Section 5.2

We now lay the foundation for the definition of the *definite integral.* In our development of the definite integral, we shall employ sums of many numbers. To express such sums compactly, it is convenient to use **summation notation**. Given a collection of numbers $\{a_1,\ a_2,.......,a_n\ \}$, the symbol $\sum_{k=1}^{n} a_k$ represents their sums as follows.

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + ... + a_n.$$

The Greek capital letter $\sum$ (sigma) indicates a sum, and a_k represents the kth term of the sum. The letter k is the **index of summation**, or the **summation variable**, and assumes successive integer values. The integers 1 and n indicate the extreme values of the summation variable.

Exercise 1. Evaluate: $\sum_{k=1}^{4} k^2(k-3)$.

Exercise 2. Evaluate: $\sum_{k=0}^{3} \frac{2^k}{k+1}$.

Class Exercise 1. Evaluate the sum.
(a) $\sum_{j=1}^{4} (j^2+1)$ (b) $\sum_{j=1}^{4} (2^j+1)$ (c) $\sum_{k=0}^{5} k(k-1)$
(d) $\sum_{k=0}^{4} (k-2)(k-3)$ (e) $\sum_{n=1}^{10} [1+(-1)^n]$ (f) $\sum_{n=1}^{4} (-1)^n(\frac{1}{n})$

Theorem: $\sum_{k=1}^{n} c = nc.$

Exercise 3. Evaluate $\sum_{k=1}^{4} 5$.

Theorem: If n is any positive integer and $\{a_1, a_2, ..., a_n\}$ and $\{b_1, b_2, ..., b_n\}$ are sets of real numbers, then

(i) $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

Exercise 4. Evaluate $\sum_{k=1}^{5} (\frac{1}{2})^k + 1$.

(ii) $\sum_{k=1}^{n} ca_k = c(\sum_{k=1}^{n} a_k)$ for every real number c

(iii) $\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

Theorem:

(i) $\sum_{k=1}^{n} k = 1 + 2 + ... + n = \frac{n(n+1)}{2}$

(ii) $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 +.......... + n^2 = \frac{n(n+1)(2n+1)}{6}$

(iii) $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 +......... + n^3 = [\frac{n(n+1)}{2}]^2$

Exercise 5. Evaluate $\sum_{k=1}^{90} k$ and $\sum_{k=1}^{15} k^2$.

Class Exercise 2. Evaluate $\sum_{k=1}^{100} k$ and $\sum_{k=1}^{20} k^2$.

Class Exercise 3. Evaluate $\sum_{k=1}^{n} (k^2-4k+3)$ in terms of n.

Exercise 6. Express 1 + 5 + 9 + 13 + 17 in summation notation.

Exercise 7. Express 2 + 5 + 8 + 11 + 14 in summation notation.

Class Exercise 4. Express the following sums in summation notation.
(a) $\frac{1}{2} + \frac{2}{5} + \frac{3}{8} + \frac{4}{11}$
(b) $\frac{1}{4} + \frac{2}{9} + \frac{3}{14} + \frac{4}{19}$
(c) $1 - \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6} + ... + (-1)^n \frac{x^{2n}}{2n}$
(d) $1 + x + \frac{x^2}{2} + \frac{x^3}{3} +...... + \frac{x^n}{n}$

Definition of a Definite Integral: If f is a function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\triangle\, x = (b-a)/n$. We let x_0 $(= a)$, x_1, x_2, ... x_n $(= b)$ be endpoints of these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then **the definite integral of f from a to b** is

$$\int_a^b f(x)\; dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \triangle\, x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable on** $[a, b]$.

Notation: The symbol $\int$ was introduced by Leibniz and is called an **integral sign**. It is an elongated S and was chosen because an integral is a limit of sums. In the notation $\int_a^b f(x)\; dx$, $f(x)$ is called the **integrand** and a and b are called the **limits of integration**; a is the **lower limit** and b is the **upper limit**. For now, the symbol dx has no meaning by itself; $\int_a^b f(x)\; dx$ is all one symbol. The dx simply indicates that the independent variable is x. The procedure of calculating an integral is called **integration**.

Notation: The sum

$$\sum_{i=1}^{n} f(x_i^*) \triangle\, x$$

that occurs in the above definition is called a **Riemann sum**.

Theorem: If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)\; dx$ exists.

