-1\} \times\left[\frac{1}{3}\frac{\langle T\rangle}{M}+\frac{q_0^2}{4M^2}(\mu_p^2+\mu_n^2) +\frac{2}{3}\frac{q_0^2}{4M^2}\mu_p\mu_n\langle\exp(i\mathbf{q}_0\cdot\boldsymbol{\rho})\rangle\right]+X\right), \quad (18)$$

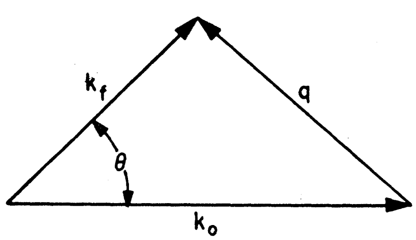


FIG. 1. Vector diagram relating incident electron momentum $\mathbf{k}_0$, final electron momentum $\mathbf{k}_f$, momentum transfer $\mathbf{q}$, and scattering angle $\theta$.

[11] L. Rosenfeld, *Nuclear Forces* (North-Holland Publishing Company, Amsterdam, 1948), p. 234.

[12] V. Jankus, Phys. Rev. **102**, 1586 (1956).

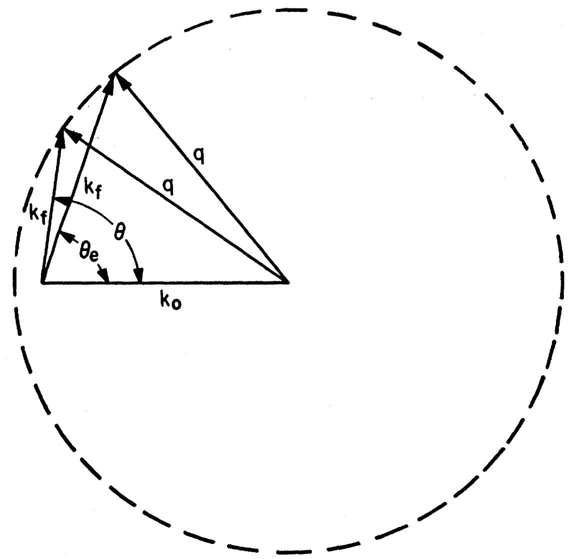


FIG. 2. Vector diagram for fixed $|\mathbf{q}|$ and fixed $k_0$.

where $X$ contains all correction terms to the Jankus result due to use of proper kinematics and due to gauge currents, and provides a correction of less than 3% in the energy range under discussion here. We display $X$ in Appendix A. With neglect of the correction term $X$ the sum rule for a light nucleus of $Z$ protons and $N=A-Z$ neutrons reads

$$\sigma_0 f^2(q_0) d\Omega_f\left(\mathfrak{T}+\frac{Zq_0^2}{2AM^2}+\{2\sec^2(\theta/2)-1\} \times\left[\frac{2}{3}\frac{Z}{AM}\langle T\rangle+\frac{q_0^2}{4M^2}(Z\mu_p^2+N\mu_n^2) +\frac{1}{3}\sum_{i\neq j}\sum\langle\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j\mu_i\mu_j\exp(i\mathbf{q}_0\cdot\boldsymbol{\rho}_{ij})\rangle\right]\right). \quad (19)$$

In concluding this section, we remark that low-resolution studies in this intermediate energy region do not promise to provide any new nuclear information.

## IV. HIGH-RESOLUTION STUDIES

With the dual aim of constructing a sum rule which contains information on the structure of the nuclear ground state, and which avoids the complications encountered in the previous section when we had to expand $\exp(i\mathbf{q}\cdot\boldsymbol{\rho}_i)$ about the elastic $\mathbf{q}$ value, we focus our attention on high-resolution cross sections. To this end, we consider cross sections summed in such a way that the momentum transferred by the electron is held constant. For a fixed incident electron momentum $\mathbf{k}_0$, the final momentum and scattering angle $\theta$ in the laboratory system are related by $\mathbf{k}_f=\mathbf{k}_0+\mathbf{q}$ as shown in the vector diagram, Fig. 1. In order to establish a sum rule for fixed $|\mathbf{q}|$ we shall consider cross sections for these angles $\theta$ and energies $k_f$ for which $\mathbf{q}$ maps out a circle, Fig. 2. In Fig. 2, the smallest $\theta=\theta_e$ for fixed $|\mathbf{q}|$ value corresponds to elastic scattering and the larger angles to smaller $k_f$ values and inelastic scattering.

We construct the sum for the total cross section at constant $q$ in the form

$$\int dk_f \frac{d^2\sigma}{d\Omega_f dk_f}\mathcal{W}^{-1}, \tag{20}$$

where the weighting factor is

$$\mathcal{W}=\sigma_0'(\theta,k_0,k_f)f^2(q), \tag{21}$$

with

$$\sigma_0'(\theta,k_0,k_f)=\frac{e^4}{4k_0^2}\frac{\cos^2(\theta/2)}{\sin^4(\theta/2)}\frac{1}{1+(k_f-k_0\cos\theta)/AM},$$

and serves to remove the phase-space and form-factor dependence on the inelasticity $\epsilon$ as expressed in (14) from the final-state sum in order to allow closure. $\mathcal{W}$ is known as a function of scattering angle $\theta$ for fixed $|\mathbf{q}|$ through the kinematic relations (11),

$$\begin{aligned}\cos\theta&=(k_f^2+k_0^2-|\mathbf{q}|^2)/2k_fk_0,\\ \epsilon&=k_0-k_f-|\mathbf{q}|^2/2AM.\end{aligned} \tag{22}$$

Here the incident electron energy $k_0$ is kept fixed. The more general case in which both $k_0$ and $k_f$ are varied with constant $|\mathbf{q}|$ is discussed in Appendix B.

