Indefinite Integrals

To find a solution to an equation of the form

$$\frac{dF}{dx} = f(x)$$

we are looking for a function $F(x)$ such that $F'(x) = f(x)$. We formalize this notion with the following definition

Definition: Antiderivative
A function $F(x)$ is an antiderivative of $f(x)$ if

$$F'(x) = f(x)$$

for all x in the domain of f.

If we have a single antiderivative $F(x)$ for a function $f(x)$, it turns out that every other antiderivative of $f(x)$ can be written as $F(x) + c$ for some constant c. Because we can choose any constant value for c it follows that this family of antiderivatives contains infinite members. This realization motivates the following definition.

Definition: Indefinite Integral
We call the set of all antiderivatives of f the indefinite integral of f, denoted by

$$\int f(x)dx$$

The symbol $\int$ is an integral sign. The function f is the called the integrand and x is the variable of integration. We say that dx is a differential of x.

Based on our previous discussion we can say that

$$\int f(x)dx = F(x) + c$$

because the expression on the right-hand side represents all possible antiderivatives of $f(x)$.

When we find the indefinite integral of a function $f(x)$ we say that we integrate the integrand $f(x)$. Thus, the process of evaluating an integral is referred to as integration. The resemblence of $\int$ and an enlongated S is not accidental. This notation reflects the relationship between integration and summation. Integration is essentially the summation of the area underneath the integrand $f(x)$ over intervals of infintessimal length. The differential dx represents an infintessimal change in x, which represents the intervals of infintessimal length over which the summation involved in integration occurs.

When it will not lead to confusion we will refer to the indefinite integral as simply the integral. The term indefinite integral is used to distinguish the process of indefinite and definite integration. Although both of these concepts are related to the area underneath a function, the indefinite integral is a function, whereas the definite integral is a constant, which is given by the area underneath a function over a set interval (defined by the limits of integration, which are not present in an indefinite integral).

Since the process of (indefinite) integration is an inverse to differentiation, we can derive many rules for integration using rules we already know for differentiation. For instance

$$\frac{d}{dx}x^{n+1} = (n+1)x^n$$

so we see that

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n \quad \text{for} x \neq -1$$

This leads us to the product rule for integrals.

Power Rule for Integrals

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

for every $n \neq -1$ ($n \in \mathbf{R} \setminus \{-1\}$).

Example 1 Evaluate the indefinite integral of x^2.
Solution In this case we use the product rule, to see that

$$\int x^2 = \frac{x^{2+1}}{2+1} + c = \frac{x^3}{3} + c$$

Example 2 Evaluate the indefinite integral of $\sqrt{x}$.
Solution Once we rewrite $\sqrt{x} = x^{1/2}$ we see that

$$\int x^{1/2} = \frac{x^{1/2+1}}{1/2+1} + c = \frac{2}{3}x^{3/2} + c$$

Example 3 Evaluate the indefinite integral of x^{-3}.
Solution We find that

$$\int x^{-3} = \frac{x^{-3+1}}{-3+1} + c = -\frac{1}{2}x^{-2} + c$$

Example 4 Evaluate $\int dt$.
Solution Rewriting the integrand we find

$$\int dt = \int t^0 dt = \frac{t^{0+1}}{0+1} + c = t + c$$

Just like differentiation, integration is a linear operation. What this means is that integration satisfies the following two properties.

Sum Rule for Integrals

$$\int \big(f(x) + g(x)\big)dx = \int f(x)dx + \int g(x)dx$$

Constant Product Rule for Integrals

$$\int af(x)dx = a\int f(x)dx$$

for every $a \in \mathbf{R}$

Example 5 Evaluate the indefinite integral of $-2x^{-3} + 4x^{1/2}$.
Solution We can use the constant product rule and sum rules in conjunction to find

$$\int -2x^{-3} + 4x^{1/2} = -2\int x^{-3} + 4\int x^{1/2} = -2(-\frac{1}{2}x^{-2}) + 4(\frac{2}{3}x^{3/2}) + c = x^{-2} + \frac{8}{3}x^{3/2} + c$$

In the above analysis we do not write the result as $-2c + 4d$, because we can easily enough choose c and d so that we have any value for the arbitrary constant. Thus, it is cleaner to just replace $-2c + 4d$ with a single c, as both are capable of representing any arbitrary constant.

