difference between this 0^+ level and the ground state (0^+) of Ne^{22} leads to a value $r_0=1.40\times10^{-13}$ cm, in good agreement with those obtained from the other members of this series.[17]

The 592-kev transition connecting the 0^+ state with the $I=3^+$ ground state of Na^{22} (pure $M3$) has an expected lifetime[18] of 0.06 second. The internal conversion coefficient gives a negligible correction to this value.[19] Our upper limit of ∼0.1 second is consistent with this prediction.[||] Considering the direct positron decay from the 592-kev state in Na^{22} to the ground state of $Ne^{22}(0^+\rightarrow0^+)$, a lifetime of ∼15 seconds is calculated on the basis of $\log ft=3.44$.[16] We see therefore that a theoretical branching ratio of about 190÷1 in favor of the electromagnetic transition should obtain.

Another missing level in this series occurs in P^{30}. Our attempts to detect this level in the reaction $Al^{27}(\alpha,n)P^{30*}$ have not been conclusive to date, although we were able to detect the 2.5-minute positron activity from the ground state. These experiments are continuing.

[18] M. Goldhaber and A. W. Sunyar, Phys. Rev. **83**, 906 (1951).
[19] Rose, Goertzel, Spinrad, Harr, and Strong, Phys. Rev. **83**, 79 (1951).
[||] *Note added in proof.*—See, however, note added in proof, galley 8.

D. Energy Levels of Na^{23} and B^{11}

The energy levels in the compound nucleus Na^{23} are listed in Table I. A level diagram illustrating all states reached in the alpha-particle bombardment of F^{19} is shown in Fig. 10. No levels in this region of excitation in Na^{23} have been previously reported. New excited states in B^{11} at 9.88, 10.24, and 10.62 Mev have been found by the inelastic scattering of alpha particles in lithium, as discussed in Sec. IV (A).

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Many-Particle Configurations in a Central Field*

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Closed formulas are obtained for the fractional parentage coefficients of j-j coupled configurations of three and four equivalent particles. In states of low seniority, these formulas can be used to simplify the calculation of familiar types of matrix elements. Some extension is made to the study of more complex configurations. In particular, it is shown that near the ground state the energy level spectrum of an even-even nucleus should be independent of the number of particles in the unfilled shell.

INTRODUCTION

THERE are two well-known ways of describing an antisymmetric configuration of several equivalent particles bound in a central field. The first[1] is the permutational construction in which the wave functions of the several particles with the several sets of quantum numbers are arrayed in a Slater determinant to give directly an antisymmetric form. However, this structure proves to be quite unwieldy when one tries to calculate matrix elements of various operators, since a great many cross terms come into the expression.

The second[2,3] is the method of fractional parentage in which one considers just one particle of the configuration separated from all the others. The antisymmetry requirement is satisfied by leaving unspecified all the quantum numbers pertaining to the combination of the "other" particles and coupling the angular momentum of the one separated particle to some particular linear combination of all the allowed states of the "others." The fractional parentage coefficients which determine this particular linear combination for some given configuration are usually found as the solutions to a set of simultaneous algebraic equations and are tabulated for any particular problem.

However, it is clear that a direct (though formal) determination of the coefficients can be achieved by equating the wave function as written in the fractional parentage way to the permutational form mentioned first. The solution generally gives the fractional parentage coefficient in terms of the transformations which interrelate all the different vector coupling schemes of the several angular momenta. Using the techniques of Racah,[4] we have obtained formulas for the fractional parentage coefficients for configurations of three and four particles.

Using recursion relations, first derived by Racah[3] and extended in our Appendix, which are developed around the seniority concept, we can deduce formulas for the fractional parentage coefficients for some states of con-

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[1] E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, Cambridge, 1951), p. 162 et seq.
[2] S. Goudsmit and R. F. Bacher, Phys. Rev. **46**, 948 (1934).
[3] G. Racah, Phys. Rev. **63**, 367 (1943).
[4] G. Racah, Phys. Rev. **62**, 438 (1942).

figurations of any number of particles. In particular, we consider states of low seniority number, which are presumed according to the shell model to describe the low-lying levels of nuclei.

With the fractional parentage coefficients thus simplified, we are able in many cases to carry out explicitly the sums involved in calculating matrix elements of one- and two-particle operators. Considering a short-range attractive two-body interaction with an arbitrary mixture of ordinary and spin-exchange forces, we find that to first order the energy difference between the ground state of spin zero and the states of spin 2, 4, etc. with seniority number 2 is independent of the (even) number of particles in the last unfilled shell.

In addition for the shell $j=7/2$, it is found that this same conclusion about the level structure for 2 and 4 particles holds for any two-body interaction whatsoever.

ANTISYMMETRIC WAVE FUNCTIONS

It is well known that an antisymmetric wave function for two particles can be written in the form

$$\psi_{12}=N[\phi_1\chi_2-\phi_2\chi_1], \quad (1)$$

where the subscripts denote the particles described, and the normalization factor N is $1/\sqrt{2}$ if ϕ and χ are orthogonal functions.

In describing the bound states in a central field, we say that two particles are equivalent if the wave functions ϕ and χ are identical in their radial dependence. If we wish to describe the state in which two equivalent particles, each of spin j,[5] are coupled so that the only good magnetic quantum number is that belonging to their total angular momentum J, we would write

$$\psi_{12}(j^2JM)=\sum_{m_1m_2}(jm_1jm_2|jjJM)N \times[\phi_1(jm_1)\phi_2(jm_2)-\phi_2(jm_1)\phi_1(jm_2)]. \quad (2)$$

However, due to the symmetry of the Clebsch-Gordon coefficient,

$$(j_1m_1j_2m_2|j_1j_2JM) =(-1)^{j_1+j_2-J}(j_2m_2j_1m_1|j_2j_1JM), \quad (3)$$

we can rename m_1 and m_2 (which are only summation variables) in the second term of (2) and reduce (2) to

$$\psi_{12}(j^2JM)=N[1-(-1)^{2j+J}] \times\sum_{m_1m_2}(jm_1jm_2|jjJM)\phi_1(jm_1)\phi_2(jm_2).$$

Here is the familiar result that the only allowed states for two equivalent particles of half-integral j are those with an even total spin J.

For any number of particles an antisymmetric wave function can be written in the form[1]

$$N\sum_p(-1)^p\phi_1\phi_2'\phi_3''\cdots\equiv\mathcal{A}\phi_1\phi_2'\phi_3''\cdots, \quad (4)$$

[5] We will work here in the j-j coupling scheme for simplicity of notation.

where the sum extends over all the permutations of the arguments of the functions ϕ, and $(-1)^p$ is the parity of each permutation. If 1, 2, 3, $\cdots$, n are all equivalent particles coupled to a resultant spin J, at least half of the $n!$ terms in the sum (4) are redundant because of (3), and in general some values of J may not be allowed.

The expansion (4) allows the specification of all the intermediate resultant spins of the coupling sequence in an arbitrary way. However, these intermediate spins may not be good quantum numbers in a given problem, and a sum of several wave functions of the type (4) may be necessary to describe an eigenstate.