The experimental total cross section corresponding to this sum rule is constructed as follows. First values for $k_0$ and $q$ are specified. Then for each value of $k_f$, the corresponding scattering angle $\theta$ is determined from (22) and the experimental differential cross section $d^2\sigma/dk_fd\Omega_f$ is measured for each pair $k_f$, $\theta$. The weighting factor $\mathcal{W}(\theta,k_f)$ is calculated by (21) and (22) for the same $k_f$, $\theta$ values and the ratio $(d^2\sigma/dk_fd\Omega_f)/\mathcal{W}$ is plotted as a function of $k_f$. The area under this curve is the desired total cross section for the specified momentum transfer $q$ and incident energy $k_0$.

The virtue of this method of constructing a high-resolution sum rule lies in the fact that the closure sum for the charge interaction terms can be effected directly with no expansions since the $\epsilon$ dependence has been completely removed from them for fixed $q$. The current interactions, which are smaller than the leading order charge terms in the energy range considered here, vary with the scattering angle for fixed $q$. The expansion of the scattering angle about the value of $\theta$ corresponding to $\epsilon=0$ corresponds here to the $q$ expansion of the low-resolution studies. However, now the correction terms appear only for the current interactions and not for the dominant charge terms. This is because we have kept $q$ constant whereas the current interactions depend on the direction as well as magnitude of $\mathbf{q}$. If we replace the actual angular dependence by its value at elastic scattering, the closure result follows from our calculations of the preceding section and is given by $1/\sigma_0f^2(q)\times$Eq. (19) with $X\rightarrow0$ and with the angular factor given by ($q^2\equiv|\mathbf{q}|^2$)

$$2\sec^2(\theta/2)-1$$
$$=\frac{3-\cos\theta}{1+\cos\theta}=\frac{3-(1-q^2/2k_0^2)}{1+(1-q^2/2k_0^2)}\equiv3Y(q,k_0) \tag{23}$$

to leading order. $Y$ remains close to unity for scattering angles near 90°. The correction terms of order $(\epsilon/k_0)$ to $Y$ can be handled by expansion as in the previous section, but they provide small corrections to terms that are already small for the range of physical parameters under discussion here. One can actually avoid this expansion by considering an experimental program which varies both incident and final energies, $k_0$ and $k_f$, in such a way as to keep both scattering angle $\theta$ and $\mathbf{q}$, and hence $Y$, fixed; this method is discussed in Appendix B.

Whereas the high-resolution sum rule for constant $q$ appears to be the more natural sum rule on theoretical grounds of simplicity than the low-resolution one for constant scattering angle, it contains little nuclear information of interest. As applied to the deuteron, the structure of the ground state manifests itself only through the average kinetic energy in the term $(\langle T\rangle/M)Y$, and through the spin correlation term $(q^2/2M)\mu_p\mu_n\langle\exp(i\mathbf{q}\cdot\boldsymbol{\rho})\rangle$. The first of these terms gives rise to a contribution of 3% in comparison with the leading-order charge scattering for $\langle T\rangle=30$ Mev; and the second one, $-5$% for a Hulthén ground-state wave function. It is clearly then impossible to study variations of these terms with assumed deuteron model in the present sum rule. For other light nuclei containing at least two protons, there appears in the sum rule (20) also a two-body charge form factor as seen in Eq. (10) for $\mathfrak{F}$. Whereas this term provides an important contribution to the total cross section, its actual value is quite well tied down by the requirement of matching the observed elastic form factor. This is because we are dealing with momentum transfers $\sim$200 Mev/c corresponding to a reduced wavelength $\lambda\!\!\!^{-}=1/q\sim10^{-13}$ cm and it takes the higher resolving power of a shorter wavelength to probe variations with nuclear model of the two-body correlation function $f_2$, Eq. (1), for fixed observed elastic scattering form factor $f_{\text{el}}$, Eq. (2). We give here typical numbers for $He^4$ and $O^{16}$. For $He^4$, we fit the observed rms radius with a product of Gaussian wave functions $\exp(-\nu r^2/2)$, so that one has

$$f_{\text{el}}=\exp(-\tfrac{3}{4}q^2/4\nu),$$

where the factor $\frac{3}{4}=1-\frac{1}{4}$ is the center-of-mass correction. Taking into account the finite proton size, the rms radius observed by Blankenbecler and Hofstadter[13] is 1.4 f [where 1 fermi(f)$=10^{-13}$ cm], so that $\nu=(9/8)(1/\langle r^2\rangle)=0.57\times10^{26}$ cm$^{-2}$ and $f_2=\exp(-q^2/2\nu)=0.41$, for 90° scattering of 140-Mev electrons ($q=200$

[13] R. Blankenbecler and R. Hofstadter, Bull. Am. Phys. Soc. Ser. II, 1, 10 (1956).

