Improper Integrals

There are two types of improper integrals - those with infinite limits of integration, and those with integrands that approach ∞ at some point within the limits of integration. First we will consider integrals with infinite limits of integration.

Infinite Limits of Integration

Suppose chemical production is governed by the differential equation

$$\frac{dP}{dt} = e^{-t}$$

moles per second. If we want to find out how much chemical would be produced were the experiment allowed to run forever, we would like to calculate

$$\int_0^\infty e^{-t} dt$$

However, since ∞ is not a number, we cannot just plug it in as one of the bounds after evaluating the indefinite integral. What we can do, is look at an indefinite integral with an upper limit T rather than ∞. This is something we can evaluate. Afterwards, we can evaluate the result in the limit $\lim_{T\to\infty}$. Thus, the first step in a problem of infinite limits of integration, is to rewrite the problem in the form of a limit. Formally, we write

$$\int_a^\infty f(t)dt = \lim_{T\to\infty} \int_a^T f(t)dt$$

Example 1 Evaluate $\displaystyle\int_0^\infty e^{-t} dt$
Solution First we rewrite the problem

$$\int_0^\infty e^{-t} dt = \lim_{T\to\infty} \int_0^T e^{-t} dt$$

We evaluate the integral

$$\int_0^T e^{-t} dt = -e^{-t}\big|_0^T = -e^{-T} - (-e^{-0}) = 1 - e^{-T}$$

Evaluating the limit we find

$$\int_0^\infty e^{-t} dt = \lim_{T\to\infty} (1 - e^{-T}) = 1$$

Thus, at this rate of production, if production continued indefinitely, only 1 mol of chemical would be produced!

Example 2 Suppose chemical production is governed by

$$\frac{dQ}{dt} = \frac{1}{1+t}$$

moles per second. How much chemical is generated if production continues indefinitely, beginning from $t = 0$?

Solution Once again, we write

$$\int_0^\infty \frac{1}{1+t} dt = \lim_{T \to \infty} \int_0^T \frac{1}{1+t} dt$$

Using u substitution, with $u = 1 + t$, so $du = dt$, we find

$$\int_0^T \frac{1}{1+t} dt = \int_1^{T+1} \frac{1}{u} du = \ln(|u|)|_1^{T+1} = \ln(T+1) - \ln(1) = \ln(T+1)$$

Thus,

$$\int_0^\infty \frac{1}{1+t} dt = \lim_{T \to \infty} \ln(T+1) = \infty$$

In this situation, if production is allowed to continue indefinitely, the amount produced grows without bound. When the result of an integral is $\pm\infty$, we say that the integral diverges, because it does not reach any real value. When the limit as $T \to \infty$ is a real value, we say that the integral converges. When evaluating improper integrals, it is important to state whether or not there is convergence or divergence, and if there is convergence, to what value.

What is it that differs between the integrands of these two integrals that causes one to converge and the other to diverge? In both of these cases the integrands are always positive over the limits of integration. Furthermore, they both approach 0 as T approaches ∞. The difference is that e^{-t} decays much more quickly than $\frac{1}{1+t}$. Based on this observation, we should be able to generalize whether some simple functions will converge or diverge, based on the rate at which their integrands approach 0 (for positive functions). We should emphasize that if the intregrand does not decay to zero, then it is guaranteed the integral will diverge. For instance

$$\int_0^\infty dt = \lim_{T \to \infty} t|_0^T = \lim_{T \to \infty} T = \infty$$

For an arbitrary decaying exponential function, $e^{-\alpha t}$, we have

$$\int_0^T e^{-\alpha t} dt = \frac{e^{-\alpha t}}{-\alpha}|_0^T = \frac{e^{-\alpha T}}{-\alpha} + \frac{1}{\alpha}$$

We find that

$$\int_0^\infty e^{-\alpha t} dt = \lim_{T \to \infty} \frac{1}{\alpha} - \frac{e^{-\alpha T}}{\alpha} = \frac{1}{\alpha}$$

The conclusion is that any decaying exponential decays fast enough so that its integral to ∞ converges.

