Section 2.5

Definition: A function f is **continuous** at a number c if the following conditions are satisfied:
(i) $f(c)$ is defined
(ii) $\lim_{x\to c} f(x)$ exists
(iii) $\lim_{x\to c} f(x) = f(c)$

Definition: If one (or more) of the three conditions in the definition is not satisfied, we say that f is **discontinuous** at c, or that f has a **discontinuity** at c.

There are three types of discontinuities: removable discontinuities, jump discontinuities, and infinite discontinuities.

Example 1: The function $f(x) = [|x|]$ has jump discontinuities.

Example 2: The function $f(x) = \frac{1}{x^2}$ has an infinite discontinuity.

Example 3: The function $f(x) = \frac{x^2}{x}$ has a removable discontinuity.

Exercise 1. Find the points of discontinuity of the function. Identify each type of discontinuity.
(a) $y = \frac{1}{(x+2)^2}$. (Waits Sec 2.3 #1)
(b) $y = |x-1|$. (Wait Sec 2.3 #4)
(c) $g(x) = \begin{cases} x+6, & x \le -4 \\ \frac{1}{2}x - 4 & x > -4 \end{cases}$ (d) $h(x) = \frac{x^2-1}{x-1}$.

Class Exercise 1. Explain why the function is discontinuous at the given number a. Sketch the graph of the function. (#18, 20, 22)
(a) a = -2
$f(x) = \begin{cases} \frac{1}{x+2} \text{ if } x \ne -2 \\ 1 \text{ if } x = -2 \end{cases}$
(b) $a = 1$
$f(x) = \begin{cases} \frac{x^2-x}{x^2+1} \text{ if } x \ne 1 \\ 1 \text{ if } x = 1 \end{cases}$
(c) $a = 3$
$f(x) = \begin{cases} \frac{2x^2-5x-3}{x-3} \text{ if } x \ne 3 \\ 6 \text{ if } x = 3 \end{cases}$

Class Exercise 2. Find the points of discontinuity of the function. Identify each type of discontinuity. (Waits Sec 2.3 #2,3, 5-10)
(a) $y = \frac{x+1}{x^2-4x+3}$ (b) $y = \frac{1}{x^2+1}$ (c) $y = \sqrt{2x+3}$
(d) $y = \sqrt[3]{2x-1}$ (e) $y = |x|/x$ (f) $y = \cot x$ (g) $y = e^{1/x}$
(h) $y = \ln(x+1)$

Theorem: (i) A polynomial function f is continuous at every real number c.
(ii) A rational function $q = f/g$ is continuous at every number c except the numbers c such that $g(c) = 0$.

Exercise 2. If $f(x) = |x|$, show that f is continuous at every real number c. (Swok Sec 2.5 Ex 1)

Class Exercise 3. If $f(x) = \frac{x^2-1}{x^3+x^2-2x}$, find the discontinuities of f. (Swok Sec 2.5 Ex 2)

Definition: If f is continuous at every number in the open interval (a, b), we say that **f is continuous on (a, b)**.

Definition: Let a function f be defined on a closed interval $[a, b]$.
The **function f is continuous on $[a, b]$** if it is continuous on (a, b) and if, in addition,

$$\lim_{x\to a^+} f(x) = f(a) \text{ and } \lim_{x\to b^-} f(x) = f(b).$$

If a function f has either a right-hand or a left-hand limit of the type indicated in the definition above, we say that **f is continuous from the right at a** or that **f is continuous from the left at b**, respectively.

Theorem: The following types of functions are continuous at every number in their domains: polynomials, rational functions, root functions, trigonometric functions, inverse trigonometric functions, exponential functions, and logarithmic functions.

Exercise 3. If $f(x) = \sqrt{9 - x^2}$, sketch the graph of f and prove that f is continuous on the closed interval $[-3, 3]$. (Swok Sec 2.5 Ex 3)

Theorem: If f and g are continuous at c, then the following are also continuous at c:
(i) the sum $f + g$ (ii) the difference $f - g$
(iii) the product fg (iv) the quotient f/g, provided $g(c) \neq 0$

Exercise 4. If $k(x) = \frac{\sqrt{9-x^2}}{3x^4+5x^2+1}$, prove that k is continuous on the closed interval $[-3, 3]$. (Swok Sec 2.5 Ex 4)

Theorem: If $\lim_{x\to c} g(x) = b$ and if f is continuous at b, then

$$\lim_{x\to c} f(g(x)) = f(b) = f(\lim_{x\to c} g(x)).$$

Theorem: If g is continuous at c and if f is continuous at $b = g(c)$, then

(i) $\lim_{x\to c} f(g(x)) = f(\lim_{x\to c} g(x)) = f(g(c))$

(ii) the composite function $f \circ g$ is continuous at c

Exercise 5. If $k(x) = |3x^2 - 7x - 12|$, show that k is continuous at every real number. (Skow Sec 2.5 Ex 5)

Exercise 6. Show that the following functions are continuous on their natural domains. (Hass Sec 2.5 Ex 8)
(a) $y = \sqrt{x^2 - 2x - 5}$ (b) $y = \frac{x^{2/3}}{1+x^4}$
(c) $y = \left| \frac{x-2}{x^2-x} \right|$ (d) $y = \left| \frac{x \sin x}{x^2+2} \right|$

Class Exercise 4. Explain why the function is continuous at every number in its domain. State the domain. (#26-32 even)
(a) $G(x) = \frac{x^2+1}{2x^2-x-1}$ (b) $R(t) = \frac{e^{\sin t}}{2+\cos \pi t}$
(c) $B(x) = \frac{\tan x}{\sqrt{4-x^2}}$ (d) $N(t) = \tan^{-1}(1 + e^{-t^2})$

Intermediate Value Theorem: If f is continuous on a closed interval $[a, b]$ and if w is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = w$.

Exercise 7. Show that $f(x) = x^5 + 2x^4 - 6x^3 + 2x - 3$ has a zero between 1 and 2. (Swok Sec 2.5 Ex 6)

Exercise 8. Show that there is a root of the equation $x^3 - x - 1 = 0$ between 1 and 2. (Hass Sec 2.5 Ex 10)

Class Exercise 5. Use the Intermediate Value Theorem to show that there is a root of the equation, $\sqrt[3]{x} = 1 - x$ in the interval (0,1). (#52)

Class Exercise 6. Use the Intermediate Value Theorem to show that there is a root of the equation, $\sin x = x^2 - x$ in the interval (1,2). (#54)

Homework: 1, 3, 7-11 ODD, 13-41 (every 4th), 43, 45, 53, 55, 57