Mev/c). The two-body correlation term thus contributes about 30% of the total cross section, but its calculated value is completely tied down at these $q$ values by the elastic scattering requirement. For larger $q$ values $f_2$ begins to depend on other factors than $f_{el}$, such as repulsive cores between pairs of particles for separations $<\frac{1}{2}\times 10^{-13}$, but then the over-all contribution of $f_2$ ceases to be important for these large $q$ values. The upper limit of the short-range correlation contribution to $f_2$ is the ratio of the volume of the excluded region to the volume $4\pi/3q^3$ and this is less than 10% in the present case, or $\lesssim 3\%$ of the total cross section.

For $O^{16}$, we fit the observed rms charge radius of $2.66\times 10^{-13}$ cm[14] with a Gaussian shell-model wave function, obtaining for the elastic form factor[15]

$$f_{el}=(1-\tfrac{1}{4}x)e^{-\frac{1}{2}x(1-1/A)},$$

and for the two-body correlation function

$$f_2=[1-\tfrac{1}{2}x+(5/84)x^2]e^{-x}, \quad \text{with} \quad x=q^2/2\nu.$$

Taking out the proton size, we have $\nu=0.33_4\times 10^{26}$ cm$^{-2}$. For the same momentum transfer of $\simeq$200 Mev/c, we have $x=1.51$ and $f_2=0.084$, so that the two-body charge correlation term contributes about 37% of the total cross section. Although sum-rule measurements at these energies in $O^{16}$ will be valuable as a consistency check on the observed elastic form factor, they do not appear to offer much hope of providing new clues as to the structure of the ground state. Interesting short-range internuclear correlations will show up prominently in $f_2$ at higher $q$ values but then the total contribution of $f_2$ will be reduced relative to the single-nucleon term in $\mathfrak{F}$. Also, at higher $q$ values, it will not be clear how to disentangle the various effects arising from the relativistic motion of the nucleus.

We turn finally to the construction of the energy-weighted total cross section which removes the contribution of elastic scattering from the sum rule and thereby emphasizes the role of the fluctuating nonstatic properties of the nuclear ground state relative to its average properties. Such a sum rule is a sensitive probe of the charge exchange process in nuclei since these give rise to oscillating contributions of the charge and spin densities which can take up energy from the scattered electron.

The specific sum rule we construct is

$$\sigma_E=\int dk_f\left\{\frac{d^2\sigma}{d\Omega_f dk_f}\mathcal{W}^{-1}\epsilon\right\}_{q=\text{const}}, \tag{24}$$

[14] U. Meyer-Berkhout (private communication).

[15] For the harmonic-oscillator shell model, it can be shown quite generally that the inclusion of the center-of-mass correction simply introduces the factor $(1-1/A)$ in the Gaussian exponent of the elastic scattering form factor. A private discussion with L. Tassie has been very helpful in this matter.

with $\mathcal{W}$ defined by (21). This cross section corresponds to weighting the curve of the preceding high-resolution study by the energy of excitation of the nuclear system for each value of $\theta$, $k_f$ for fixed $q$, as expressed in (22) and then computing the area under this weighted curve.

Comparing with (15), and using (12), we have

$$\begin{aligned}\sigma_E&=\sum_f \epsilon\left|\sum_{i=1}^{A}\langle\varphi_f|\Theta_i|\varphi_0\rangle\right|^2\\&=\langle\varphi_0|\sum_{i,j}\Theta_j^*H\Theta_i|\varphi_0\rangle,\end{aligned} \tag{25}$$

where $\langle\,\rangle$ indicates sum over final and average over initial electron spins, as in (9), as well as integration over nuclear coordinates, and where

$$\begin{aligned}e\Theta_i=&\langle u(k_f)|u(k_0)\rangle e_i\\&\times\exp(i\mathbf{q}\cdot\boldsymbol{\rho}_i)-\langle u(k_f)|\boldsymbol{\alpha}|u(k_0)\rangle\\&\cdot\Bigg[[\boldsymbol{\pi}_i\exp(i\mathbf{q}\cdot\boldsymbol{\rho}_i)+\exp(i\mathbf{q}\cdot\boldsymbol{\rho}_i)\boldsymbol{\pi}_i]\frac{e_i}{2M}\\&+\frac{i\mu_i}{2M}\boldsymbol{\sigma}_i\times\mathbf{q}\exp(i\mathbf{q}\cdot\boldsymbol{\rho}_i)+2e\sum_{j(\neq i)}(\boldsymbol{\tau}_j\times\boldsymbol{\tau}_i)_3\\&\times\int_{r_j}^{r_i}\exp(i\mathbf{q}\cdot\mathbf{s})d\mathbf{s}\;V_\tau(ij)\Bigg].\end{aligned}$$

We are once again led to the commutators of the Hamiltonian which appeared in the earlier low-resolution calculation, (16), but now these are the main terms. In particular, as in (16), isotopic-spin-dependent interactions give rise to important contributions.