We can also consider a decaying function of the type $\frac{1}{t^p}$, where $p > 0$. We will begin integrating at $t = 1$, to avoid division by 0. If $p = 1$, then we are considering the integral

$$\int_1^\infty \frac{1}{t} dt$$

which we already observed diverges. For $p \neq 1$, we use the power rule so

$$\int_1^T \frac{1}{t^p} dt = \frac{t^{1-p}}{1-p}\Big|_1^T = \frac{T^{1-p}}{1-p} - \frac{1}{1-p}$$

When we evaluate in the limit that $T \to \infty$, we must consider whether $p > 1$ or $p < 1$. First recall that

$$\lim_{T\to\infty} T^p = \infty \quad \text{for } p \geq 0$$

and

$$\lim_{T\to\infty} T^p = 0 \quad \text{for } p < 0$$

For $p < 1$

$$\lim_{T\to\infty} \left(\frac{T^{1-p}}{1-p} - \frac{1}{1-p}\right) = \infty$$

because $1 - p > 0$ so that the first term grows without bound. In the case that $p > 1$

$$\lim_{T\to\infty} \left(\frac{T^{1-p}}{1-p} - \frac{1}{1-p}\right) = \frac{1}{p-1}$$

because $1 - p < 0$ so that the first term goes to zero. In conclusion

$$\int_1^\infty \frac{1}{t^p} = \begin{cases} \infty & 0 \leq p \leq 1 \\ \frac{1}{p-1} & p > 1 \end{cases}$$

It is noteworthy that for an improper integral, moving the lower limit of integration a finite amount will not alter the integral's convergence or divergence, as long as it does not introduce divison by zero into the limits of integration. This means that we can already gather a lot of information about the convergence and divergence of other improper integrals. For example,

$$\int_5^\infty \frac{1}{\sqrt{t}} dt = \int_1^\infty \frac{1}{\sqrt{t}} dt - \int_1^5 \frac{1}{\sqrt{t}} dt$$

using the summation property for integrals. We know that

$$\int_1^\infty \frac{1}{\sqrt{t}} dt$$

diverges, and that

$$\int_1^5 \frac{1}{\sqrt{t}} dt$$

is just some finite amount. If we subtract some finite amount from a diverging integral, the result will still be something that diverges. Thus, without any computation we can deduce that

$$\int_5^\infty \frac{1}{\sqrt{t}} dt$$

diverges. In a sense, this means that the function's behavior for small input values has no influence over the convergence of such an integral - convergence is related solely to the rate at which the function decays to zero as inputs grow larger.

Now suppose that we are faced with a more complicated integral, something like

$$\int_1^\infty \left(\frac{1}{t} + e^{-t}\right) dt$$

We can write

$$\int_1^\infty \left(\frac{1}{t} + e^{-t}\right) dt = \int_1^\infty \frac{1}{t} dt + \int_1^\infty e^{-t} dt$$

Since $\int_1^\infty e^{-t} dt$ is just a positive number, we can deduce that

$$\int_1^\infty \left(\frac{1}{t} + e^{-t}\right) dt > \int_1^\infty \frac{1}{t} dt = \infty$$

Thus, we can conclude that our integral diverges, since it is larger than an integral diverging to ∞. A very similar idea to this one leads us to the comparison test.

The Comparison Test
Suppose $0 \leq f(x) \leq g(x)$ for all $x \geq a$.

1. $\displaystyle\int_a^\infty f(x)dx$ converges if $\displaystyle\int_a^\infty g(x)dx$ converges.
2. $\displaystyle\int_a^\infty g(x)dx$ diverges if $\displaystyle\int_a^\infty f(x)dx$ diverges.

In summary, if some positive function $f(x)$ is always less than or equal to another positive function $g(x)$, then its integral will be less, so if $g(x)$ converges, then $f(x)$ must converge as well. Similarly, if $f(x)$ diverges, and $g(x)$ is greater than or equal to it, then it must also diverge, as the integral will be greater.

Example 3 Determine whether or not $\displaystyle\int_1^\infty \frac{1}{t + e^t} dt$ converges or diverges.
Solution First we can note that because the integrand is always positive, the integral must be greater than 0. Also, for all $t > 0$ we have

$$\frac{1}{t + e^t} < \frac{1}{e^t}$$

so by the comparison test, we have that

$$0 < \int_1^\infty \frac{1}{t+e^t} < \int_1^\infty \frac{1}{e^t} = 1$$

Although we don't know to what exact value, we can conclude that this integral converges.

Example 4 Determine whether or not

$$\int_0^\infty \frac{1}{2x+2} dx$$

converges or diverges. If it converges, find to what value.
Solution This function looks like $\frac{1}{x}$, which is divergent, so we suspect that this integral should also diverge. However, it is not clear how to use the comparison test in this case, so let us rewrite

$$\int_0^\infty \frac{1}{2x+2} dx = \lim_{T\to\infty} \int_0^T \frac{1}{2x+2} dx$$

Now, using substitution $u = 2x+2$, so $\frac{du}{2} = dx$, and

$$\int_0^T \frac{1}{2x+2} dx = \frac{1}{2} \int_2^{2T+2} \frac{1}{u} du$$

Now that this integral is written in a more familiar form, we can see that it diverges, using the fact that

$$\int_1^\infty \frac{1}{u} du = \infty$$

Example 5 Determine whether or not

$$\int_2^\infty \frac{1}{(x-1)^{1/2}} dx$$

converges or diverges. If it converges, find to what value.
Solution Using the comparison test,

$$\frac{1}{(x-1)^{1/2}} > \frac{1}{x^{1/2}}$$

where the latter of these terms is the integrand for a divergent integral under these limits. Thus, the integral in question diverges.