THE METHOD OF FRACTIONAL PARENTAGES

An alternative procedure for describing the antisymmetrically vector-coupled state of n equivalent particles is the method of fractional parentage.[2,3] In this approach it is not the identity of the particles which is used as the basis of an expansion as in (4), but instead one couples the nth particle to some linear combination of all the allowed states of the $n-1$ other particles.

Thus, for example, with three equivalent particles one would write

$$\psi_{123}(j^3JM)=\sum_{J'}(j^2(J')jJ]\!]j^3J) \times\sum_{m_1m_2m_3M'}\phi_1(jm_1)\phi_2(jm_2)\phi_3(jm_3) \times(jm_1jm_2|jjJ'M')(J'M'jm_3|J'jJM). \quad (5)$$

The coefficients of fractional parentage $(j^2(J')jJ]\!]j^3J)$ are, of course, zero for odd values of J', and they are usually normalized according to

$$\sum_{J'}|(j^2(J')jJ]\!]j^3J)|^2=1. \quad (6)$$

Now since the evenness of J' assures the antisymmetry of (5) with respect to interchange of particles 1 and 2, we need only require that (5) be also antisymmetric under an interchange of particles 2 and 3.[3]

Thus we must have

$$\sum_{J'}(j^2(J')jJ]\!]j^3J)\sum_{m_1'm_2'm_3'M'}\phi_1(jm_1')\phi_2(jm_2')\phi_3(jm_3') \times(jm_1'jm_2'|jjJ'M')(J'M'jm_3'|J'jJM)$$
$$=-\sum_{J''}(j^2(J'')jJ]\!]j^3J)\sum_{m_1''m_2''m_3''M''}\phi_1(jm_1'')\phi_2(jm_2'') \times\phi_3(jm_3'')(jm_1''jm_3''|jjJ''M'')(J''M''jm_2''|J''jJM).$$

Multiplying by

$$\phi_1^*(jm_1)\phi_2^*(jm_2)\phi_3^*(jm_3)(jm_1jm_2|jjJ'''M''') \times(J'''M'''jm_3|J'''jJM),$$

integrating over the coordinates of the three particles, and summing over the m's, we have from the orthogonality of the ϕ's and the Clebsch-Gordon coefficients:

$$(j^2(J')jJ]\!]j^3J)=-\sum_{J''}(j^2(J'')jJ]\!]j^3J)\times\sum_{m_1m_2m_3M'M''}(jm_1jm_2|jjJ'M')(jm_1jm_3|jjJ''M'')\times(J'M'jm_3|J'jJM)(J''M''jm_2|J''jJM).$$

This sum over four Clebsch-Gordon coefficients has been defined by Racah[4] in terms of his W coefficient. So,

$$(j^2(J')jJ]\!]j^3J)=-\sum_{J''}(j^2(J'')jJ]\!]j^3J)(2J'+1)^{\frac{1}{2}}\times(2J''+1)^{\frac{1}{2}}W(jjjJ;J'J'')(-1)^{j+J+J'+J''}. \quad (7)$$

For convenience we shall use Wigner's form[6] of the Racah coefficient

$$\begin{Bmatrix} abe \\ dcf \end{Bmatrix}=(-1)^{a+b+c+d}W(abcd;ef).$$

Thus,

$$(j^2(J')jJ]\!]j^3J)=\sum_{J''}(j^2(J'')jJ]\!]j^3J)(2J'+1)^{\frac{1}{2}}(2J''+1)^{\frac{1}{2}}\times\begin{Bmatrix} jjJ' \\ jJJ'' \end{Bmatrix}(-1)^{J'+J''}, \quad (J' \text{ even only}). \quad (8)$$

The coefficients $(j^2(J')jJ]\!]j^3J)$ are thus given as the solution to the system of Eqs. (8). However, this solution is rather involved, a more direct formula would be much desired.

We can write an antisymmetric wave function for the three particles in the form (4) [remembering that only half of the 3! terms are necessary since the evenness of J_0 leaves all pairs differing only in the permutation of particles 1 and 2 equivalent according to (3)]

$$\psi_{123}(j^2J_0jJM)=N\sum_{m_1m_2m_3M_0}(jm_1jm_2|jjJ_0M_0)\times(J_0M_0jm_3|J_0jJM)[\phi_1(jm_1)\phi_2(jm_2)\phi_3(jm_3)-\phi_1(jm_1)\phi_2(jm_3)\phi_3(jm_2)-\phi_1(jm_3)\phi_2(jm_2)\phi_3(jm_1)]. \quad (9)$$

We can compare this expression with the expansion (5) to solve directly for the coefficients $(j^2(J')jJ]\!]j^3J)$.

One gets then§

$$(j^2(J')jJ]\!]j^3J)=C(J_0)\left[\delta(J',J_0)(jjJ_0)(jJJ_0)+(2J_0+1)^{\frac{1}{2}}(2J'+1)^{\frac{1}{2}}(1+(-1)^{J'})\times\begin{Bmatrix} jjJ' \\ jJJ_0 \end{Bmatrix}\right]\left(\frac{1+(-1)^{J_0}}{2}\right), \quad (10)$$

and in accordance with (6),

$$C(J_0)=\pm\left[3(jjJ_0)(jJJ_0)+6(2J_0+1)\begin{Bmatrix} jjJ_0 \\ jJJ_0 \end{Bmatrix}\right]^{-\frac{1}{2}}, \quad (11)$$

where the phases may be chosen to agree with the conventions of Racah.[3] The symbols (jjJ_0) and (jJJ_0) are "triangular delta functions,"

$$(abc)=\begin{cases}0 & \text{if } a, b, c \text{ cannot form a triangle}\\ 1 & \text{if } a, b, c \text{ can form a triangle.}\end{cases}$$

This solution can also be easily verified by direct substitution in (8) with the help of the following well-known relations[4,6] of the Racah coefficients§

$$\sum_{J'}\begin{Bmatrix} jjJ' \\ jJJ_0 \end{Bmatrix}\begin{Bmatrix} jjJ' \\ jJJ'' \end{Bmatrix}(2J'+1)=\frac{\delta(J_0,J'')}{2J_0+1}(jjJ_0)(jJJ_0),$$

$$\sum_{J'}\begin{Bmatrix} jjJ' \\ jJJ_0 \end{Bmatrix}\begin{Bmatrix} jjJ' \\ jJJ'' \end{Bmatrix}(2J'+1)(-1)^{J'}=(-1)^{J_0+J''}\begin{Bmatrix} jjJ'' \\ jJJ_0 \end{Bmatrix}.$$

The triangular conditions are also implied in the Racah coefficient; and although they will not always be written expressly, they will always be understood as a selection on the permissible values of the parameter J_0.

