The Precise Definition of a Limit

Previously we stated that intuitively the notion of a limit is the value a function approaches at a given point. We refined this notion in terms of approximations, stating that

$$\lim_{x \to x_0} f(x) = L$$

if for any error tolerance around L, we can find an interval around x_0 (excluding x_0) such that for all x values within that interval, the distance between $f(x)$ and L is within the error tolerance. In order to prove mathematically that a limit does or does not exist, it is much more useful to cast this definition in the language of mathematics. Without further ado, the formal definition of a limit.

Definition: The Limit
Suppose $f(x)$ is defined on an open interval about x_0, not necessarily containing x_0. We say that L is the limit of $f(x)$ as x approaches x_0, written

$$\lim_{x \to x_0} f(x) = L$$

if for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x with $0 < |x - x_0| < \delta$ we have

$$|f(x) - L| < \epsilon.$$

This definition is the same as the definition we stated above, but we have rewritten some of the verbal terms such as 'error tolerance' in terms of variables. We can describe the mathematical definition in words as follows: for any error tolerance ($\epsilon > 0$), we can restrict the values of $f(x)$ to be within that error tolerance of L ($|f(x) - L| < \epsilon$) by restricting the values of x to be sufficiently close but not equal to x_0 (by choosing $\delta > 0$ and considering x with $0 < |x - x_0| < \delta$). We require $0 < |x - x_0|$ in order to exclude the point x_0 from the interval of interest, because the limit does not depend on the function value at that single point. It is also noteworthy that when we use the phrase 'for every $\epsilon > 0$,' it is sufficient to show that it is true for an arbitrary ϵ (because we could just as well replace 'this' ϵ with any and 'every' number ϵ and the analysis would still hold true). We have recasted the verbal definition so that we can manipulate the variables ϵ and δ in order to make precise calculations to prove whether or not a limit exists. Let us consider the following examples in order to further understand this definition.

Example 1 Given

$$f(x) = 2x + 1$$

is it possible to find a δ-interval around $x_0 = 2$ so that for x with $0 < |x - 2| < \delta$ it follows that

$$|f(x) - 5| < \epsilon$$

if $\epsilon = 0.5$? Using this information can you conclude whether or not

$$\lim_{x \to 2} f(x) = 5$$

is a true statement?

Solution We will begin with the inequality we want to satisfy, and try to deduce a restriction on x in order to satisfy it.

$$\begin{aligned}
|f(x) - 5| &< \epsilon \\
|2x + 1 - 5| &< 0.5 \\
|2x - 4| &< 0.5 \\
2|x - 2| &< 0.5 \\
|x - 2| &< 0.25
\end{aligned}$$

Since the inequality $|f(x) - 5| < \epsilon$ is true for $|x - 2| < 0.25$, it follows that $\delta = 0.25$ will define an interval sufficiently small. Note that any smaller value for δ will also work, so we could just as well let $\delta = 0.1$. Either way, we conclude that it is possible to find such a δ-interval. Although

$$\lim_{x \to 2} f(x) = 5$$

is in fact a true statement, being able to find δ for a single error tolerance (as we have just done) is insufficient information to make this conclusion (we would need to find the width of the δ-interval for *every* ϵ.)

Example 2 Given

$$f(x) = \sin(\frac{1}{x})$$

is it possible to find a δ-interval around $x_0 = 0$ so that for x with $0 < |x| < \delta$ it follows that

$$|f(x) - 0| < \epsilon$$

if $\epsilon = 0.1$? Using this information can you conclude whether or not

$$\lim_{x \to 0} f(x) = 0$$

is a true statement?

Solution We will need to approach this problem differently, because there is no way for us to algebraically manipulate the inequality

$$|\sin(\frac{1}{x}) - 0| < 0.1$$

in order to reach a restriction for x. Instead, let us consider the problem graphically. We see that the function values are oscillating between -1 and 1, and the oscillations become more rapid as $x \to 0$. Because these oscillations continue indefinitely close to $x = 0$, no matter how small a δ-interval we define around $x = 0$ we will be able to find a point x within it such that $f(x) = 1$, which is outside of the error tolerance around 0. Recalling that

$$\sin(x) = 1 \quad \text{when} \quad x = \frac{\pi}{2} + 2n\pi \quad n \in \mathbb{Z}$$

we find that

$$\sin(\frac{1}{x}) = 1 \quad \text{when} \quad x = \frac{1}{\frac{\pi}{2} + 2n\pi} \quad n \in \mathbb{Z}$$