Example 6 Determine whether or not

$$\int_2^\infty \frac{2}{(3x-5)^2} dx$$

converges or diverges. If it converges, find to what value.

Solution This integrand has the form of $\frac{1}{x^2}$, so we suspect that it should converge. Using substituion $u = 3x - 5$ so $\frac{du}{3} = dx$. Evaluating the improper integral

$$\int \frac{2}{(3x-5)^2} dx = \frac{2}{3} \int u^{-2} du = -\frac{2}{3} \cdot \frac{1}{u} + c = -\frac{2}{3} \cdot \frac{1}{3x-5} + c$$

Now we see that

$$\int_2^T \frac{2}{(3x-5)^2} dx = -\frac{2}{3} \cdot \frac{1}{3x-5} |_2^T = -\frac{2}{3} \cdot \frac{1}{3T-5} + \frac{2}{3}$$

Thus, we finally conclude

$$\int_2^\infty \frac{2}{(3x-5)^2} dx = \lim_{T\to\infty} \Big(-\frac{2}{3} \cdot \frac{1}{3T-5} + \frac{2}{3} \Big) = \frac{2}{3}$$

Infinite Integrands

We can also consider functions the approach infinity at some point, and look at their definite integral on an interval containing that point. For now we will restrict ourselves to functions that approach ∞ as $x \to 0$, but there is no reason we cannot generalize and consider any point where the functions approach ∞. Once again we'll rewrite the integral using a limit, casting it in a form we can evaluate. We will look at the limit as the lower limit of integration approaches zero from the right, so that we can evaluate an integral where the integrand is always finite, but approaches the original integral. Write

$$\int_0^a f(x) dx = \lim_{\epsilon\to 0^+} \int_\epsilon^a f(x) dx$$

Example 1 Evaluate $\displaystyle\int_0^1 x^{-1/2} dx$

Solution We find that

$$\int_0^1 x^{-1/2} dx = \lim_{\epsilon\to 0^+} \int_\epsilon^1 x^{-1/2} dx = \lim_{\epsilon\to 0^+} 2\sqrt{x}|_\epsilon^1 = \lim_{\epsilon\to 0^+} (2 - 2\sqrt{\epsilon}) = 2$$

Example 2 Evaluate $\displaystyle\int_0^1 x^{-1} dx$

Solution We find that

$$\int_0^1 t^{-1} dt = \lim_{\epsilon\to 0^+} \int_\epsilon^1 t^{-1} dt = \lim_{\epsilon\to 0^+} \ln(|t|)|_\epsilon^1 = \lim_{\epsilon\to 0^+} (\ln(1) - \ln(\epsilon) = \infty$$

Just as with infinite limits of integration, we can generalize the result for

$$\int_0^a \frac{1}{t^p} dt$$

We've seen that if $p = 1$ then this integral diverges. If $p \neq 1$ then

$$\int_0^a \frac{1}{t^p} dt = \lim_{\epsilon\to 0^+} \int_\epsilon^a \frac{1}{t^p} dt = \lim_{\epsilon\to 0^+} \frac{t^{1-p}}{1-p} |_\epsilon^a = \lim_{\epsilon\to 0^+} \Big(\frac{a^{1-p}}{1-p} - \frac{\epsilon^{1-p}}{1-p}$$

If $p < 1$, the integral converges, and if $p > 1$, the integral diverges. We can summarize these results as

$$\int_0^a \frac{1}{t^p} = \begin{cases} \infty & p \geq 1 \\ \frac{a^{1-p}}{1-p} & 0 \leq p < 1 \end{cases}$$

Just as before, we can use the comparison test to evaluate the convergence or divergence of integrals with integrands that approach ∞. Consider the following example

Example 3 Determine whether or not $\displaystyle\int_0^1 \frac{1}{\sqrt{t} + \sqrt[3]{t}}dt$ converges or diverges

Solution We will use the comparison test. For $0 \leq t \leq 1$ we have

$$\sqrt[3]{t} > \sqrt{t}$$

so that

$$\frac{1}{\sqrt{t} + \sqrt[3]{t}} < \frac{1}{\sqrt{t} + \sqrt{t}} = \frac{1}{2\sqrt{t}}$$

Thus

$$\int_0^1 \frac{1}{\sqrt{t} + \sqrt[3]{t}}dt < \frac{1}{2}\int_0^1 \frac{1}{\sqrt{t}}dt = \frac{1}{2} \cdot 2 = 1$$