It remains to interpret the role of J_0 in Eq. (10). Equation (9) for all the different allowed values of J_0 yields the totality of wave functions which can be constructed for three particles which are antisymmetric in all three particles and which are eigenfunctions of j_1^2, j_2^2, j_3^2 $(j_1=j_2=j_3=j)$, J^2 and J_z. It is seen that (9) can be written in the symbolic form:

$$\psi_{123}(j^3JM)=N[\phi(j_1j_2(J_0)j_3JM)-\phi(j_1j_3(J_0)j_2JM)-\phi(j_3j_2(J_0)j_1JM)], \quad (12)$$

which can be interpreted as describing a state in which there is always a pair of angular momenta coupled to J_0. To say that the definite pair (12) is coupled to J_0 would be inconsistent with the symmetry requirement among all three, but the statement that a pair is coupled to J_0 is sensible and may serve as an additional labeling of the state.

Since the different ϕ's in (12) are not necessarily orthogonal, one cannot say whether different choices of J_0 will lead to different ψ's or not. The ψ's defined by (9) will thus constitute generally a redundant basis for all the permissible states. For $j<9/2$ one knows that there is no further degeneracy in the states $\psi(j^3JM)$; thus different choices of J_0 should lead to identical sets of coefficients (10, 11). This implies a number of identities among the Racah coefficients with four half-integers (three of them identical) and two even integers.

In general we may consider the symmetric matrix,

$$(J_0|j^3J|J')=[1+(-1)^{J_0}]\delta(J',J_0)+(2J_0+1)^{\frac{1}{2}}(2J'+1)^{\frac{1}{2}}\times[1+(-1)^{J'}][1+(-1)^{J_0}]\begin{Bmatrix} jjJ' \\ jJJ_0 \end{Bmatrix}, \quad (13)$$

§ This formula has also been derived recently by P. J. Redmond of Birmingham.

[6] A. de-Shalit, Phys. Rev. 91, 1479 (1953).

which satisfies the eigenvalue equation

$$\sum_{J'}(J_0|j^3J|J')\left[(2J'+1)^{\frac{1}{2}}(2J''+1)^{\frac{1}{2}}\begin{Bmatrix} jjJ' \\ jJJ'' \end{Bmatrix} -\delta(J',J'')\right]=0,$$

and so the coefficients (13) can describe at most m independent states, if the matrix,

$$\lambda_{J'J''}=(2J'+1)^{\frac{1}{2}}(2J''+1)^{\frac{1}{2}}\begin{Bmatrix} jjJ' \\ jJJ'' \end{Bmatrix},\quad (J',\, J''\text{ even only}),$$

has m eigenvalues $+1$.

It should be noted that if we take a linear combination of sets of coefficients (10) (given by different J_0's), the normalization (11) is not correct.

THE SENIORITY NUMBER

An additional quantum number which is useful in classifying the states of a number of equivalent particles, and one which is very practical in nuclear spectroscopy, is the seniority number.[3] The seniority number v for the configuration j^nvJ is given by[7]

$$\tfrac{1}{2}(n-v)(2j+3-n-v)=Q(n,v), \tag{14}$$

where $Q(n,v)$ is the eigenvalue of the operator $\sum_{i<j}q_{ij}$, and

$$(j^2J'|q_{ij}|j^2J')=(2j+1)\delta(J',0). \tag{15}$$

For three particles the seniority number v is one if $(j^2(0)jJ]\!]j^3J)\neq 0$ (therefore $J=j$) and three otherwise. There is never more than one state with $v=1$, though with larger values of j the multiplicity of other states may increase.

We shall now show that $J_0=0$ and $J=j$ in Eq. (10), with the normalization (11), always represents just this state with seniority $v=1$.

Using the relation,

$$\begin{Bmatrix} jjJ' \\ jj0 \end{Bmatrix}=(-1)^{2j+J'}\frac{1}{(2j+1)}, \tag{16}$$

we have from (10) and (11) with $J_0=0$:

$$(j^2(J')jJ]\!]j^3J)=\begin{cases}\left(\dfrac{2j-1}{3(2j+1)}\right)^{\frac{1}{2}}, & J'=0\\[2ex] -\left(\dfrac{(2J'+1)}{3(2j+1)(2j-1)}\right)^{\frac{1}{2}}[1+(-1)^{J'}], & J'>0.\end{cases} \tag{17}$$

Calculating the eigenvalue $Q(3,v)$ we have [see reference 3, Eq. (42)]

$$\tfrac{1}{2}(3-v)(2j-v)=3(2j+1)|(j^2(0)jJ]\!]j^3J)|^2,$$

or using the above value (17),

$$\tfrac{1}{2}(3-v)(2j-v)=(2j-1),$$

which is satisfied by $v=1$ only [v can never be greater than $\frac{1}{2}(2j+1)$].

Using the general recursion relations first derived by Racah[3] [his Eqs. (58) here translated into the j-j scheme], we can calculate explicitly the fractional parentage coefficients for the seniority one configuration of any (odd) number of particles

$$\begin{aligned}(j^{n-1}(v'=0\;\;J'=0)jJ=j]\!]j^nv=1\;\;J=j)&=\left(\frac{2j+2-n}{n(2j+1)}\right)^{\frac{1}{2}},\\ (j^{n-1}(v'=2J'\neq 0)jJ=j]\!]j^nv=1\;\;J=j)&=-\left(\frac{2(n-1)(2J'+1)}{n(2j-1)(2j+1)}\right)^{\frac{1}{2}},\quad J'\text{ even only.}\end{aligned} \tag{18}$$

MATRIX ELEMENTS IN STATES OF SENIORITY ONE

The reduced matrix element of an operator of the type $F=\sum_i f_i$, where f_i is a tensor operator of rank r, can be written:[3,4]

$$\begin{aligned}(j^n\alpha J\|F\|j^n{}_{\alpha'}J')=n\sum_{J''\alpha''}&(j^n{}_\alpha J[\![j^{n-1}(\alpha''J'')jJ)\\ &\times(j^{n-1}(\alpha''J'')jJ']\!]j^n\alpha'J')(-1)^{j+J''+J+r}(2J+1)^{\frac{1}{2}}\\ &\times(2J'+1)^{\frac{1}{2}}\begin{Bmatrix} jJJ'' \\ J'jr \end{Bmatrix}(j\|f\|j).\end{aligned} \tag{19}$$

With the configurations $j^n\alpha J$, $j^n\alpha'J'$, both of seniority $v=v'=1$ $(J=J'=j)$. Substituting (18) in (19) and using the relations

$$\begin{aligned}&\sum_{J'}\begin{Bmatrix} jjJ' \\ jjr \end{Bmatrix}(2J'+1)=1,\\ &\sum_{J'}\begin{Bmatrix} jjJ' \\ jjr \end{Bmatrix}(2J'+1)(-1)^{J'}=(-1)^{2j}\delta(r,0)(2j+1),\end{aligned} \tag{20}$$

we have

$$\begin{aligned}(j^nv=1\;\;J=j\|F\|j^nv=1\;\;J=j)&=\frac{(j\|f\|j)}{(2j-1)}\{(n-1)(2j+1)\delta(r,0)+2j-1-(n-1)[1+(-1)^r]\}\\ &=(j\|f\|j)\begin{cases}\times 1, & r\text{ odd}\\ \times\dfrac{2j+1-2n}{2j-1}, & r\text{ even}\neq 0\\ \times n, & r=0\end{cases}\end{aligned} \tag{21}$$

which is identical with Racah's equations (69).[3]

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[7] All the formulas of Racah (see reference 3) on fractional parentage and seniority can be translated from the LS scheme to the j-j scheme by replacing $4l+2$ with $2j+1$ everywhere except in Eq. (15) above.

