Physics 8B Notes - Part I, Electricity

Karthik Siva

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Contents

1 Logistics

See class website on bCourses. Discussion notes will be posted at

`https://ocf.io/~ksiva/physics8_fa17/`.

Attendance at discussion is mandatory for the first two weeks to secure your spot. Participation in discussion is not mandatory nor will it directly affect your grade. However, asking questions and getting clarification will definitely help you get a better grade, so in an indirect way, participation does affect your grade. So, please ask any questions you have. I will try to post discussion notes, but do not expect them to be self-contained or complete, especially since this is the first time I am serving as a GSI for this course.

2 Preliminaries

- What is a vector? What is a scalar? What are examples of each?
- What is a basis? What are unit vectors?
- How do we represent an arbitrary vector? What happens to the representation as we change the basis?
- What is a dot product? How do we represent it? What happens to its representation as we change the basis? Important cases.
- What is a cross product? How do we represent it? What happens to its representation as we change basis? Important cases. (Won't need the cross product until we discuss magnetism.)

We will see how these objects arise in this topic in just a moment.

2.1 Notation

In these notes, I will use the following convention when typesetting: boldface indicates that a quantity is a vector (such as $\boldsymbol{\mathscr{r}}$, a hat on top of a boldface character (such as $\hat{\boldsymbol{\mathscr{r}}}$), and $|\ldots|$ around a vector indicate the magnitude of the vector (such as $|\boldsymbol{\mathscr{r}}|$). On the chalkboard, I cannot really boldface a character, so I will indicate a vector using an arrow over the letter (such as $\vec{r}$).

3 Electrostatics

Much of the discussion of electricity and magnetism begins with studying the behavior of charged particles, such as electrons, protons, ions, a balloon after you rub it on your hair...

3.1 Charge

Charge comes in quantized chunks, with SI units of *Coulombs (C)*. Familiar particles like the proton and electron have charge 1.602×10^{-19}C and -1.602×10^{-19}. They have equal magnitude (henceforth labeled e) but opposite sign. In fact, in general, since objects have an integer number of fundamental particles, each carrying an integer amount of charge, can only come with integer numbers of charge

$$\text{Total charge} = \sum_i q_i = N q_e$$

where $q_e = 1.602 \times 10^{-19}$C. Experience tells us that two objects with charges of opposite sign attract and two objects with the charge of the same repel. It also tells us that the farther away two objects are, the less they attract/repel. This leads us to our first law.

3.2 Coulomb's Law

Coulomb's law states that the force due to a charge at a location $\mathbf{r}'$ ON a charge at location $\mathbf{r}$ is given by

$$\mathbf{F}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q_r q_{r'}}{|\mathbf{r} - \mathbf{r}'|^2} \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \tag{1}$$

where ε_0 is a fundamental constant of nature, which, in SI units, takes on the value

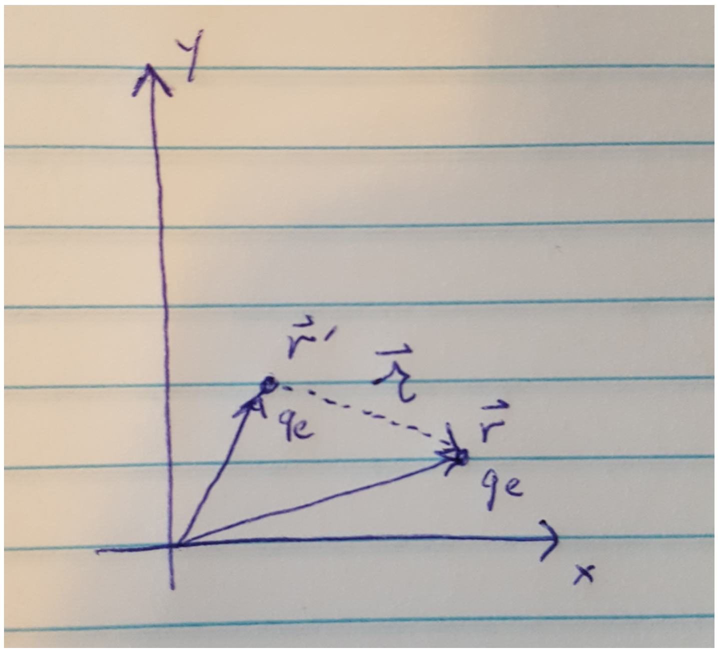
$$\varepsilon_0 = 8.85418781 \times 10^{-12} \mathrm{F} \cdot \mathrm{m}^{-1} \tag{2}$$

