Section 1.5

One-to-One Functions

Definition: A function $f(x)$ is **one-on-one** on a domain D if $f(a) \neq f(b)$ whenever $a \neq b$.

Horizontal Line Test: Every horizontal line intersects the graph of a one-to-one function at most once.

Exercise 1. Is $f(x) = x^2$ a one-to-one function?

Class Exercise 1. Is $g(x) = x^3$ a one-to-one function? Is $h(x) = |x|$ a one-to-one function?

Since each output of a one-to-one function comes from just one input, a one-to-one function can be reversed to send outputs back to the inputs which they came. The function defined by reversing a one-to-one function f is the inverse of f.

Inverse Functions

Definition of Inverse Function: Let f and g be two functions such that $f(g(x)) = x$ for every x in the domain of g and $g(f(x)) = x$ for every x in the domain of f. Under these conditions, the function g is the **inverse function** of the function f. The function g is denoted by f^{-1} (read "f-inverse"). So,

$$f(f^{-1}(x)) = x \text{ and } f^{-1}(f(x)) = x.$$

The domain of f must equal to the range of f^{-1}, and the range of f must be equal to the domain of f^{-1}.

Exercise 2. Show the functions are inverse functions of each other. (Larson Sec 1.7 Ex 3)

$$f(x) = 2x^3 - 1 \text{ and } g(x) = \sqrt[3]{\tfrac{x+1}{2}}$$

Class Exercise 2. Which of the functions is the inverse function of $f(x) = \frac{5}{x-2}$? (Larson Sec 1.7 Ex 4)

$$g(x) = \frac{x-2}{5} \qquad \text{or} \qquad h(x) = \frac{5}{x} + 2$$

Graph of an Inverse Function

The graph of a function and its inverse are closely related. To read the value of a function from its graph, we start at a point x on the x-axis, go vertically to the graph, and then move horizontally to the y-axis to read the value of y. The inverse function can be read from the graph by reversing the process. Start with a point y on the y-axis, go horizontally to the graph of $y = f(x)$, and then move vertically to the x-axis to read the value of $x = f^{-1}(y)$.

Exercise 3. Plot $f(x) = \frac{1}{2}x + 1$ and $f^{-1}(x)$ on the same coordinate axes. (Hass Sec 1.6 Ex 1)

Exercise 4. Plot $f(x) = x^2$, $x \geq 0$, and $f^{-1}(x)$ on the same coordinate axes. (Hass Sec 1.6 Ex 2)

Class Exercise 3. Plot $f(x) = 2x + 6$ and $f^{-1}(x)$ on the same coordinate axes. (Briggs Sec 1.3 Ex 4(a))

Class Exercise 4. Plot $f(x) = \sqrt{x-1}$ and $f^{-1}(x)$ on the same coordinate axes. (Briggs Sec 1.3 Ex 4(b))

Finding Inverse Function Algebraically

1. Use the Horizontal Line Test to decide whether f has an inverse function.

2. In the equation for $f(x)$, replace $f(x)$ by y.

3. Interchange the roles of x and y, and solve for y.

4. Replace y by $f^{-1}(x)$ in the new equation.

Exercise 5. Find the inverse function of $f(x) = \frac{5-3x}{2}$. (Larson Sec 1.7 Ex 7)

Exercise 6. Find the inverse function of $f(x) = x^3 - 4$. (Larson Sec 1.7 Ex 8)

Exercise 7. Find the inverse function of $f(x) = \sqrt{2x-3}$. (Larson Sec 1.7 Ex 9)

Class Exercise 5. Find a formula for the inverse of the function. (#22, 24, 26)
(a) $f(x) = \frac{4x-1}{2x+3}$ (b) $y = x^2 - x$, $x \geq \frac{1}{2}$ (c) $y = \frac{e^x}{1+2e^x}$

Definition For $x > 0$, $a > 0$, and $a \neq 1$,

$$y = \log_a x \text{ if and only if } x = a^y.$$

The function given by

$$f(x) = \log_a x$$

is called the **logarithmic function with base** a.

Exercise 8. Use the definition of logarithmic function to evaluate $f(x) = \log_2 x$ at $x = 32$. (Sec 4.2 Ex 1)

Class Exercise 6. Use the definition of logarithmic function to evaluate:
(a) $f(x) = \log_3 x$ at $x = 1$ (b) $f(x) = \log_4 x$ at $x = 2$. (c) $f(x) = \log_{10} x$ at $x = \frac{1}{100}$.