The matrix element of the two-particle scalar operator $G=\sum_{i<j} g_{ij}$ in the state $j^3\alpha J$ is given by

$$(j^3\alpha J|G|j^3\alpha J)=3\sum_{\alpha' J'}|(j^2(\alpha' J')jJ]j^3\alpha J)|^2 \times(j^2J'|g_{ij}|j^2J').$$

So for the state of seniority $v=1$ using (17), we have

$$(j^3v=1\ J=j|G|j^3v=1\ J=j)$$
$$=\frac{2j-3}{2j-1}(j^2J=0|g_{ij}|j^2J=0)+\frac{2}{(2j+1)(2j-1)} \times\sum_{J'}(2J'+1)(1+(-1)^{J'})(j^2J'|g_{ij}|j^2J').$$

However, the sum

$$\sum_{J'}(2J'+1)\tfrac{1}{2}[1+(-1)^{J'}](j^2J'|g_{ij}|j^2J')$$

can be recognized as simply [see Appendix, Eq. (A5) *et seq.*]

$$(j^{2j+1}0|G|j^{2j+1}0),$$

the interaction in the filled shell.

For the configuration $j^nv=1\ J=j$ we have, using Eq. (A8)

$$(j^nv=1\ J=j|G|j^nv=1\ J=j)$$
$$=\frac{(n-1)(n-3)}{(2j-3)(2j+1)}(j^{2j+1}J=0|G|j^{2j+1}J=0)+\frac{(n-1)(2j-n)}{2(2j-3)}(j^3v=1\ J=j|G|j^3v=1\ J=j).$$

THE CASE OF SENIORITY THREE

To calculate fractional parentage coefficients from Eq. (10) for states of seniority three when the state is not uniquely determined by the value of J alone, we must take some linear combination of states described by different values of J_0. There are, for example, two states of the configuration $(9/2)^3$ with total spin $J=9/2$. The one of seniority $v=1$ is described by (17). With $J_0=2$ in (10), we get the values (not normalized):

$$-\frac{1}{10}(J'=0),\quad \frac{23}{132\sqrt{5}}(J'=2),\quad \frac{-1}{220}(J'=4),$$
$$\frac{8\sqrt{13}}{165}(J'=6),\quad \frac{\sqrt{17}}{55}(J'=8),$$

and with $J_0=8$

$$-\frac{1}{10}(J'=0),\quad \frac{1}{11\sqrt{5}}(J'=2),\quad \frac{93}{55\cdot 13}(J'=4),$$
$$\frac{5}{44\sqrt{13}}(J'=6),\quad \frac{289}{26\cdot 22\sqrt{17}}(J'=8).$$

The desired set of coefficients is that belonging to the state $v=3$, which should have $(j^2(0)jJ]j^3v=3J)=0$. So subtracting the two above sets and normalizing, we get

$$0(J'=0),\quad -\frac{\sqrt{13}}{6\sqrt{11}}(J'=2),\quad \frac{7\sqrt{5}}{2\sqrt{143}}(J'=4),$$
$$\frac{-31}{6\sqrt{55}}(J'=6),\quad \frac{3\sqrt{17}}{2\sqrt{715}}(J'=8),$$

identical with the results of Flowers.[8]

In general, if two different values of J_0, say J_0 and J_0', give two independent sets of coefficients (10), the state described by the linear combination of these with the coefficient for $J'=J_0$ equal to zero is orthogonal to the state described by J_0 alone.

To see this, consider one matrix $(J_0|j^3J|J')$ of the type defined in (13) and any other $F(J')$ formed from a linear combination of terms $(J_0'|j^3J|J')$. We then get for the cross sum

$$\sum_{J'}(J_0|j^3J|J')F(J')$$
$$=[1+(-1)^{J_0}]\left[F(J_0)+\sum_{J'}F(J')(2J_0+1)^{\frac{1}{2}}\times(2J'+1)^{\frac{1}{2}}[1+(-1)^{J'}]\begin{Bmatrix} jjJ' \\ jJJ_0\end{Bmatrix}\right],$$

but this second sum can be performed by the defining Eq. (8)

$$[1+(-1)^{J_0}]\sum_{J'}F(J')(2J_0+1)^{\frac{1}{2}}(2J''+1)^{\frac{1}{2}}\times[1+(-1)^{J'}]\begin{Bmatrix} jjJ' \\ jJJ_0\end{Bmatrix}=2[1+(-1)^{J_0}]F(J_0),$$

since $(-1)^J F(J')=F(J')$. So,

$$\sum_{J'}(J_0|j^3J|J')F(J')=3[1+(-1)^{J_0}]F(J_0),$$

which is zero if the linear combination F has the coefficient for $J'=J_0$ equal to zero.

Looking for more complicated configurations, we find that for $(11/2)^3$ $J=9/2$ there are two independent states, both of seniority $v=3$. For $J_0=2$ we get the (normalized) values

I: $0.687(J'=2)$, $0.352(J'=4)$, $0.282(J'=6)$, $-0.105(J'=8)$, $-0.560(J'=10)$.

With $J_0=10$ we get another set of values, and taking the linear combination of these two described above, we get the orthogonal (normalized) set:

II: $0(J'=2)$, $-0.500(J'=4)$, $0.726(J'=6)$, $-0.452(J'=8)$, $0.141(J'=10)$.

In contrast with the convenient characterization of nuclear states afforded by the usual concept of seniority,

[8] B. H. Flowers, Proc. Roy. Soc. (London) **A215**, 398 (1952).

this division of the two states of the configuration $(11/2)^3{}_{9/2}$ does not appear to be very practical. That is, if we consider the effect of a perturbing short-range attractive force between all pairs of particles, the state of lowest energy in this configuration is not described by I alone or II alone, but by a mixture of both.

MORE THAN THREE EQUIVALENT PARTICLES

In the fractional parentage description of a state of equivalent particles, the wave function is written

$$\psi(j^n\alpha_nJ_nM_n)=\sum_{\substack{J\cdots\\ \alpha\cdots}}(j^{n-1}(\alpha_{n-1}J_{n-1})jJ_n]\!]j^n\alpha_nJ_n)$$
$$\times(j^{n-2}(\alpha_{n-2}J_{n-2})jJ_{n-1}]\!]j^{n-1}\alpha_{n-1}J_{n-1})\cdots$$
$$\times(j^2(\alpha_2J_2)jJ_3]\!]j^3\alpha_3J_3)\delta(J_2\text{ even})$$
$$\times\sum_{m\cdots}\phi_1(jm_1)\phi_2(jm_2)\cdots$$
$$\times\phi_n(jm_n)(jm_1jm_2|jjJ_2M_2)\cdots$$
$$\times(J_{n-1}M_{n-1}jm_n|J_{n-1}jJ_nM_n). \quad (22)$$

If the fractional parentage coefficients for 3, 4, $\cdots n-1$ particles are known from earlier work, we can get the coefficients for n particles by comparing (22) with (4). The antisymmetrization (4) can be interpreted as a series of changes in the coupling order of the several angular momenta. Thus, it is seen that the general coefficient $(j^{n-1}(\alpha'J')jJ]\!]j^n\alpha J)$ is given as sums over products of all the fractional parentage coefficients for less than n particles and coefficients for all the transformations in the coupling scheme of n angular momenta. These transformation coefficients can always be expressed as sums of products of Racah coefficients. There will be $n-2$ arbitrary parameters analogous to J_0 used before and these will help to describe the state.