A common convention is to write the constant $1/(4\pi\varepsilon_0)$ as k. For convenience we will denote $\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}'$ and $\hat{\boldsymbol{\mathscr{r}}}$ as the unit vector in the direction of $\boldsymbol{\mathscr{r}}$. That is,

$$\hat{\boldsymbol{\mathscr{r}}} = \frac{1}{|\mathbf{r} - \mathbf{r}'|}(\mathbf{r} - \mathbf{r}') \tag{3}$$

which is just the last term in Coulomb's law above. Next we do an example of how to use this law and interpret it:

Example 3.1 (Two charges of same sign)**.** Suppose you have two particles each of charge q_e and you know their positions. How do we compute the force on one particle due to the other particle?



In the example above, we use the information as follows:

$$\mathbf{r} = 3\hat{\mathbf{i}} + 1\hat{\mathbf{j}} \tag{4}$$

$$\mathbf{r}' = 1\hat{\mathbf{i}} + 2\hat{\mathbf{j}} \tag{5}$$

$$\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}' = (3-1)\hat{\mathbf{i}} + (1-2)\hat{\mathbf{j}} = 2\hat{\mathbf{i}} - 1\hat{\mathbf{j}} \tag{6}$$

$$|\boldsymbol{\mathscr{r}}| = \sqrt{2^2 + (-1)^2} = \sqrt{5} \tag{7}$$

$$\hat{\boldsymbol{\mathscr{r}}} = \frac{1}{\sqrt{5}}(2\hat{\mathbf{i}} - 1\hat{\mathbf{j}}) = \frac{2}{\sqrt{5}}\hat{\mathbf{i}} - \frac{1}{\sqrt{5}}\hat{\mathbf{j}} \tag{8}$$

$$\mathbf{F}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q_e^2}{|\boldsymbol{\mathscr{r}}|^2}\hat{\boldsymbol{\mathscr{r}}} = \frac{1}{4\pi\varepsilon_0}\frac{q_e^2}{5}\left(\frac{2}{\sqrt{5}}\hat{\mathbf{i}} - \frac{1}{\sqrt{5}}\hat{\mathbf{j}}\right) \tag{9}$$

The result says that the force on the particle at $\mathbf{r}$ points down and to the right–away from the particle at $\mathbf{r}'$, which agrees with our intuition that the two charges of the same sign should be forced away from each other. Check to see that if you had instead swapped $\mathbf{r}'$ and $\mathbf{r}$ and thus asked what the force on the charge at $1\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$ due to the charge at $3\hat{\mathbf{i}} + 1\hat{\mathbf{j}}$ that you would get the force vector with the same magnitude but pointing up and to the right (opposite of the direction obtained in this example).

The first part of this law tells us the following important facts:

