One-Sided Limits

In order to calculate a limit at a point, we need to have an interval around that point; that is, we consider values of the function for x values on both sides of the point. Since we are considering values on both sides of the point, this type of limit is sometimes referred to as a two-sided limit. At some points, such as end points, it is not possible to find an interval on both sides of the point; for endpoints we can only find an interval on one side of the point. Instead, we can use the information that we are provided on that interval, in order to calculate a one-sided limit. In this way, we can define left-hand and right-hand limits, looking at the function from the left or right side of the point, respectively. We write the left-hand limit of $f(x)$, or the limit as x approaches x_0 from the left-hand side as

$$\lim_{x \to x_0^-} f(x)$$

and we write the right-hand limit as

$$\lim_{x \to x_0^+} f(x)$$

where the $-$ and $+$ denote whether it is approaching from the left or right hand side, respectively. More formally, we have the following definitions.

Definition: Right-hand Limit
We say that L is the right-hand limit of $f(x)$ at x_0, written

$$\lim_{x \to x_0^+} f(x) = L$$

if for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 < x < x_0 + \delta \Longrightarrow |f(x) - L| < \epsilon$$

Definition: Left-hand Limit
We say that L is the left-hand limit of $f(x)$ at x_0, written

$$\lim_{x \to x_0^-} f(x) = L$$

if for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 - \delta < x < x_0 \Longrightarrow |f(x) - L| < \epsilon$$

It is noteworthy that all of the rules for combining two-sided limits also apply for combining one-sided limits.

Example 1 Find $\lim_{x \to 0^+} f(x)$ and $\lim_{x \to 0^-} f(x)$ for $f(x) = \frac{|x|}{x}$.
Solution The solution to this problem becomes much more evident if we rewrite $f(x)$ as

$$f(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$$

Now we can see that looking from just the left or right side of the point $x = 0$, we have two constant functions. Since the limit of a constant is just that constant, it follows that

$$\lim_{x \to 0^+} f(x) = 1 \quad \text{and} \quad \lim_{x \to 0^-} f(x) = -1$$

The following theorem is a useful tool for relating one-sided and two-sided limits.

> *Theorem: One-sided and Two-sided Limits*
> A function $f(x)$ has a limit L at x_0 if and only if it has right-hand and left-hand limits at x_0, and both of those limits are L.

If both of the one-sided limits have the same value L, then we can certainly construct a δ-interval on both sides of x_0 by combining both of the one-sided intervals, which implies the two-sided limit exists. If the one-sided limits exist but disagree, then it is impossible for the function to approach a single value as $x \to x_0$, which implies that the two-sided limit does not exist. From this we can conclude that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist. This is a much more efficient way to prove a limit does not exist than proving that it does not exist for all possible values L.

Example 2 Prove that

$$\lim_{x \to 0^+} \sqrt{x} = 0$$

Solution Consider $\epsilon > 0$, arbitrary. We need to find $\delta > 0$ so that for all x with $0 < x < \delta$ we have $|\sqrt{x} - 0| < \epsilon$ or $\sqrt{x} < \epsilon$. Manipulating this inequality

$$\begin{aligned} \sqrt{x} &< \epsilon \\ 0 \leq x &< \epsilon^2 \end{aligned}$$

Thus, if we set $\delta = \epsilon^2$, for any x with $0 < x < \delta = \epsilon^2$ we have

$$\sqrt{x} < \sqrt{\epsilon^2} = \epsilon$$

and the conclusion follows.

Example 3 Let $f(x)$ be given by

$$f(x) = \sqrt{4 - x^2}$$

Find the one-sided limits at the endpoints of the domain of this function. Using the definition of left and right-hand limits, prove that these limits exist, for some values L.

Solution First we must recall that $\sqrt{x}$ is not definied on $\mathbb{R}$ for $x < 0$. In this way, we can determine that if $|x| > 2$ then $f(x) = \sqrt{4 - x^2}$ is not defined. Thus, we can see that the domain of this function is $[-2, 2]$. On this domain our function is a semicircle. At the left endpoint we must consider the right-hand limit, and at the right endpoint we consider the left-hand limit. Using the rules for combining limits,

$$\lim_{x \to -2^+} \sqrt{4-x^2} = 0 \quad \text{and} \quad \lim_{x \to 2^-} \sqrt{4-x^2} = 0$$

Now our task is to prove that these limits exist as written above, using the definition of one-sided limits. We will prove that the limit as $x \to 2^-$ is 0, and leave the analagous proof at the left endpoint to the reader.

Consider $\epsilon > 0$, arbitrary. We need to find a $\delta > 0$ so that for all x with $2 - \delta < x < 2$ we have

$$|\sqrt{4-x^2} - 0| < \epsilon$$

In this problem it will be difficult to directly manipulate the second inequality in order to find a sufficiently small value for δ. We will need to take a slightly more creative approach. Notice that

$$\sqrt{4-x^2}$$

will be at its largest when x is the smallest, or when x is at the farthest left point of the interval $2 - \delta < x < 2$. Thus, if we can find a value for δ such that

$$\sqrt{4-(2-\delta)^2} < \epsilon$$

we will have a δ such that for all x in the interval $2 - \delta < x < 2$ the function values are within the error tolerance ϵ of 0. Thus, for all x in the interval

$$\sqrt{4-x^2} < \sqrt{4-(2-\delta)^2} = \sqrt{4-(4-4\delta+\delta^2)} = \sqrt{4-4+4\delta-\delta^2} = \sqrt{4\delta-\delta^2} = \sqrt{\delta(4-\delta)}$$

The first thing to note is that because we cannot have a negative input to $\sqrt{\delta(4-\delta)}$ we need to have $(4-\delta) \geq 0$ which means that

$$\delta \leq 4$$

Now, to have

$$\sqrt{\delta(4-\delta)} < \epsilon$$

we need

$$\delta(4-\delta) < 4\delta < \epsilon^2$$

From this inequality, we obtain the restriction

$$\delta < \frac{\epsilon^2}{4}$$

Thus, we set

$$\delta = min(\frac{\epsilon^2}{16}, 1)$$

in order to encapsulate both of the previous restrictions on δ we found (there is nothing unique about the values we chose, just that they satisfy $\delta \leq 4$ and $\delta < \epsilon^2/4$). Now consider arbitrary x with $2 - \delta < x < 2$. It follows

$$\sqrt{4-x^2} < \sqrt{4-(2-\delta)^2} = \sqrt{\delta(4-\delta)} < \sqrt{\frac{\epsilon^2}{16} \cdot 4} = \frac{\epsilon}{2} < \epsilon$$

After this long and arduous analysis, we have managed to prove the one-sided limit exists, and is equal to 0.