We compute (25), first for the deuteron for simplicity in order to establish which are the large terms. Dividing $\Theta_i$ into charge, moment, and recoil plus gauge current terms, $\Theta_i^e$, $\Theta_i^\mu$, $\Theta_i^r$, respectively, we find that the major contributions come from the charge-charge and the moment-moment terms. Thus,

$$\begin{aligned}&\langle\varphi_0|\sum_i\sum_j\Theta_j^{e*}H\Theta_i^e|\varphi_0\rangle\\&=\tfrac{1}{2}\langle\varphi_0|\sum_{i,j}\{\Theta_j^{e*},[H,\Theta_i^e]\}|\varphi_0\rangle\\&=\frac{q^2}{4M}+2\langle\varphi_0|[V_a(\rho)+V_b(\rho)][1-\exp(i\mathbf{q}\cdot\boldsymbol{\rho})]|\varphi_0\rangle\end{aligned}$$

for a general two-body Hamiltonian

$$\begin{aligned}H=\frac{P^2}{4M}+\frac{\pi^2}{M}+V_w(\rho)+\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2V_a(\rho)\\+\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2V_b(\rho)+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2V_c(\rho),\end{aligned}$$

and

$$\tfrac{1}{2}\langle\varphi_0|\sum_i\sum_j\{\Theta_i^{\mu*},[H,\Theta_j^{\mu}]\}|\varphi_0\rangle$$

$$=\frac{q^2}{4M}Y(\theta)\left(\frac{g_p-g_n}{2}\right)^2$$

$$\times\left\{\frac{3}{4}\frac{q^2}{2M}[1+\tfrac{1}{3}\langle\varphi_0|\exp(i\mathbf{q}\cdot\boldsymbol{\rho})|\varphi_0\rangle]\right.$$

$$+2\langle\varphi_0|V_a(\rho)[3-\exp(i\mathbf{q}\cdot\boldsymbol{\rho})]+2V_b(\rho)$$

$$\left.\times[1-\exp(i\mathbf{q}\cdot\boldsymbol{\rho})]-V_c(\rho)[1+\exp(i\mathbf{q}\cdot\boldsymbol{\rho})]|\varphi_0\rangle\right\}.$$

For simplicity in writing these terms, we have replaced $\mu_p^2+\mu_n^2=11.4$ and $-2\mu_p\mu_n=10.7$ by $\frac{1}{2}[(g_p-g_n)/2]^2=11.0$; the resulting error in the final sum rule is $<1\%$.

Before evaluating these terms further, we look at additional contributions to (25). The $\Theta^e\Theta^\mu$ and $\Theta^r\Theta^\mu$ cross terms vanish upon spin average since they form vectors in spin space. The charge-current cross terms $\Theta^e\Theta^r$ are calculated straightforwardly to give

$$\frac{q^2}{4M}\left\{\left\langle\frac{\pi^2}{3M^2}+\frac{V_a+V_b}{M}[1+\exp(i\mathbf{q}\cdot\boldsymbol{\rho})]\right\rangle\right.$$

$$-\frac{q^2}{8M^2}\left\langle\frac{2\mathbf{q}\cdot\boldsymbol{\rho}\mathbf{q}\cdot\boldsymbol{\pi}}{q^2}(V_a+V_b)\frac{\exp(i\mathbf{q}\cdot\boldsymbol{\rho})-1}{\mathbf{q}\cdot\boldsymbol{\rho}}\right\rangle$$

$$-\frac{2}{M}\left\langle\left(\frac{\mathbf{q}\cdot\boldsymbol{\rho}\sin\mathbf{q}\cdot\boldsymbol{\rho}}{q\rho}\frac{d}{d(q\rho)}+\frac{\sin\mathbf{q}\cdot\boldsymbol{\rho}}{\mathbf{q}\cdot\boldsymbol{\rho}}\right)\right.$$

$$\left.\left.\times(V_a+V_b)\right\rangle\right\}.\quad(26)$$

In order to estimate these and succeeding minor terms in (25), we consider three deuteron models. Model (I) assumes a Hulthén ground-state function ($s$ state only) and a Rosenfeld[11] two-body interaction of Yukawa spatial dependence

$$V_{\mathrm{I}}=\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2(0.1+0.23\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\frac{e^{-\mu r}}{\mu r}V_0,$$

with $\mu^{-1}=1.4$f, $V_0=40$ Mev. Model (II) differs from Model (I) in that a Gaussian shape is assumed for the two-body interaction in order to remove the singularity of the interaction at the origin. Thus,

$$V_{\mathrm{II}}=\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2(0.1+0.23\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\exp(-0.46\mu^2r^2)(0.7V_0),$$

with the depth and range chosen to give the correct effective range for low-energy scattering, and binding energy. Model (III) is the Gartenhaus model[16] for the deuteron $s$ states.

[16] S. Gartenhaus, Phys. Rev. **100**, 900 (1955).

According to all of these models (26) gives a correction of $<1\%$ to $\sigma_E$. It is felt that Models (I) and (II) provide a liberal estimate of this contribution since the entire interaction is of form $V_a$ and $V_b$ in these cases, being $\propto\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2$. In the Gartenhaus potential there is a near cancellation[17] of $V_a$ and $V_b$.