For $n=4$ we compare the two developments

$$\psi(j^4JM)=\sum_{J'J''}(j^3(J')jJ]\!]j^4J)(j^2(J'')jJ']\!]j^3J')$$
$$\times\sum_{\substack{m_1m_2m_3\\ m_4M'M''}}\phi_1(jm_1)\phi_2(jm_2)\phi_3(jm_3)\phi_4(jm_4)$$
$$\times(jm_1jm_2|jjJ''M'')(J''M''jm_3|J''jJ'M')$$
$$\times(J'M'jm_4|J'jJM), \quad (23)$$

and

$$\psi(j^4JM)=N\sum_{\substack{m_1m_2m_3\\ m_4M_2M_3}}(jm_1jm_2|jjJ_2M_2)$$
$$\times(J_2M_2jm_3|J_2jJ_3M_3)(J_3M_3jm_4|J_3jJM)$$
$$\times\phi_1(jm)\phi_2(jm)\phi_3(jm)\phi_4(jm),$$
$$[1234-1243+1423-1432+1342-1324$$
$$-2341+2314+2431-2413+3412-3421], \quad (24)$$

where the permutations have been written out symbolically. Then

$$(j^3(J')jJ]\!]j^4J)=\sum_{J''}(j^3J'[\![j^2(J'')jJ')X(J''), \quad (25)$$

where $X(J'')$ is a series of sums of products of six Clebsch-Gordon coefficients. This can be evaluated in terms of Racah coefficients and $9-j$ symbols[6]

$$\begin{Bmatrix}j_{11}j_{12}j_{13}\\ j_{21}j_{22}j_{23}\\ j_{31}j_{32}j_{33}\end{Bmatrix}=\sum_j(2j+1)(-1)^{2j}\begin{Bmatrix}j_{11}j_{12}j_{13}\\ j_{23}j_{33}j\end{Bmatrix}$$
$$\times\begin{Bmatrix}j_{21}j_{22}j_{23}\\ j_{12}j\ \ j_{32}\end{Bmatrix}\begin{Bmatrix}j_{31}j_{32}j_{33}\\ j\ \ j_{11}j_{21}\end{Bmatrix}, \quad (26)$$

giving

$$X(J'')=N(2J'+1)^{\frac{1}{2}}(2J''+1)^{\frac{1}{2}}(2J_2+1)^{\frac{1}{2}}(2J_3+1)^{\frac{1}{2}}$$
$$\times(-1)^{J'-J''-J_3}\Bigg\{-\frac{\delta(J_2,J'')}{2J_2+1}\frac{\delta(J_3,J')}{2J_3+1}$$
$$+\frac{\delta(J_2,J'')}{2J_2+1}\begin{Bmatrix}jJ_2J_3\\ jJJ'\end{Bmatrix}-2\frac{\delta(J_3,J')}{2J_3+1}\begin{Bmatrix}jjJ_2\\ jJ_3J''\end{Bmatrix}$$
$$+2\begin{Bmatrix}jjJ_2\\ jJ'J''\end{Bmatrix}\begin{Bmatrix}jJ_3J_2\\ jJ'J\end{Bmatrix}+2\begin{Bmatrix}jjJ''\\ jJ_3J_2\end{Bmatrix}\begin{Bmatrix}jJ_3J''\\ jJ'J\end{Bmatrix}$$
$$+\begin{Bmatrix}jjJ_2\\ J'J''j\\ JjJ_3\end{Bmatrix}[1+(-1)^{j+J_3+J}]\Bigg\}. \quad (27)$$

Now substituting (10) and (27) in (25) and summing over J'' we obtain

$$[j^3(J')jJ]\!]j^4J]=N(J_2,J_3)(2J'+1)^{\frac{1}{2}}(2J_2+1)^{\frac{1}{2}}$$
$$\times(2J_3+1)^{\frac{1}{2}}(-1)^{J'+J_3}\Bigg\{\frac{-3\delta(J'J_3)}{2J_3+1}$$
$$\times\left[2\begin{Bmatrix}jjJ_2\\ jJ_3J_2\end{Bmatrix}+\frac{1}{2J_2+1}\right]+3\begin{Bmatrix}jJ_3J_2\\ jJ'J\end{Bmatrix}$$
$$\times\left[2\begin{Bmatrix}jjJ_2\\ jJ'J_2\end{Bmatrix}+\frac{1}{2J_2+1}\right]+[3+(-1)^{j+J_3+J}]$$
$$\times\Bigg[\begin{Bmatrix}J_2J_2J\\ jJ_3j\end{Bmatrix}\begin{Bmatrix}J_2J_2J\\ jJ'j\end{Bmatrix}(-1)^{J'+J_3+J}$$
$$+\begin{Bmatrix}jjJ_2\\ jJ_3J_2\end{Bmatrix}\begin{Bmatrix}jJ'J_2\\ jJ_3J\end{Bmatrix}+\begin{Bmatrix}jjJ_2\\ J'J_2j\\ JjJ_3\end{Bmatrix}\Bigg]\Bigg\}. \quad (28)$$

Here J_2 is equivalent to the earlier J_0, and J_3 is a new parameter of similar character. In general, to describe some particular state a linear combination of coefficients (28) given by different values of J_2, J_3 may be needed.