When we are faced with a differential equation of the form

$$\frac{dm}{dt} = f(t)$$

we can use integration to find a solution $m(t)$, as an antiderivative of $f(t)$ will be a function with derivative of $f(t)$, so it will satisfy the differential equation. The fact that the indefinite integral is a family of antiderivatives corresponds directly with the fact that there are a family of solutions to the above differential equation. It follows that we will need to choose c appropriately to satisfy the initial conditions of a given initial value problem.

One major application in which polynomial differential equations arise is to find a velocity given an acceleration, or position given velocity and acceleration. We have the following relationships

$$\frac{dv}{dt} = a$$

and

$$\frac{dy}{dt} = v$$

where a denotes acceleration, v denotes velocity, and y denotes position.

Example 6 Using the above relationships we can solve the initial value problem related to a free-falling object (ignoring wind resistance). Suppose a shoe is thrown from a height of 100m at an initial velocity of -5 m/s. Find the velocity and position of the object as functions of time.

Solution To solve this problem we will assume a constant gravitational force, which causes the shoe to accelerate at a constant rate of 9.8 m/s after it is released at time $t = 0$. To solve the equation

$$\frac{dv}{dt} = a = -9.8$$

we integrate finding

$$v(t) = \int -9.8dt = -9.8t + c$$

Using the initial condition we find

$$v(0) = -9.8t + c = c = -5$$

so $v(t) = -9.8t - 5$ is our exact solution for velocity as a function of time. Note that the velocity is negative because the object is traveling downward (in free fall). Now we must solve

$$\frac{dx}{dt} = v = -9.8t - 5$$

to find x. Integrating we find

$$y(t) = \int (-9.8t - 5)dt = -9.8\int tdt - 5\int dt = -4.9t^2 - 5t + c$$

Using the initial condition

$$y(0) = -4.9 \cdot 0^2 - 5 \cdot 0 + c = 100$$

implies that $y(t) = -4.9t^2 - 5t + 100$ is our solution for position as a function of time. If we solve for $y(t) = 0$ we find that the shoe hits the ground after about 4 seconds. When it hits the ground its velocity is about -45 m/s.

Corresponding to other differentiation rules we have learned, we can evaluate the following integrals.

$$\int e^x dx = e^x + c$$

$$\int \cos(x)dx = \sin(x) + c$$

$$\int \sin(x)dx = -\cos(x) + c$$

Note that the negative sign corresponds to the integral of the sine function, just as the negative sign corresponds to the derivative of cosine. We'd also like to note

$$\int \frac{1}{x}dx = \ln(|x|) + c$$

In the above formula we use the magnitude of x, $|x|$, because although $\frac{1}{x}$ is defined for negative x, $\ln(x)$ is not. We can verify this integral through differentiation. For $x > 0$

$$\frac{d}{dx}\ln(|x|) = \frac{d}{dx}\ln(x) = \frac{1}{x}$$

If we consider $x < 0$ then

$$\frac{d}{dx}\ln(|x|) = \frac{d}{dx}\ln(-x) = \frac{1}{-x} \cdot (-1) = \frac{1}{x}$$

We cannot evaluate the above derivative at $x = 0$, because $\ln(x)$ is undefined for $x = 0$.

Example 7 Evaluate $\int \Big(\frac{1}{x} + 2e^x + 3\cos(x)\Big)dx$

Solution We can evaluate the above integral using the sum and constant product rules in conjunction with the integrals we have just discussed.

$$\int \Big(\frac{1}{x} + 2e^x + 3\cos(x)\Big)dx = \int \frac{1}{x}dx + 2\int e^x dx + 3\int \cos(x)dx = \ln(|x|) + 2e^x + 3\sin(x) + c$$