For simplicity, we divide the contributions of the current-current terms into parts due to recoil and due to gauge currents. The result by direct calculation of the Hamiltonian commutators in (25) for the recoil currents alone is

$$\frac{1}{2M^2}Y\left\langle\frac{q^2}{2M}\boldsymbol{\pi}^2+4[1+\exp(i\mathbf{q}\cdot\boldsymbol{\rho})](V_a+V_b)\boldsymbol{\pi}^2\right.$$

$$\left.-[\boldsymbol{\pi}^2(V_w+V_c)]-[\boldsymbol{\pi}\cdot\exp(i\mathbf{q}\cdot\boldsymbol{\rho})\boldsymbol{\pi}(V_a+V_b)]\right\rangle.\quad(27)$$

The first term of (27) can be written as $(q^2/4M)Y\langle T\rangle/M$ and provides a $\sim3\%$ correction to the sum rule. The appearance of second derivatives in the remaining terms of (27) makes their values sensitive to details of the shape of the wave function and potential at small distances. For example, in the second term of (27), we can replace $\pi^2/M$ operating on the deuteron ground state by $-2.2$ Mev$-V$ and thereby obtain an expression quadratic in the interaction potential. For Model (I), according to which $\lim_{\rho\to0}\{\rho^2\psi^2V^2\}>0$, an appreciable contribution to (27) results from the region of $\rho\to0$. Setting aside the first term of (27) for the moment, the contribution of the remaining terms of (27) is $\cong5\%$ according to Model (I). These terms contribute negligibly according to Models (II) and (III) which remove the singularity at $\rho\to0$ with an assumed finite potential in case (II) and a "repulsive core" in the case (III).

Similar results obtain for all contributions of the gauge current terms. Writing, for simplicity, the approximate upper limit of these terms for unit deuteron form factor $[\exp(i\mathbf{q}\cdot\boldsymbol{\rho})\to1]$, we have

$$16Y\left\langle\frac{1}{M}\left\{\left(6+\tfrac{3}{2}\rho\frac{d}{d\rho}\right)(V_a+V_b)^2\right.\right.$$

$$\left.+\rho^2\left[\frac{d}{d\rho}(V_a+V_b)\right]^2\right\}+2\rho^2(V_a+V_b)^3$$

$$\left.+(V_a+V_b)\rho\frac{d}{d\rho}(V_w+V_c)\right\rangle,$$

which is again negligible for Models (II) and (III) but contributes $\sim15\%$ due to the $\rho\to0$ contributions of Model (I). Since we believe these terms to be of unphysical origin, we drop them for future discussions and confine our attentions to the charge and moment contributions plus the recoil kinetic energy term of (27).

[17] S. D. Drell, Phys. Rev. **100**, 97 (1955) (see Fig. 2, p. 102).

In this approximation, for the deuteron

$$\sigma_E=\Bigg[\frac{q^2}{4M}\bigg\{\bigg(1+\frac{\langle T\rangle}{M}\bigg)+\frac{3}{4}\bigg(\frac{g_p-g_n}{2}\bigg)^2 Y\frac{q^2}{2M^2}[1+\tfrac{1}{3}\langle\exp(i\mathbf{q}\cdot\boldsymbol{\rho})\rangle]\bigg\}+2\langle(V_a+V_b)[1-\exp(i\mathbf{q}\cdot\boldsymbol{\rho})]\rangle+\frac{q^2}{4M}Y\bigg(\frac{g_p-g_n}{2}\bigg)^2\times\bigg\langle\frac{2V_a}{M}[3-\exp(i\mathbf{q}\cdot\boldsymbol{\rho})]+\frac{4V_b}{M}[1-\exp(i\mathbf{q}\cdot\boldsymbol{\rho})]-\frac{2V_c}{M}[1+\exp(i\mathbf{q}\cdot\boldsymbol{\rho})]\bigg\rangle\Bigg]. \quad (28)$$

Extending this formula to an arbitrary nucleus with $Z=A/2$, we have

$$\sigma_E=\Bigg[Z\bigg(1-\frac{1}{A}\bigg)\frac{q^2}{2M}\bigg(1+\frac{\langle T\rangle}{M}\bigg)-\frac{q^2}{8AM}\sum_{i\neq j}\sum\langle(1+\tfrac{1}{3}\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)\exp(i\mathbf{q}\cdot\boldsymbol{\rho}_{ij})\rangle+\frac{Zq^2}{2M}Y\frac{3q^2}{8M^2}\bigg(\frac{g_p-g_n}{2}\bigg)^2\bigg(1-\frac{1}{A}\bigg)-\frac{q^2}{8AM}Y\frac{q^2}{4M^2}\times\bigg(\frac{g_p-g_n}{2}\bigg)^2\sum_{i\neq j}\sum\bigg\langle\frac{\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j}{3}\exp(i\mathbf{q}\cdot\boldsymbol{\rho}_{ij})\bigg\rangle-\tfrac{1}{3}\sum_{i\neq j}\sum\langle(V_a+V_b\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j\times[1-\exp(i\mathbf{q}\cdot\boldsymbol{\rho}_{ij})]\rangle-\frac{q^2}{4M^2}Y\bigg(\frac{g_p-g_n}{2}\bigg)^2\times\sum_{i\neq j}\sum\langle V_a\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j[1-\tfrac{1}{3}\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j\exp(i\mathbf{q}\cdot\boldsymbol{\rho}_{ij})]+V_b[\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j+\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j-\tfrac{2}{3}\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j\exp(i\mathbf{q}\cdot\boldsymbol{\rho}_{ij})]+V_c\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j[1-\tfrac{1}{3}\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j\exp(i\mathbf{q}\cdot\boldsymbol{\rho}_{ij})]\rangle\Bigg]. \quad (29)$$