While the above method of breaking off a single particle is what one needs for calculating matrix elements of one-particle (F-type) operators, the two-particle operators (G-type) are best handled with the following description (for four particles).

$$\psi(j^4JM)=\sum_{J_{12}J_{34}}(j^2(J_{12})j^2(J_{34})J]\!]j^4J)\sum_{m\cdots}\phi_1(jm_1)\times\phi_2(jm_2)\phi_3(jm_3)\phi_4(jm_4)(jm_1jm_2|jjJ_{12}M_{12})\times(jm_3jm_4|jjJ_{34}M_{34})(J_{12}M_{12}J_{34}M_{34}|J_{12}J_{34}JM),\quad(29)$$

where the particles are grouped in pairs. With J_{12}, J_{34} even, this description is antisymmetric in the pairs of particles (1,2) and (3,4). To determine the coefficients $(j^2(J_{12})j^2(J_{34})J]\!]j^4J)$ we shall exchange particles two and three and require the wave function to be -1 times (29)

$$(29)=-\sum_{J_{13}J_{24}}(j^2(J_{13})j^2(J_{24})J]\!]j^4J)\sum_{m\cdots}\phi_1(jm_1)\times\phi_2(jm_2)\phi_3(jm_3)\phi_4(jm_4)(jm_1jm_3|jjJ_{13}M_{13})\times(jm_2jm_4|jjJ_{24}M_{24})(J_{13}M_{13}J_{24}M_{24}|J_{13}J_{24}JM).\quad(30)$$

Comparing (29) and (30), we have the requirement

$$\sum_{J_{13}J_{24}}(j^2(J_{13})j^2(J_{24})J]\!]j^4J)[(2J_{13}+1)(2J_{24}+1)\times(2J_{12}+1)(2J_{34}+1)]^{\frac{1}{2}}\begin{Bmatrix}jjJ_{12}\\ jjJ_{34}\\ J_{13}J_{24}J\end{Bmatrix}=-(j^2(J_{12})j^2(J_{34})J]\!]j^4J),\quad(J_{12},\ J_{34}\text{ even only}).\quad(31)$$

This is the 4-particle analog of Eq. (7).

By comparing (29) with (24), we could solve for the coefficient directly; but with the help of the relations,

$$\sum_{J_{12}J_{34}}\begin{Bmatrix}jjJ_{12}\\ jjJ_{34}\\ KLJ\end{Bmatrix}\begin{Bmatrix}jjJ_{12}\\ jjJ_{34}\\ J_{13}J_{24}J\end{Bmatrix}(2J_{12}+1)(2J_{34}+1)=\frac{\delta(K,J_{13})\delta(L,J_{24})}{(2K+1)(2L+1)}(jjJ_{13})(jjJ_{24})(J_{13}J_{24}J),$$

$$\sum_{J_{12}J_{34}}\begin{Bmatrix}jjJ_{12}\\ jjJ_{34}\\ KLJ\end{Bmatrix}\begin{Bmatrix}jjJ_{12}\\ jjJ_{34}\\ J_{13}J_{24}J\end{Bmatrix}(2J_{12}+1)(2J_{34}+1)(-1)^{J_{34}}=(-1)^{2j+J_{24}-L}\begin{Bmatrix}jjJ_{13}\\ jjJ_{24}\\ KLJ\end{Bmatrix},\quad(32)$$

it is easy to show that a general solution to (31) is

$$(j^2(J_{12})j^2(J_{34})J]\!]j^4J)=C(K,L)\Big\{\delta(J_{12},K)\delta(J_{34},L)+(-1)^J\delta(J_{12},L)\delta(J_{34},K)-[1+(-1)^{J_{12}}][1+(-1)^{J_{34}}][(2J_{12}+1)\times(2J_{34}+1)(2K+1)(2L+1)]^{\frac{1}{2}}\begin{Bmatrix}jjJ_{12}\\ jjJ_{34}\\ KLJ\end{Bmatrix}\Big\},\quad(33)$$

where K, L are two (even) parameters similar to J_0 used before. The six triangular conditions implied in the $9-j$ symbols are understood to apply to the entire expression (33) although not written out expressly.

Equation (33) takes on a simple form when we set one of the parameters, say L, equal to zero; then, by the triangular conditions, we must have $K=J$ (even), and we get

$$(j^2(J_{12})j^2(J_{34})J]\!]j^4J)=C\Big[[\delta(J_{12},J)\delta(J_{34},0)+\delta(J_{12},0)\delta(J_{34},J)+[1+(-1)^{J_{12}}][1+(-1)^{J_{34}}]\times\frac{(2J_{12}+1)^{\frac{1}{2}}(2J_{34}+1)^{\frac{1}{2}}}{(2j+1)^{\frac{1}{2}}}\begin{Bmatrix}J_{12}J_{34}J\\ j\ \ j\ \ j\end{Bmatrix}\Big].\quad(34)$$

If we require the normalization

$$\sum_{J_{12}J_{34}}|(j^2(J_{12})j^2(J_{34})J]\!]j^4J)|^2=1,$$

we have

$$C=\pm\left[6\delta(J,0)+6\frac{2j-3}{2j+1}\right]^{-\frac{1}{2}}.\quad(35)$$

Testing these values (34), (35) in Eqs. (14) and (15), we have

$$\tfrac{1}{2}(4-v)(2j-1-v)=6(2j+1)\sum_{J_{12}}|(j^2(J_{12})j^2(0)J]\!]j^4J)|^2=6(2j+1)\left[6\delta(J,0)+6\frac{2j-3}{2j+1}\right]^{-1}\left[1+\frac{-4\delta(J,0)}{2j+1}\right]^2=\begin{cases}2(2j-1) & J=0\\ (2j-3) & J>0.\end{cases}$$

So that the fractional parentage coefficient (34), (35) describes the state of j^4 with seniority zero or two as J is zero or greater. Using the recursion relations (A_3), we have then for n particles of seniority two:

$$(j^{n-2}(v'=0J')j^2(J'')J]\!]j^nv=2J)$$
$$=\left[\frac{2(2j+1-n)(2j+3-n)}{n(n-1)(2j-1)(2j+1)}\right]^{\frac{1}{2}}\delta(J',0)\delta(J'',J),$$
$$(j^{n-2}(v'=2J'\neq 0)j^2(J'')J]\!]j^nv=2J)$$
$$=\left[\frac{3\cdot 4(n-2)(2j+1-n)}{2n(n-1)(2j-3)}\right]^{\frac{1}{2}}\left[6\frac{2j-3}{2j+1}\right]^{-\frac{1}{2}}$$
$$\times\left[\delta(J',J)\delta(J'',0)+[1+(-1)^{J'}][1+(-1)^{J''}]\right.$$
$$\left.\times\frac{(2J'+1)^{\frac{1}{2}}(2J''+1)^{\frac{1}{2}}}{(2j+1)^{\frac{1}{2}}}\begin{Bmatrix}J'J''J\\ jjj\end{Bmatrix}\right]. \quad (36)$$

To calculate the matrix element of an operator of the type G in a configuration of four equivalent particles with seniority two, we have

$$(j^4v=2J|G|j^4v=2J)$$
$$=6\sum_{J'J''}|(j^2(J')j^2(J'')J]\!]j^4v=2J)|^2(j^2J'|g_{ij}|j^2J')$$
$$=\textstyle\sum_{J'}Z(J')(j^2J'|G|j^2J'), \quad (37)$$

where

$$Z(J')=6\sum_{J''}\left[6\frac{2j-3}{2j+1}\right]^{-1}\left[\delta(J',J)\delta(J'',0)\right.$$
$$+\delta(J',0)\delta(J'',J)-\frac{8}{(2j+1)}[\delta(J',J)\delta(J'',0)$$
$$+\delta(J',0)\delta(J'',J)]+4[1+(-1)^{J'}][1+(-1)^{J''}]$$
$$\left.\times\frac{(2J'+1)(2J''+1)}{(2j+1)}\begin{Bmatrix}J'J''J\\ jjj\end{Bmatrix}^2\right]$$
$$=\frac{1}{2j-3}\left\{(2j-7)[\delta(J',J)+\delta(J',0)]\right.$$
$$+4(2J'+1)[1+(-1)^{J'}]$$
$$\left.\times\left[\frac{1}{2j+1}-\begin{Bmatrix}jjJ'\\ jjJ\end{Bmatrix}\right]\right\}. \quad (38)$$