No matter how small of a δ-interval we consider, we will always find a point x as defined above, by considering sufficiently large values of n. Thus, in any interval around $x_0 = 0$ there will be values of x such that

$$|\sin(\frac{1}{x}) - 0| = 1 > 0.1$$

We conclude that it is impossible to find a δ-interval so that the inequality $|\sin(\frac{1}{x}) - 0| < 0.1$ is satisfied for all x in the interval. Using this information we can conclude that

$$\lim_{x\to 0} \sin(\frac{1}{x}) \neq 0$$

because we have found a single error tolerance $\epsilon = 0.1$ for which we cannot find a corresponding δ-interval to satisfy the definition of the limit. In fact, it is not just that the limit does not equal 0, but that there is no value L that satisfies the limit (so the limit does not exist), which can be shown using a technique similar to the one we used above.

Now that we have more familiarity with the definition of the limit, let us apply the definition in order to abstractly prove the existence of a limit in question. We will reiterate that in order to prove a limit exists using the formal definition of a limit, we must consider an arbitrary $\epsilon > 0$. For this arbitrary ϵ, we need to find a δ so that if we have any x with $0 < |x - x_0| < \delta$ it follows that $|f(x) - L| < \epsilon$, that is, the values of the function within this δ-interval are within the error tolerance ϵ of the limit L. In general, δ will depend on ϵ; a case where it does not is if $f(x) = c$. Written algorithmically, the process to proving a limit exists is as follows:

1. Consider an arbitrary $\epsilon > 0$.
2. Analyze the inequality $|f(x) - L| < \epsilon$ to try and find a restriction on x, so that if x meets the given restriction, this inequality will be satisfied.
3. Choose a value for $\delta > 0$ which encapsulates the restriction found in the previous step. In other words, for any x with $0 < |x - x_0| < \delta$ it should follow that x satisfies the above restriction.
4. Consider an arbitrary x satisfying $0 < |x - x_0| < \delta$. Using the fact that x satisfies $0 < |x - x_0| < \delta$, show that for this x, $|f(x) - L| < \epsilon$ is satisfied. From here, the conclusion follows.

Example 3 Prove that $\lim_{x \to 1} f(x) = 3$ if $f(x) = 2x + 1$.

Solution Consider $\epsilon > 0$, arbitrary. We must find a corresponding $\delta > 0$ such that for $0 < |x-1| < \delta$ we have $|f(x) - 3| < \epsilon$. Our strategy will be to analyze the second inequality to try to determine δ.

$$|f(x) - 3| = |(2x+1) - 3| = |2x - 2| = 2|x - 1|$$

and

$$2|x-1| < \epsilon \quad \text{if} \quad |x-1| < \frac{\epsilon}{2}$$

Thus, if we let $\delta = \frac{\epsilon}{2}$ we find that for arbitrary x with $0 < |x-1| < \delta$ it follows

$$|f(x) - 3| = 2|x-1| < 2 \cdot \frac{\epsilon}{2} = \epsilon$$

From the definition of the limit, it follows that

$$\lim_{x \to 1} f(x) = 3$$

Example 4 Prove that $\lim_{x \to 3} f(x) = 9$ for

$$f(x) = \begin{cases} x^2 & x \neq 3 \\ 3 & x = 3 \end{cases}$$

Solution Consider $\epsilon > 0$, arbitrary. We must find a corresponding $\delta > 0$ such that for $0 < |x-3| < \delta$ we have $|f(x) - 9| < \epsilon$. Since $0 < |x-3|$, it follows $x \neq 3$, so $f(x) = x^2$. Thus,

$$|f(x) - 9| = |x^2 - 9| = |(x+3) \cdot (x-3)| = |x+3| \cdot |x-3|$$

We can control the value of $|x-3|$ directly with δ, but we will have to control $|x+3|$ indirectly. If we knew that δ would be less than 1, then we'd know from $0 < |x-3| < \delta$ that $3 < x < 5$. Now the expression

$$|f(x) - 9| = |x+3| \cdot |x-3| < \epsilon$$

consists of two terms we can control. If $\delta < 1$, then the maximum value of x is 5, so the maximum value of $|x+3|$ is 8. It follows, granted $\delta < 1$, that if $|x-3| < \epsilon/8$, then $|f(x) - 9| < \epsilon$. Thus, choose $\delta = min(1, \epsilon/8)$.

