April 6th DIFFERENTIATION, MVT

Definition

$$f'(x) = \lim_{x \to p} \frac{f(x)-f(p)}{x-p}$$ for some $f:[a,b] \to \mathbb{R}$ and $p \in [a,b]$

A function is differentiable at p if $f'(p)$ exists

Theorem

If f is differentiable $\rightarrow$ f is continuous

↳ Show that $\lim_{x \to p} f(x) = f(p)$

Theorem

If $f(x)$ is differentiable at $p \rightarrow \exists u(x)$ where

$$u(x) = \begin{cases} \frac{f(x)-f(p)}{x-p} - f'(p) & x \neq p \\ 0 & x = p \end{cases}$$

$$f(x) = f(p) + (x-p) \cdot f'(p) + \overbrace{(x-p)\, u(x)}^{\text{remainder}}$$

theorem

$f, g : [a,b] \to \mathbb{R}$ and f and g are differentiable at p

$$\Rightarrow (f+g)'(p) = f'(p) + g'(p)$$

$$\Rightarrow (f \cdot g)'(p) = f'(p)\, g(p) + f(p)\, g'(p)$$

$$\Rightarrow \text{if } g(p) \neq 0 \quad (f/g)'(p) = \frac{f'g - fg'}{g^2}$$

$$\Rightarrow g(f(p))' = g'(f(p))\, f'(p)$$ Chain Rule

to prove, use prev theorem

Definitions

Local Maximum

$\exists \delta > 0$ s.t. $\forall x \in [a,b] \cap B_\delta(p)$ $\quad f(x) \leq f(p)$

Local Min

$\exists \delta > 0$ s.t. $\forall x \in [a,b] \cap B_\delta(p)$ $\quad f(x) \geq f(p)$

Lemma

$f: [a,b] \to \mathbb{R}$ If p is a local max, $p \in (a,b)$, and $f'(p)$ exists

$\Rightarrow f'(p) = 0$

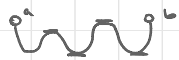
to ignore local maxima at boundary

to ignore 

Rolle's Theorem

$f: [a,b] \to \mathbb{R}$ is continuous, $f'(x)$ exists for all $x \in (a,b)$, $f(a) = f(b)$

$\Rightarrow \exists c \in (a,b)$ s.t. $f'(c) = 0$



Use prev lemma for min, max $[a,b]$

follows from

Generalized MVT

$f, g: [a,b] \to \mathbb{R}$ are continuous, differentiable on (a,b)

$\Rightarrow \exists c \in (a,b)$ s.t. $[f(b) - f(a)]\, g'(c) = [g(b) - g(a)]\, f'(c)$

$$\approx \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

follows from

MVT

$f(b) - f(a) = (b-a)\, f'(c)$