Definition: The function defined by

$$f(x) = \log_{10} x = \log x, \; x > 0$$

is called the **common logarithmic function**.

Definition: The function defined by

$$f(x) = \log_e x = \ln x, \; x > 0$$

is called the **natural logarithmic function**.

Properties of Logarithms
Let a be a positive number such that $a \neq 1$, let n be a real number. If u and v are positive real numbers, the following properties are true.

1. **Product Property**: $\log_a(uv) = \log_a u + \log_a v$

2. **Quotient Property**: $\log_a \frac{u}{v} = \log_a u$ - $\log_a v$

3. **Power Property**: $\log_a u^n = n \log_a u$

Exercise 9. Express the given quantity as a single logarithm: ln 5 + 5 ln 3. (#39)

Class Exercise 7. Express the given quantity as a single logarithm: $\ln(a+b) + \ln(a-b)$ - 2 ln c. (#40)

Exercise 10. Use the properties of logarithms to expand the expression: $\log_4 5x^3y$. (Larson Sec 4.3 Ex 5a)

Class Exercise 8. Use the properties of logarithms to expand the expression: $\ln\frac{\sqrt{3x-5}}{7}$. (Larson Sec Ex 5b)

Exercise 11. Use the properties of logarithms to condense the logarithmic expression. (Larson Sec 4.3 Ex 6a)

$$\frac{1}{2}\log_{10}x + 3\log_{10}(x+1)$$

Class Exercise 9. Use the properties of logarithms to condense the logarithmic expressions. (Larson Sec 4.3 Ex 6b, c)
(a) $2 \ln (x+2) - \ln x$ (b) $\frac{1}{3}[\log_2 x + \log_2(x-4)]$

We now move on to solving exponential and logarithmic equations. Here are some strategies:

1. Rewrite the original equation in a form that allows the use of the One-to-One Properties of exponential or logarithmic functions.

2. Rewrite an *exponential* equation in logarithmic form and apply the Inverse Property of logarithmic functions.

3. Rewrite a *logarithmic* equation in exponential form and apply the Inverse Property of exponential functions.

Exercise 12. Solve the exponential equation: (a) $e^x = 72$ (b) $3(2^x) = 42$. (Larson Sec 4.4 Ex 2)

Class Exercise 10. Solve the exponential equation: (a) $4e^{2x} - 3 = 2$ (b) $2(3^{2t-5}) - 4 = 11$
(c) $e^{2x} - 3e^x + 2 = 0$ (Larson Sec 4.4 Ex 3, 4, 5)

Exercise 13. Solve the logarithmic equation: (a) $\ln 3x = 2$
(b) $\log_3(5x-1) = \log_3(x+7)$. (Larson Sec 4.4 Ex 6)

Class Exercise 11. Solve the logarithmic equation: (a) $5 + 2 \ln x = 4$
(b) $2 \log_5 3x = 4$ (c) $\ln (x-2) + \ln (2x-3) = 2 \ln x$. (Larson Sec 4.4 Ex 7, 8, 9)

Inverse Trigonometric Functions

The **inverse sine function** is defined by

$$y = \sin^{-1}x \text{ if and only if } \sin y = x$$

where $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$. The domain of $y = \sin^{-1}x$ is [-1,1], and the range is $[-\pi/2, \pi/2]$.

The **inverse cosine function** is defined by

$$y = \cos^{-1}x \text{ if and only if } \cos y = x$$

where $-1 \le x \le 1$ and $0 \le y \le \pi$. The domain of $y = \cos^{-1}x$ is [-1,1], and the range is $[0, \pi]$.

The **inverse tangent function** is defined by

$$y = \tan^{-1}x \text{ if and only if } \tan y = x$$

where $-\infty < x < \infty$ and $-\pi/2 < y < \pi/2$. The domain of $y = \tan^{-1}x$ is $(-\infty, \infty)$, and the range is $(-\pi/2, \pi/2)$.

Exercise 14. Find the exact value of each expression: (a) $\tan^{-1}(1/\sqrt{3})$ (b) $\sec^{-1}2$ (#64)

Class Exercise 12. Find the exact value of each expression. (#66, 68)
(a) $\cot^{-1}(-\sqrt{3})$ (b) $\arccos(-\frac{1}{2})$ (c) $\tan(\sec^{-1}4)$ (d) $\sin(2\sin^{-1}(\frac{3}{5}))$

Homework: 1, 5, 9, 15, 19, 23, 27, 33, 41, 45, 51-71 (every 4th)