For numerical results, we consider D, $He^4$, and $O^{16}$. Applying (28) to the deuteron, we find that the first or kinematic terms are insensitive to the assumed deuteron model and add up to

$$(1+0.39)q^2/4M,$$

for scattering of 140-Mev electrons with a 190 Mev/c momentum transfer. Both Models (I) and (II) give essentially the same results for the second terms since the volume of the interaction is the same in these two cases. In these cases the charge-exchange terms add 0.19 $(q^2/4M)$ and the moment exchange terms 0.16 $(q^2/4M)$. The added terms in Model (III) are $-0.10$ $(q^2/4M)$, coming mainly from the spin exchange term. Summarizing

$$(\sigma_E)_{\rm I}\cong(\sigma_E)_{\rm II}=\frac{q^2}{4M}(1.39+0.35),$$

$$(\sigma_E)_{\rm III}=\frac{q^2}{4M}(1.39-0.10).$$

These numbers are intended to indicate primarily the expected orders of magnitude of the exchange current effects on the average energy loss. Clearly inclusion of $D$-state and tensor-force terms, together with a better understanding of the forces and wave functions themselves, are required in order to achieve a good theoretical number. On the other hand, it is just this sensitivity to the nuclear ground-state properties, as indicated in the above estimates, that makes $\sigma_E$ an interesting quantity for experimental study.[18]

In applying (29) to $He^4$, we have assumed $He^4$ to consist of 4 nucleons in an $s$-state with central forces, so that $\langle\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j\rangle\to-1$, $\langle\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j\rangle\to-1$, and $\langle\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j\rangle\to-3$ and the average energy loss formula simplifies to

$$\sigma_E=\bigg(\frac{3q^2}{4M}\bigg)\Bigg[1+\frac{\langle T\rangle}{M}-\tfrac{1}{3}\langle\exp(i\mathbf{q}\cdot\boldsymbol{\rho}_{ij})\rangle+\frac{3q^2}{8M^2}\bigg(\frac{g_p-g_n}{2}\bigg)^2 Y[1+\tfrac{1}{3}\langle\exp(i\mathbf{q}\cdot\boldsymbol{\rho}_{ij})\rangle]+\frac{8}{3}\bigg\langle\frac{V_a+3V_b}{(q^2/2M)}[1-\exp(i\mathbf{q}\cdot\boldsymbol{\rho}_{ij})]\bigg\rangle+4Y\bigg(\frac{g_p-g_n}{2}\bigg)^2\times\bigg\langle\frac{V_a+2V_b+V_c}{M}[1-\exp(i\mathbf{q}\cdot\boldsymbol{\rho}_{ij})]\bigg\rangle\Bigg]$$

$$=\bigg(\frac{3q^2}{4M}\bigg)[1.23+\delta],$$

where $\delta=0.4$ for Model (I), and $\delta=0.5$ for Model (II) of the interaction potential, for the same $q$ and $k_0$ values as above. In both cases the $\alpha$-particle ground-state wave function is taken to be a Gaussian of observed rms radius 1.4 f (allowing for finite nucleon size).

Finally, we have carried through a shell-model calculation of these exchange contributions to $\sigma_E$ for $O^{16}$ according to Model (II), assuming gaussian wave functions with the same exponent for both $s$ and $p$ shells. The pertinent reduction formulas are in Appendix C. Here, we give just the results; again for $k_0=140$ Mev;

[18] Calculations of the photo effect sum rules by Levinger and collaborators [see, for example, M. L. Rustgi and J. S. Levinger, Phys. Rev. **106**, 530 (1957)] show a similar sensitivity to exchange forces which play a comparable role in the dipole sum rule $\int\sigma dW$.

$q=190$ Mev/c:

$$\sigma_E = Z(1-1/A)(q^2/2M) \times [1.02-0.03+0.30+0.03+0.20+0.01] = 1.53(15q^2/4M),$$

where the terms are in the order of those of (29).

## V. CONCLUSIONS

To summarize, we have constructed sum rules for the analysis of inelastic scattering of high-energy electrons from light nuclei. Corrections to previous sum rules arising from nucleon recoil and spin currents, from exchange currents operating in nuclei, from finite nucleon size and from kinematical factors have been considered.

The analysis has been limited to light nuclei so that the interaction of the electron with the nucleus can be treated in first Born approximation.[19] It has also been limited to an energy range corresponding to momentum transfers $\sim 200$ Mev/c in order to allow description of the nuclear system as a statically bound collection of "Pauli" nucleons.[20]

This work has shown that sum rules for total cross sections at a fixed scattering angle (low resolution) or at a fixed magnitude of momentum transfer (high resolution) are insensitive to nuclear ground-state properties. However, the energy-weighted cross section

$$\sigma_E = \int_{q=\text{const}} \epsilon\sigma(\epsilon,q)d\epsilon$$

appears to provide a sensitive test of the role of exchange currents in the nuclear ground state. Typical numbers for $\sigma_E$ have been given in Sec. IV for simple models of the D, $He^4$, and $O^{16}$ where it was seen that exchange forces arising in the Rosenfeld model of the nucleon interaction increase the average energy loss by as much as 40%.

