Section 4.4

In this chapter, we have been discussing applications of derivatives. In this section, we will use derivatives to find special limits, including those of the form $\frac{0}{0}$.

Definition: If we have a limit of the form

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, the limit may or may not exist and is called an **indeterminate form of type $\frac{0}{0}$**.

Exercise 1. Find $\lim_{x\to 3} \frac{x^2-5x+6}{x-3}$.

Exercise 2. Find $\lim_{x\to 0} \frac{1-\cos x}{x+x^2}$.

To tackle Exercise 2, we will use l'Hospital's Rule.

History of L'Hospital's Rule: L'Hospital's Rule is named after the French mathematician Guillaume de l'Hospital, but was first discovered by the Swiss mathematician Johann Bernoulli.

L'Hospital's Rule: Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0.$$

Then,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Exercise 3. Find $\lim_{x\to 0} \frac{e^x+e^{-x}-2}{1-\cos 2x}$.

Remark: Sometimes, it is necessary to employ l'Hospital's Rule several times in the same problem.

Remark: L'Hospital's Rule is also valid for one-sided limits.

Exercise 4. Find $\lim_{x\to 0^+} \frac{\sin x}{x^2}$.

Remark: Note that we got different limits in each of the last three exercises.

Exercise 5. Find $\lim_{x\to 2} \frac{2x-4}{2x}$.

Remark: Don't blindly apply l'Hospital's Rule!

Remark: A common error made by my students in the past is that they try to apply the Quotient Rule to the limit. The Quotient Rule is only used for derivatives.

Class Exercise 1. Evaluate the limit. (Waits Sec 8.1 #15-18)
(a) $\lim_{\theta\to 0} \frac{\sin \theta^2}{\theta}$ (b) $\lim_{\theta\to \pi/2} \frac{1-\sin\theta}{1+\cos 2\theta}$ (c) $\lim_{t\to 0} \frac{\cos t-1}{e^t-t-1}$ (d) $\lim_{x\to 0} \frac{e^x+e^{-x}}{x^2}$

In the previous exercises, the limits were of the indeterminate form $\frac{0}{0}$. L'Hospital's Rule can also be applied to limits of the indeterminate form $\frac{\infty}{\infty}$.

Definition: If we have a limit of the form

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to \infty$ (or $-\infty$) and $g(x) \to \infty$ (or $-\infty$), then the limit may or may not exist and is called an **indeterminate form of type ∞/∞**.

L'Hospital's Rule (∞/∞): Suppose that f and g are differentiable on an open interval I containing a, with $g'(x) \neq 0$ on I when $x \neq a$. If $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right exists (or is $\pm\infty$).

Exercise 6. Find $\lim_{x\to(\pi/2)^-} \frac{4\tan x}{1+\sec x}$.

Remark: Sometimes, it is better to simplify an expression algebraically as much as possible before using l'Hospital's Rule!

L'Hospital's Rule is also valid for limits at infinity or negative infinity; that is, "$x \to a$" can be replaced by either of the symbols $x \to \infty$, or $x \to -\infty$.

Exercise 7. Find $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$. (Swok Sec 10.1 Ex 4)

Exercise 8. Find $\lim_{x\to\infty} \frac{e^{3x}}{x^2}$, if it exists. (Swok Sec 10.1 Ex 5)

Class Exercise 2. Find (a) $\lim_{x\to\infty} \frac{\ln(x+1)}{\log_2 x}$ (b) $\lim_{x\to\infty} \frac{\log_2 x}{\log_3(x+3)}$ (Waits Sec 8.1 #19, 20)

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$ (or $-\infty$), then it isn't clear what the value of $\lim_{x\to a} [f(x)g(x)]$, if any, will be. This kind of limit is called an **indeterminate form of type $0 \cdot \infty$**.

Guidelines for investigating $\lim_{x\to a} [f(x)g(x)]$ for the form $0 \cdot \infty$.

1. Write $f(x)g(x)$ as

$$\frac{f(x)}{1/g(x)} \text{ or } \frac{g(x)}{1/f(x)}.$$

2. Apply l'Hospital's Rule to the resulting indeterminate form $0/0$ or ∞/∞.

Exercise 9. Find $\lim_{x\to 0^+} x^2 \ln x$. (Swok Sec 4.4 Ex 1)

Exercise 10. Find $\lim_{x\to\infty} x \tan \frac{1}{x}$.

Class Exercise 3. Use l'Hospital's Rule to evaluate the limit.
(a) $\lim_{y\to\pi/2} (\frac{\pi}{2} - y) \tan y$ (Waits Sec 10.2 #22)
(b) $\lim_{x\to 0^+} x \ln x$ (Waits Sec 10.2 #23)
(c) $\lim_{x\to(\pi/2)^-} (2x - \pi) \sec x$. (Swok Sec 4.4 Ex 2)

Exercise 11. Find $\lim_{x\to 0^+} (1+3x)^{1/(2x)}$. (Swok Sec 10.2 Ex 3)

Class Exercise 4. Use l'Hospital's Rule to evaluate the limit. (Waits Sec 8.1 #29, 30, 37-40)
(a) $\lim_{x\to 0} (e^x + x)^{1/x}$ (b) $\lim_{x\to 0} (\frac{1}{x^2})^x$ (c) $\lim_{x\to 0^+} (1+x)^{1/x}$
(d) $\lim_{x\to 1} x^{\frac{1}{x-1}}$ (e) $\lim_{x\to 0^+} (\sin x)^x$ (f) $\lim_{x\to 0^+} (\sin x)^{\tan x}$

Definition: If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then the limit

$$\lim_{x\to a} [f(x) - g(x)]$$

is called an **indeterminate form of type $\infty - \infty$**.

Exercise 12. Find $\lim_{x\to 0^+} \frac{1}{e^x - 1} - \frac{1}{x}$. (Swok Sec 10.2 Ex 4)

Class Exercise 5. Use l'Hospital's Rule to evaluate the limit. (Waits Sec #25-28)
(a) $\lim_{x\to 0^+}$ (csc x - cot x + cos x) (b) $\lim_{x\to\infty}$ (ln $2x$ - ln $(x+1)$)
(c) $\lim_{x\to 0^+}$ (ln x - ln sin x) (d) $\text{lm}_{x\to 0^+} (\frac{1}{x} - \frac{1}{\sqrt{x}})$

Homework: 1-65 (every 4th)