Theorem: If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\; dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) \triangle\, x$$

where $\triangle\, x = \frac{b-a}{n}$ and $x_i = a + i\triangle\, x$.

Exercise 8. Express

$$\lim_{n\to\infty} \sum_{i=1}^{n} (x_i^3 + x_i \sin x_i) \triangle\, x \text{ as an integral on the interval } [0, \pi].$$

Class Exercise 5. Express the limit as a definite integral on the given interval.

(a) $\lim_{n\to\infty} \sum_{i=1}^{n} x_i \ln(1 + x_i^2) \triangle\, x$, $[2, 6]$ (Section 5.1 # 17)

(b) $\lim_{n\to\infty} \sum_{i=1}^{n} (\cos x_i)/x_i \triangle\, x$, $[\pi, 2\pi]$ (Section 5.1 # 18)

(c) $\lim_{n\to\infty} \sum_{i=1}^{n} [5(x_i^*)^3 - 4x_i^*] \triangle\, x$, $[2, 7]$ (Section 5.1 #19)

(d) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{x_i^*}{(x_i^*)^2+2} \triangle\, x$, $[1, 3]$ (Section 5.1 #20)

Exercise 9. If $f(x) = 16 - x^2$, find the area of the region under the graph of f from 0 to 3.

Class Exercise 6. Use the form of the definite integral to evaluate the integral
(a) $\int_1^4 (x^2 - 4x + 2)\; dx$.
(b) $\int_0^2 (2x - x^3)\; dx$.

Exercise 10. Evaluate the integral by interpreting it in terms of areas.
(a) $\int_{-2}^4 \frac{1}{2}(x + 3)\; dx$.
(b) $\int_{-4}^4 \sqrt{16 - x^2}\; dx$.
(c) $\int_4^4 \sqrt{16 - x^2}\; dx$.

Formula: If $f(x) \leq 0$ on $[a, b]$, $\int_a^b f(x)\; dx$ = -(area of the region between the graph of $y = f(x)$ and the x-axis).

Class Exercise 7. Evaluate the integral by interpreting it in terms of areas.
(a) $\int_0^9 (\frac{1}{3}x - 2)\ dx$.
(b) $\int_0^{10} |x-5|\ dx$.

Here are some properties of the integral:

1. $\int_a^b c\ dx = c(b-a)$, where c is any constant

2. $\int_a^b [f(x)+g(x)]\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$

3. $\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$, where c is any constant

4. $\int_a^b [f(x)-g(x)]\ dx = \int_a^b f(x)\ dx - \int_a^b g(x)\ dx$

5. $\int_a^c f(x)\ dx + \int_c^b f(x)\ dx = \int_a^b f(x)\ dx$

Exercise 11. If it is known that $\int_0^7 f(x)\ dx = 5$ and $\int_0^3 f(x)\ dx = 4$, find $\int_3^7 f(x)\ dx$.

Class Exercise 8. If $\int_1^5 f(x)\ dx = 12$ and $\int_4^5 f(x)\ dx = 3.6$, find $\int_1^4 f(x)\ dx$.

Class Exercise 9. Evaluate: $\int_{-5}^5 x - \sqrt{25-x^2}\ dx$.

Class Exercise 10. Find $\int_0^5 f(x)\ dx$ if $f(x) = \begin{cases} 3 & x < 3 \\ x & x \geq 3 \end{cases}$

Here are some comparison properties of the integral:

6. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\ dx \geq 0$.

7. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\ dx \geq \int_a^b g(x)\ dx$.

8, If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\ dx \leq M(b-a).$$

Exercise 12. Estimate the value of the integral: $\int_0^2 \frac{1}{1+x^2}\ dx$.

Class Exercise 11. Estimate the value of the integral: $\int_0^2 (x^3 - 3x + 3)\ dx$.

Class Exercise 12. Estimate the value of the integral: $\int_\pi^{2\pi} (x - 2\sin x)\ dx$.

Exercise 13. Evaluate the integral: $\int_{-2}^2 -\sqrt{4-x^2}\ dx$.

Class Exercise 13. Evaluate the integral: $\int_0^{10} 2 - x\ dx$.

Class Exercise 14. Evaluate the integral: $\int_0^6 |x-3| - 2\ dx$.

Homework: 1-11 ODD, 19-35 (every 4th), 41, 45, 51-75 (every 4th)