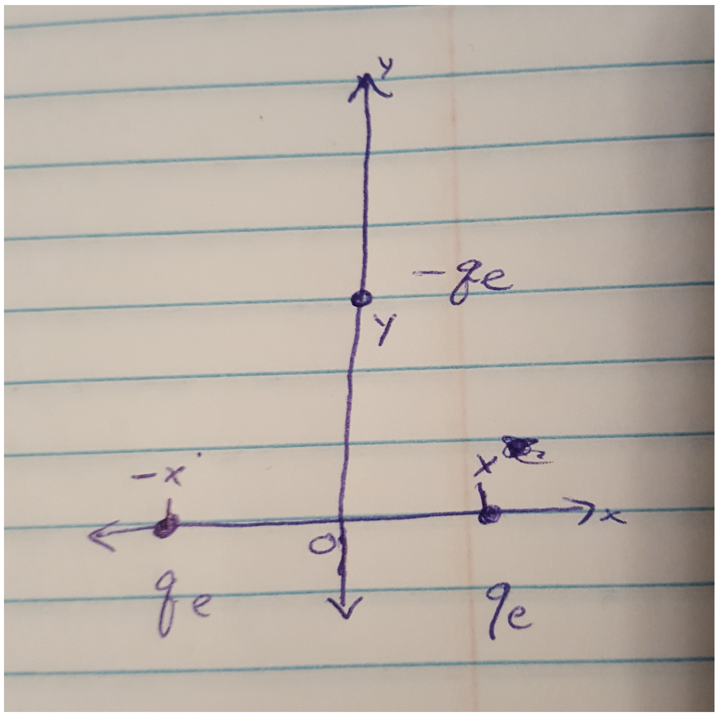
- The force acts along the vector between the two charges.

- The force is attractive if the two charges are oppositely signed and repulsive if they are same-signed.
- The magnitude of the force falls off as $1/r^2$ where r is the distance between the two particles.

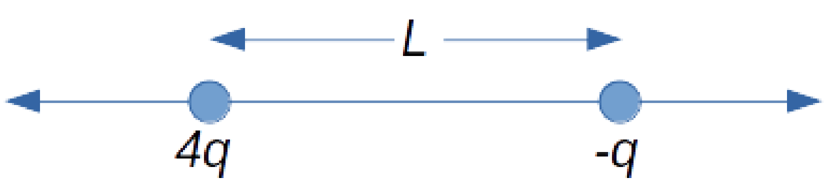
The second important (experimental) fact: the force on a charged particle due to many other charged particles is just the sum of the individual forces. This is known as the "principle of superposition".

$$\mathbf{F}_{\text{total}}(\mathbf{r}) = \sum_i \mathbf{F}_i(\mathbf{r}) = \sum_i \frac{q_r q_{r'_i}}{|\mathbf{r}-\mathbf{r}'_i|^2} \frac{\mathbf{r}-\mathbf{r}'_i}{|\mathbf{r}-\mathbf{r}'_i|} \tag{10}$$

Example 3.2 (Total force.)**.** Consider the setup of three charged particles shown below.

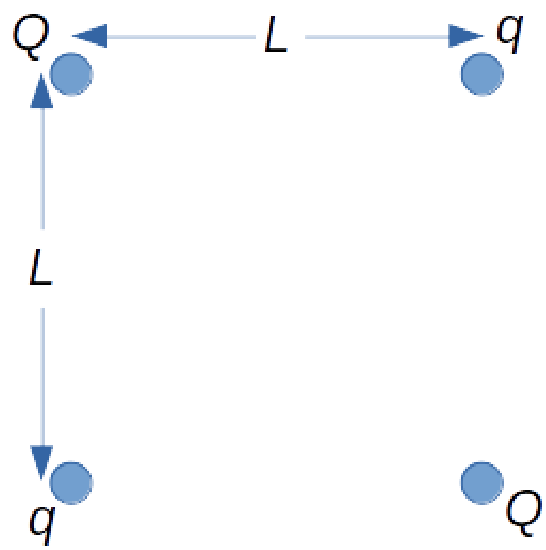


Example 3.3 (Solving for location of equilibrium (HRW 21-4 Sample).)**.** Consider the setup of two charges shown below.



Find a location along the line such that a particle of charge q at that location is at equilibrium (the total force on it is zero). Since the strength of the forces due to the two source charges above falls off as $1/x^2$, one answer is simply $x = \pm\infty$. Instead I am asking is there any *finite* location for which the total force on the test charge is zero.

Example 3.4 (Solving for unknown charge (HRW 21.10).)**.** Consider the setup of four charged particles at the vertices of a square shown below.