## APPENDIX A

$$\begin{aligned}
X = & \left\langle \frac{V_\tau}{M}\left(1-\frac{\sin q\rho}{q\rho}\right)\right\rangle + 2\sec^2(\theta/2)\left\langle 2\rho^2 V_\tau^2[1-\tfrac{2}{3}\cos^2(\theta/2)]\frac{2}{q\rho}\int_0^{q\rho}\frac{1-\cos x}{x^2}dx - \frac{1}{k_0^2}V_\tau^2\left(1-\frac{\sin q\rho}{q\rho}\right)\right\rangle \\
& + \frac{2}{M}[2\sec^2(\theta/2)-1]\left\langle V_\tau \int_0^1 dt\, j_0(q\rho t)\right\rangle + \frac{2}{3M}[1+3\tan^2(\theta/2)]\left\langle \rho V_\tau' \int_0^1 dt\, j_0(q\rho t)\right\rangle \\
& + \frac{2}{M}\sec^2(\theta/2)\left\langle V_\tau' \frac{\cos q\rho - (q\rho)^{-1}\sin q\rho}{4\rho k_0^2}\right\rangle + \left[\frac{q^2}{8M^2} + \left\langle \frac{V_\tau}{M}\left(1-\frac{\sin q\rho}{q\rho}\right)\right\rangle\right]\left(2+\frac{2M}{3k_0}q^2\langle\rho^2\rangle\right) \\
& - \left(1-\frac{k_0}{2M}\right)\frac{k_0\sin^2(\theta/2)}{M} + 2\mathbf{i}_f\cdot\nabla_q\left\langle V_\tau \frac{\sin q\rho}{q\rho}\right\rangle + \left(1+\frac{2M}{3k_0}q^2\langle r^2\rangle\right)\left[\frac{4}{M}\mathbf{i}_f\cdot\nabla_q\left\langle V_\tau^2\frac{\sin q\rho}{q\rho}\right\rangle\right. \\
& \left. - \frac{q^2}{2M^2}\mathbf{i}_f\cdot\nabla_q\left\langle V_\tau\frac{\sin q\rho}{q\rho}\right\rangle - \frac{2k_0}{M^2}\sin^2(\theta/2)\left\langle V_\tau\left(1+\frac{\sin q\rho}{q\rho}\right)\right\rangle - \frac{\mathbf{i}_f\cdot\nabla_q}{M}\frac{q}{M}\langle V_\tau' j_1(q\rho)\rangle\right] \\
& + \frac{2}{3}\frac{\langle T\rangle}{M} + \frac{1}{4M^2}[q^2+2(\mathbf{q}\cdot\mathbf{i}_f)^2] - 8(\mathbf{i}_f\cdot\nabla_q)^2\left\langle V_\tau^2\frac{\sin q\rho}{q\rho}\right\rangle + \frac{2}{M}\left\langle V_\tau\left(1+\frac{\sin q\rho}{q\rho}\right)\right\rangle \\
& + \frac{q^2}{M}(\mathbf{i}_f\cdot\nabla_q)^2\left\langle V_\tau\frac{\sin q\rho}{q\rho}\right\rangle + \frac{4(\mathbf{i}_f\cdot\mathbf{q})(\mathbf{i}_f\cdot\nabla_q)}{M}\left\langle V_\tau\frac{\sin q\rho}{q\rho}\right\rangle + \frac{2}{M}(\mathbf{i}_f\cdot\nabla_q)^2\langle qV'j_1(q\rho)\rangle.
\end{aligned}$$

$$V' \equiv \frac{d}{d\rho}V.$$

## APPENDIX B

We describe here the general form for total cross sections where the greatest possible use will be made of variation of the experimental parameters of incident energy $k_0$, final energy $k_f$, and angle of scattering $\theta$. It has already been indicated that one will want to keep $\mathbf{q}$ constant; $\mathbf{q}=\mathbf{k}_f-\mathbf{k}_0$. The dependence of the differential cross section on all the parameters as they come from the Möller potential, final-state phase-space factors and nucleon form factors is all factored out in (21). We are here interested in the relative variation

[19] Problems encountered in the construction of sum rules for high-energy electrons scattered by nuclei of large $Z$ have been considered by W. E. Drummond using the Schiff high-energy approximation. See W. E. Drummond, Ph.D. thesis, Stanford University Physics Department, 1958 (to be published), and L. I. Schiff, Phys. Rev. **103**, 443 (1956); Nuovo cimento **10**, 1223 (1957).

[20] Relativistic effects for higher energy collisions have been analyzed in the case of the deuteron by R. Blankenbecler, Phys. Rev. **111**, 1684 (1958); Bull. Am. Phys. Soc. Ser. II, **2**, 389 (1957).

with parameters other than $|\mathbf{q}|$ of the separate parts of the matrix element.

We write (8) in the form

$$\mathfrak{M}_{f0} \propto \langle u(k_f) | u(k_0) \rangle Q - \langle u(k_f) | \boldsymbol{\alpha} | u(k_0) \rangle \cdot \mathbf{J},$$

where $Q$, representing the scalar part, and $\mathbf{J}$, the vector part of the nuclear matrix element, depend on no parameter of the scattering other than $\mathbf{q}$.

