# Section 3.11

The exponential expressions

$$\frac{e^x - e^{-x}}{2} \text{ and } \frac{e^x + e^{-x}}{2}$$

occur in advanced applications of calculus. The properties are similar in many ways to those of sin $x$ and cos $x$.

**Definition**: The **hyperbolic sine function**, denoted by **sinh**, and the **hyperbolic cosine function**, denoted by **cosh**, are defined by

$$\sinh x = \frac{e^x - e^{-x}}{2} \text{ and } \cosh x = \frac{e^x + e^{-x}}{2}$$

for every real number $x$.

**Exercise 1.** Prove that $\cosh^2 x - \sinh^2 x = 1$. (Swok Page 441)

**Definition**: The **hyperbolic tangent**, **hyperbolic cotangent**, **hyperbolic secant**, and **hyperbolic cosecant** functions, denoted by tanh, coth, sech, and csch, respectively, are defined as follows.

(i) $\tanh x = (\sinh x)/(\cosh x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ (ii) $\coth x = (\cosh x)/(\sinh x) = \frac{e^x + e^{-x}}{e^x - e^{-x}}$, $x \neq 0$
(iii) $\text{sech } x = 1/(\cosh x) = \frac{2}{e^x + e^{-x}}$ (iv) $\text{csch } x = 1/(\sinh x) = \frac{2}{e^x - e^{-x}}$, $x \neq 0$

**Class Exercise 1.** Prove: (a) $\sinh(-x) = -\sinh x$ (b) $\cosh(-x) = \cosh x$
(c) $1 - \tanh^2 x = \text{sech}^2 x$ (d) $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$
(e) $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$.

**Exercise 2.** Prove: (a) $\frac{d}{dx}(\sinh x) = \cosh x$ (b) $\frac{d}{dx}(\tanh x) = \text{sech}^2 x$. (Swok Page 444)

**Class Exercise 2.** Prove: (a) $\frac{d}{dx}(\text{csch } x) = -\text{csch } x \coth x$ (b) $\frac{d}{dx}(\cosh x) = \sinh x$
(c) $\frac{d}{dx}(\text{sech } x) = -\text{sech } x \tanh x$ (d) $\frac{d}{dx}(\coth x) = -\text{csch }^2 x$.

**Exercise 3.** If $f(x) = \cosh(x^2 + 1)$, find $f'(x)$. (Swok Sec 8.3 Ex 1)

**Class Exercise 3.** Find the derivative (#30-38 even)
(a) $f(x) = \tanh(1 + e^{2x})$ (b) $g(x) = \cosh(\ln x)$ (c) $y = x \coth(1 + x^2)$
(d) $f(t) = \text{csch } t\ (1 - \ln \text{csch } t)$ (e) $y = \sinh(\cosh x)$

We now define some inverse hyperbolic functions:

$y = \sinh^{-1} x$ if and only if $\sinh y = x$

$y = \cosh^{-1} x$ if and only if $\cosh y = x$ and $y > 0$

$y = \tanh^{-1} x$ if any only if $\tanh y = x$

**Exercise 4.** Prove that $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$. (Swok Page 448)

**Class Exercise 4.** Prove that $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$.

**Class Exercise 5.** Prove that $\tanh^{-1} x = \frac{1}{2}\ln(\frac{1+x}{1-x})$.

We now find derivatives of inverse hyperbolic functions:

**Exercise 5.** Prove: $\frac{d}{dx}\ (\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$ (Swok Page 449)

**Class Exercise 6.** Prove: (a) $\frac{d}{dx}\ (\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$ (b) $\frac{d}{dx}\ (\tanh^{-1} x) = \frac{1}{1-x^2}$
(c) $\frac{d}{dx}\ (\text{csch}^{-1} x) = -\frac{1}{|x|\sqrt{x^2+1}}$ (d) $\frac{d}{dx}\ (\text{sech}^{-1} x) = -\frac{1}{x\sqrt{1-x^2}}$ (e) $\frac{d}{dx}\ (\coth^{-1} x) = \frac{1}{1-x^2}$

**Exercise 6.** If $y = \sinh^{-1}(\tan x)$, find $dy/dx$. (Swok Sec 8.4 Ex 1)

**Class Exercise 7.** Find the derivative. (#42, 44)
(a) $y = x \tanh^{-1} x + \ln\ (\sqrt{1 - x^2})$ (b) $y = \text{sech}^{-1}(e^{-x})$

Homework: 1, 5, 13-25 (every 4th), 33-49 (every 4th), 57