If g_{ij} is $-\delta(\mathbf{r}_i-\mathbf{r}_j)$, then[6]

$$(j^2J'|g_{ij}|j^2J')=\frac{-F_0}{2}(2j+1)^2\frac{(j\tfrac{1}{2}j-\tfrac{1}{2}|jjJ'0)^2}{2J'+1}, \quad J' \text{ even},$$

and we can sum (37) and get

$$(j^4,v=2,J|G|j^4,v=2,J)$$
$$=-\tfrac{1}{2}F_0(2j+1)+(j^2J|g_{ij}|j^2J). \quad (39)$$

Using Eq. (A8), we have for $j^nv=2$:

$$(j^n,v=2,J|G|j^n,v=2,J)$$
$$=-\tfrac{1}{2}F_0(2j+1)(n-2)/2+(j^2J|g_{ij}|j^2J).$$

Also for states of seniority $v=0$, $J=0$:

$$(j^n,v=0,J=0|G|j^n,v=0,J=0)=-\tfrac{1}{2}F_0(2j+1)n/2.$$

Hence, for the configuration j^n, the energy level separation from the state $v=J=0$ to the state $v=2$, J is given by

$$\Delta_{0J}^{(n)}=(j^2,v=2,J|g_{ij}|,j^2,v=2,J)+\tfrac{1}{2}F_0(2j+1)$$
$$=\Delta_{0J}^{(2)}. \quad (40)$$

Thus one would expect the $0-2-4\cdots$ energy level differences from the ground states of even-even nuclei to be independent of the number of particles in the unfilled outer shell. This conclusion may also be drawn from the more general analysis of Racah and Talmi.[9]

THE 7/2 SHELL

The configuration $(7/2)^4$ is the simplest (i.e., lowest j) instance where the seniority number is needed to classify the states. There are two states each of spin $J=2$ and $J=4$; in each pair one state has seniority number $v=2$ and the other has $v=4$. We shall now show that in this case the seniority number v is always a good quantum number. Consider any two-body symmetric interaction

$$G=\sum_{i<j}g_{ij}, \quad (41)$$

where, according to Racah, the angular and spin parts of g_{ij} can be written as a sum of scalar products of tensor operators which act on each particle separately:

$$g_{ij}=\textstyle\sum_r \tau_i^r\cdot\tau_j^r.$$

G is now conveniently written in the following form

$$G=\textstyle\sum_r\frac{1}{2}[\sum_i\tau_i^r\cdot\sum_j\tau_j^r-\sum_i\tau_i^r\cdot\tau_i^r]$$
$$=\textstyle\sum_r\frac{1}{2}[|T^r|^2-\sum_i\tau_i^r\cdot\tau_i^r], \quad (42)$$

where $T^r=\sum_i\tau_i^r$. Taking the matrix element of G between two states of the configuration j^n with different seniorities, we have from the first term

$$(j^nvJ\|T^r\cdot T^r\|j^nv'J)=\sum_{J''v''}(2J+1)^{-\frac{1}{2}}(-1)^{J-J''}$$
$$\times(j^nvJ\|T^r\|j^nv''J'')(j^nv''J''\|T^r\|j^nv'J). \quad (43)$$

Racah has shown that the matrix elements of T^r for odd r are diagonal in the seniority; since $v\neq v'$, every term in (43) has $v''\neq v$ or $v''\neq v'$ so all these matrix elements vanish.

For the even values of r we can use Racah's equation (74)[4] modified according to his discussions (76)[4] and (65).[3] For the half-shell, $n=(2j+1)/2$, the formula is

$$(j^nvJ\|T^r\|j^nv''J'')$$
$$=(-1)^{r+1+\frac{1}{2}(v''-v)}(j^nvJ\|T^r\|j^nv''J''). \quad (44)$$

[9] G. Racah and I. Talmi, Physica 18, 1097 (1952).

except for $r=0$. Hence, for even $r(\neq 0)$, these matrix elements vanish for $\Delta v=0$. In our case of $j=7/2$, the four particles constitute just half a shell, $v-v'$ is 2, and the general selection rule for these matrix elements is $|v-v''|=0, 2$. Thus, for every term in (43), either $|v-v''|$ or $|v'-v''|$ is 0 and all these terms vanish. All that is left of (42) are the terms T^0 and $\tau_i^r\cdot\tau_i^r$, both of which are scalars, i.e., independent of all angle and spin coordinates. Since, then, wave functions for states of different seniorities are orthogonal, the entire matrix element of G is zero and the seniority remains a good quantum number.

We shall now calculate the diagonal energy matrix element of G in the state $(7/2)^4$ $v=2$ $J=2$, which in nuclear spectroscopy is expected to be the first excited state of this configuration. This matrix element is given by (37):

$$((7/2)^4, v=2, J=2|G|(7/2)^4, v=2, J=2) =\sum_{J'} Z(J')((7/2)^2J'|G|(7/2)^2J'). \quad (45)$$

Equation (38) for $Z(J')$ contains the term

$$(2J'+1)[1+(-1)^{J'}]\begin{Bmatrix} jjJ' \\ jjJ \end{Bmatrix}.$$

This will be recognized as the major term in the formula for the fractional parentage coefficient for the configuration j^3 $J=j$, where J (which is even) plays the role of J_0. Since we know that for $(7/2)^3$ $J=7/2$ there is only one state, and that one is of seniority $v=1$, we can use (17) and write directly

$$(2J'+1)[1+(-1)^{J'}]\begin{Bmatrix} jjJ' \\ jjJ \end{Bmatrix} =-\delta(J',J)+C\left[\delta(J',0)-\frac{(2J'+1)}{2j+1}(1+(-1)^{J'})\right], \quad (46)$$

and setting $J'=0$ we have $C=-2/(2j-1)$. Substituting (46) into (38) we have

$$Z(J')=\delta(J',J)+\frac{8}{(2j-1)(2j-3)}\delta(J',0) +\frac{4}{(2j+1)(2j-1)}(2J'+1)[1+(-1)^{J'}], \quad (47)$$

or

$$((7/2)^422|G|(7/2)^422) =((7/2)^22|G|(7/2)^22)+\tfrac{1}{3}((7/2)^20|G|(7/2)^20) +\tfrac{1}{12}\sum_{J'}(2J'+1)[1+(-1)^{J'}]((7/2)^2J'|G|(7/2)^2J').$$

For the state $(7/2)^4$ $v=0$ $J=0$, we get, using (A8), the matrix element

$$((7/2)^400|G|(7/2)^400)=(4/3)((7/2)^20|G|(7/2)^20)+\tfrac{1}{6}D,$$

but

$$D=\tfrac{1}{2}\sum_{J'}(2J'+1)[1+(-1)^{J'}](j^2J'|G|j^2J'). \quad (48)$$

Finally, for the $0-2$ energy difference we have

$$((7/2)^400|G|(7/2)^400)-((7/2)^422|G|(7/2)^422) =((7/2)^20|G|(7/2)^20)-((7/2)^22|G|(7/2)^22). \quad (49)$$

Here we have the general result that for any two-body interaction the excitation energy from the ground to the first excited state is the same for two as for four particles in the $j=7/2$ shell.