Squaring and summing over the electron spin orientations yields

$$\sigma \propto QQ(1+\cos\theta) + (\hat{k}_0 + \hat{k}_f) \cdot (\mathbf{J}Q + Q\mathbf{J}) + \mathbf{J} \cdot \mathbf{J}(1 - \tfrac{1}{3}\cos\theta) + 2(\hat{k}_0 \cdot \mathbf{J}\hat{k}_f \cdot \mathbf{J} - \tfrac{1}{3}\hat{k}_0 \cdot \hat{k}_f \mathbf{J} \cdot \mathbf{J}),$$

where the carets signify unit vectors. When we sum over the complete set of nuclear final states with any weighting $f(\epsilon)$, and average over the orientations of the original angular momentum of the nucleus, we must be left with only scalar quantities. Since $\mathbf{q}$ is the only vector in the function $\mathbf{J}$, we may then make the replacements

$$\mathbf{J} \rightarrow \mathbf{q}(\mathbf{q} \cdot \mathbf{J})/q^2$$

$$\mathbf{J}\mathbf{J} - \tfrac{1}{3}\mathbf{J} \cdot \mathbf{J} \rightarrow (\mathbf{q}\mathbf{q} - \tfrac{1}{3}\mathbf{q} \cdot \mathbf{q})(\mathbf{q} \cdot \mathbf{J}\mathbf{q} \cdot \mathbf{J} - \tfrac{1}{3}q^2 \mathbf{J} \cdot \mathbf{J}) / \tfrac{2}{3}(q^2)^2.$$

Factoring out $(1+\cos\theta)$ which belongs in $\mathfrak{W}$, we finally have

$$\sigma \propto QQ + (\Delta/q)(Q\mathbf{q} \cdot \mathbf{J} + \mathbf{q} \cdot \mathbf{J}Q) + \mathbf{J} \cdot \mathbf{J}Y - \tfrac{3}{2}(Y - \Delta^2/q^2)(\mathbf{q} \cdot \mathbf{J}\mathbf{q} \cdot \mathbf{J} - \tfrac{1}{3}\mathbf{J} \cdot \mathbf{J}).$$

Here $\Delta = k_f - k_0$ is almost equal to $\epsilon$, the nuclear excitation, and $Y$ is the familiar angular function $(1 - \frac{1}{3}\cos\theta)/(1+\cos\theta)$. In measuring a total cross section one will certainly *not* want to keep $\Delta$ fixed (we have in fact dropped these terms with $\Delta$ in our calculations as they are small); but we could keep $Y$ constant and thus avoid the necessary approximation made in Sec. IV that $Y$ was given some average value and its variation (for fixed $k_0$) with $\epsilon$ was neglected. This result is achieved by the simultaneous variation of $k_0$ and $k_f$ described by the vector diagram Fig. 3.

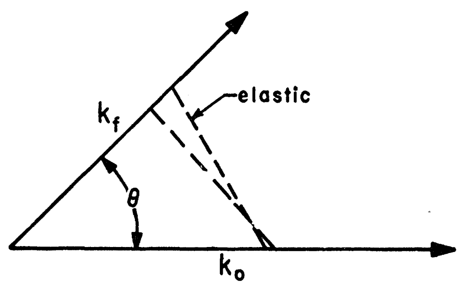

FIG. 3. Vector diagram for fixed $|\mathbf{q}|$ and fixed $\theta$.

## APPENDIX C

The techniques of atomic and nuclear shell-model spectroscopy are well known and we give here the results for the several matrix elements in (29) for the $1s^4$, $2p^{12}$ ground state of $O^{16}$, with harmonic-oscillator radial wave functions

$$R_{1s} = N_{1s} \exp(-\tfrac{1}{2}\nu r^2), \quad R_{2p} = N_{2p} r \exp(-\tfrac{1}{2}\nu r^2),$$

$$\langle T \rangle = (9/8)(\hbar^2 \nu / M).$$

All the other operators can be summarized in the form

$$\sum_{i \neq j} \sum Q_{ij},$$

$$Q_{12} = (a + b\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + c\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 + d\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2) V(r_{12}) e^{(i\mathbf{q} \cdot \mathbf{r}_{12})}$$

where $a$, $b$, $c$, and $d$ can have any numerical value; $V$ can be put equal to one, $\mathbf{q}$ can be put equal to zero to give all desired terms. Since the nucleus has no orientation the answer cannot depend on the direction of $\mathbf{q}$, so we can average over the angles of $\mathbf{q}$, and set

$$Q(r) = V(r)(\sin qr / qr).$$

For the ground state of oxygen we get

$$\langle \sum_{i \neq j} \sum Q_{ij} \rangle = \nu^{\frac{3}{2}} \left[ \Lambda_0 - \Lambda_1 \frac{2\nu}{3} \frac{d}{d\nu} + \Lambda_2 \frac{4\nu^2}{15} \frac{d^2}{d\nu^2} \right] \times \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty dr\, r^2 \exp(-\nu r^2/2) Q(r),$$

$$\Lambda_0 = 93(a - w),$$

$$\Lambda_1 = 102a + 90w,$$

$$\Lambda_2 = 45(a - w),$$

$$w = b + c + 3d.$$