Consider arbitrary x with $0 < |x-3| < \delta$. For such an x, it follows

$$|f(x) - 9| = |x+3| \cdot |x-3| < 8 \cdot \frac{\epsilon}{8} = \epsilon$$

Example 5 Prove that $\lim_{x \to 0} f(x) \neq 3$ for

$$f(x) = \frac{1}{x}$$

Solution Unlike the previous examples, where we needed to find a δ-interval that forces the values of $f(x)$ to be sufficiently close to L for an arbitrary ϵ, our task here is to find a single value for ϵ for which it is impossible to construct any δ-interval which restricts the function to be within ϵ of $L = 3$ (in other words, for any δ-interval we need to demonstrate at least one value x for which $|f(x) - L| > \epsilon$) . Because this particular function $f(x)$ grows without bound as $x \to 0$ from the right and decreases without bound as $x \to 0$ from the left, we can definitely find such a value for ϵ; it is just a matter of coming up with one. One method would be to simply guess a very small value for ϵ, such as $\epsilon = 10^{-10}$ and check if it will work. In fact, we could in theory choose any finite value for ϵ in this case, but let us just use a simple number such as $\epsilon = 1$ (the mechanics of this problem only become a bit more difficult if we choose $\epsilon > 3$, because of the way the function is separated around the y-axis).

Now we consider $\delta > 0$, arbitrary. If we let

$$x = -\frac{\delta}{2}$$

we will have

$$0 < |x - 0| = \frac{\delta}{2} < \delta$$

yet

$$|f(x) - 3| = |\frac{1}{-\delta/2} - 3| = 3 + \frac{2}{\delta} > 1$$

Thus, we have just proven that

$$\lim_{x \to 0} \frac{1}{x} \neq 3$$

It is noteworthy that we could just as well replace 3 with any positive, finite value L and the same type of analysis would work. If we wanted to consider a negative value for L, we would just use

$$x = -\frac{\delta}{2}$$

instead. Still, if we wanted to consider L values small in magnitude, we would correspondingly need to choose smaller values for ϵ in order to show the limit does not exist (at least in this way). We are able to perform such analysis for any L value with ease because $f(x)$ is positive for positive x and negative for negative x, which conveniently separates this function around the y-axis.

Example 6 Prove that $\lim\limits_{x \to 0} f(x) \neq 10$ for

$$f(x) = \frac{1}{|x|}$$

Solution Once again we need to find a value for ϵ so that for any $\delta > 0$ we can find an x with $0 < |x - 0| < \delta$ and $|f(x) - 10| > \epsilon$. Looking at this function graphically we can once again see this must be possible, but because we do not have the same separation of positive and negative values for $f(x)$ as we cross the y-axis, the mechanics of this problem are a little bit more difficult. For the sake of illustration, let us solve this problem using $\epsilon = 0.5$

Consider $\delta > 0$, arbitrary. Suppose we try to use the same trick and define

$$x = \frac{\delta}{2}$$

Since δ is arbitrary, we don't know what value it might have. If it happens that $\delta = 0.2$ making $x = 0.1$, we would find

$$|f(x) - 10| = |\frac{1}{0.1} - 10| = 0 < \epsilon = 0.5$$

so we would have failed in our task to show the limit does not exist. There is nothing special about choosing

$$x = \frac{\delta}{2}$$

and in fact we will run into the same problem no matter how we define x directly in terms of δ (if $x = k \cdot \delta$, then we will run into problems if it happens that $\delta = 0.1/k$; we must be careful here that we are indeed defining an x such that $0 < |x| < \delta$!). Looking at the function graphically we can definitely see that the limit does not exist, but we will need to be more clever in order to prove it. No matter how we define x, there is a corresponding range of δ values that will cause us trouble. However, if we can guarantee that x will not be in this problematic range of values, *and* $0 < |x| < \delta$ will be satisfied, then we can overcome this hurdle. We will need to restrict x close to 0 in order to accomplish this, but how small must we make x be? This depends on our choice of ϵ. Since we let $\epsilon = 0.5$, in order to have

$$|f(x) - 10| = |\frac{1}{|x|} - 10| > 0.5$$

we will need to have

$$|x| < \frac{1}{10.5}$$

After considering arbitrary $\delta > 0$, let us set $x = min(\frac{1}{10.5}, \frac{\delta}{2})$ For such an x we find

$$|f(x) - 10| = |\frac{1}{|x|} - 10| > 0.5 = \epsilon$$

Now we have proven that

$$\lim_{x \to 0} \frac{1}{|x|} \neq 10$$