An example of this result can be seen in the results of the explicit calculations of Kurath.[10]

The clearest data on this subject involves the pair of isotopes Ca^{42} and Ca^{44}. Both nuclei have a magic number, 20, of protons, hence we can well ignore the effect of a mixing of neutron and proton configurations. The nuclei have, respectively, two and four neutrons in the $f_{7/2}$ shell. This shell is supposedly well isolated between the magic numbers 20 and 28 so that contributions from a second-order calculation would be very small.

The energy of the first excited state in these two nuclei are[11]

Ca^{42}—1.51 Mev,
Ca^{44}—1.16 Mev.

The large discrepancy between these two measured values indicates very strongly the limitations to which an individual-particle shell model for the nucleus must be restricted.

APPENDIX

As was mentioned earlier, for work with G-type operators it is most convenient to describe a wave function for n equivalent particles in the form:

$$\psi(j^n\alpha J)=\sum_{\alpha''J'J''}(j^{n-2}(\alpha''J'')j^2(J')J]\!]j^n\alpha J) \times\psi(j^{n-2}(\alpha''J'')j^2(J')J). \quad (A1)$$

These two-particle fractional parentage coefficients are related to the one-particle coefficients by

$$(j^{n-2}(\alpha''J'')j^2(J')J]\!]j^n\alpha J) =\sum_{\alpha'''J'''}(2J'+1)^{\frac{1}{2}}(2J'''+1)^{\frac{1}{2}}(-1)^{J+J''+2j} \times\begin{Bmatrix} jjJ' \\ JJ''J''' \end{Bmatrix}(j^{n-2}(\alpha''J'')jJ''']\!]j^{n-1}\alpha'''J''') \times(j^{n-1}(\alpha'''J''')jJ]\!]j^n\alpha J), \quad (A2)$$

and in terms of seniority they obey the selection rule $\Delta v=0, \pm 2$. Putting Racah's recursion relations [reference 3, Eq. (58)] for the one-particle coefficients into (A2) we can get recursion relations for the two-particle

[10] D. Kurath, Phys. Rev. **91**, 1430 (1953).
[11] G. Scharff-Goldhaber, Phys. Rev. **90**, 587 (1953).

coefficients

$$(j^{n-2}(v''J'')j^2(J')J]\!]j^nvJ)$$
$$=\left[\frac{v(v-1)(2j+3-n-v)(2j+5-n-v)}{n(n-1)(2j+3-2v)(2j+5-2v)}\right]^{\frac{1}{2}}\times(j^{v-2}(v-2J'')j^2(J')J]\!]j^vvJ)(v''=v-2)$$
$$=\left[\frac{(v+1)(v+2)(n-v)(2j+3-n-v)}{2n(n-1)(2j+1-2v)}\right]^{\frac{1}{2}}\times(j^v(vJ'')j^2(J')J]\!]j^{v+2}vJ)(v''=v)$$
$$=\left[\frac{(v+3)(v+4)(n-v)(n-v-2)}{8n(n-1)}\right]^{\frac{1}{2}}\times(j^{v+2}(v+2J'')j^2(J')j]\!]j^{v+4}vJ)(v''=v+2). \quad \text{(A3)}$$

We shall take matrix elements of the symmetric scalar operator $G=\sum_{i<j}g_{ij}$ with the wave function (A1) as follows:

$$(j^nvJ|G|j^nv'J)=\frac{n(n-1)}{2}\sum_{J'J''v''}(j^2J'|g_{ij}|j^2J')\times(j^nvJ[\![j^{n-2}(v''J''),\,j^2(J')J)\times(j^{n-2}(v''J''),\,j^2(J')J]\!]j^nv'J). \quad \text{(A4)}$$

We shall also need the relation

$$(j^{m-n}\alpha J|G|j^{m-n}\alpha'J)=(j^n\alpha J|G|j^n\alpha'J)+\frac{2j+1-2n}{2j+1}D, \quad \text{(A5)}$$

where $m=2j+1$; and

$$D=(j^m,\,v=0,\,J=0|G|j^m,\,v=0,\,J=0)=\tfrac{1}{2}m(m-1)\sum_{J'}(j^{m-2}(J'),\,j^2(J')0]\!]j^m0)^2\times(j^2J'|g_{ij}|j^2J').$$

From a simple generalization of Racah's equation (19),[3] we have

$$(j^{m-2}(J'),\,j^2(J')0]\!]j^m0)^2=\frac{2}{m(m-1)}(2J'+1)(j^0(0),\,j^2(J')J']\!]j^2J'),$$

so

$$D=\sum_{J'}(2J'+1)\left(\frac{1+(-1)J'}{2}\right)(j^2J'|g_{ij}|j^2J').$$

In (A4) the selection rule is $v-v'=0,\ \pm2,\ \pm4$. For $v'=v-4$, there is only one term in the sum over v'', namely $v''=v-2$; then using Eq. (A3), we can separate out the n-dependence of the matrix element and get

$$(j^nvJ|G|j^nv-4J)=\left[\frac{(2j+3-n-v)(2j+5-n-v)(n-v+2)(n-v+4)}{8(2j+3-2v)(2j+5-2v)}\right]^{\frac{1}{2}}\times(j^vvJ|G|j^vv-4J). \quad \text{(A6)}$$

For $v'=v-2$ there are two terms in the sum (A4), $v''=v,\ v-2$. Thus this matrix element for n particles can be expressed in terms of that for v particles and one other configuration. We take the other to be $2j+1-v$ particles and use (A5) to get

$$(j^nvJ|G|j^nv-2J)=\left[\frac{(2j+3-n-v)(n-v+2)}{2(2j+3-2v)}\right]^{\frac{1}{2}}\times\left[(j^vvJ|G|j^vv-2J)+\frac{(n-v)}{2j+1}D\right]. \quad \text{(A7)}$$

For $v'=v$ we shall use the three configurations of v, $2j+1-v$, and $v+2$ particles to account for the three terms in the sum (A4). Substitution of the relations (A3) gives an expression which finally reduces to the earlier quoted result

$$(j^nvJ|G|j^nvJ)=\frac{(n-v)(2j+1-n-v)}{2(2j-1-2v)}\times(j^{v+2}vJ|G|j^{v+2}vJ)-\frac{(n-v-2)(2j-1-n-v)}{2(2j-1-2v)}\times(j^vvJ|G|j^vvJ)+\frac{(n-v)(n-v-2)}{(2j-1-2v)(2j+1)}D. \quad \text{(A8)}$$

These reduction equations play the same role in the study of two-particle operators as Eqs. (67) and (69) of Racah[3] for one-particle operators.