If the total force on the top left charge (Q) is zero, what is Q/q?

3.3 Electric Field

So far, when we wanted to study the effects of charged particles, we only discussed it by analyzing what happens when we put a particle of known charge (a test charge) at a certain location. We then had to keep track of the the relative sign of the test charge and the individual source charges, and to give you any force as a function of position $\mathbf{F}(\mathbf{r})$, I had to tell you where every source charge was as well.

Instead, suppose I wanted to create an arbitrary force as a function of position, and I didn't want to tell you where all the charges that create that force are located. I also want to stop using a test charge and worrying about whether it is positive or negative every time I want to write down an equation. Here, we introduce a new idea, the *electric field*, denoted **E**. It is a vector quantity that is the *force per unit (test) charge*:

$$\vec{E}(\mathbf{r}) = \frac{\mathbf{F}_q(\mathbf{r})}{q} \tag{11}$$

which has units of N/C (Newtons per Coulomb). Note that **E** has *nothing* to do with the amount or sign of any test charge that we are used to talking about. It only depends on what the source charges are and where they are relative to $\vec{r}$. To get the electric field at a point $\mathbf{r}$ due to a charge at a point $\mathbf{r}'$, dividing Eq. 1 by q_r gives:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q_{r'}}{|\boldsymbol{\mathscr{r}}|^2}\hat{\boldsymbol{\mathscr{r}}} = \frac{1}{4\pi\varepsilon_0}\frac{q_{r'}}{|\mathbf{r}-\mathbf{r}'|^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \tag{12}$$

Since $\mathbf{F}_q$ obeys the principle of superposition, so does $\vec{E}$. If the forces come from multiple sources indexed by i:

$$\vec{E}_{\text{total}}(\mathbf{r}) = \frac{1}{q}\sum_i \mathbf{F}_i = \sum_i \frac{\mathbf{F}_i}{q} = \sum_i \mathbf{E}_i \tag{13}$$

So if we have multiple source charges, then to find **E** at **r**, we just need to add up the electric fields created by each individual source charge. Now, let's draw the electric fields of some important cases.

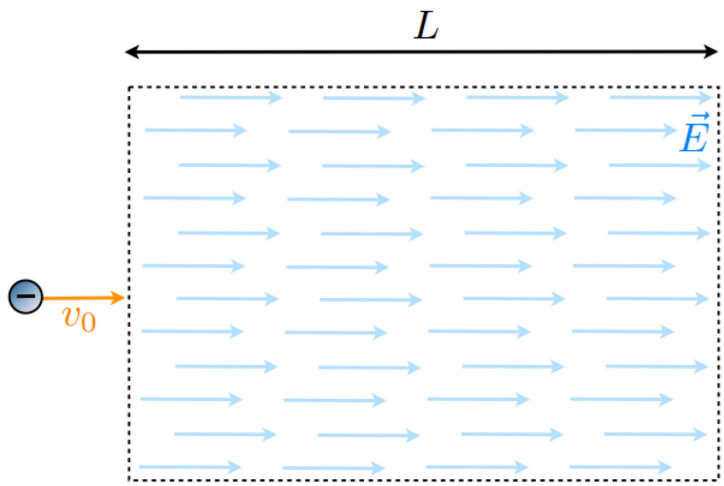
Example 3.5 (Sketching field lines of a lone positive charge.).

Example 3.6 (Sketching field lines of a lone negative charge.).

Example 3.7 (Sketching field lines of a positive charge near a negative charge.)**.**

Example 3.8 (Sketching field lines of a positive charge near a positive charge.)**.**

Example 3.9 (Velocity of a particle in uniform electric field. (Chien-I, Optional Problems 1))**.** A particle of mass m and charge $-q$ is released with initial velocity $v_0\hat{\mathbf{i}}$ in a region of space of width L which contains a uniform electric field $\vec{E} = E\hat{\mathbf{i}}$ (shown below).



What is the minimum initial velocity that the particle needs to be able to exit the region on the right side?

The first thing to do is to analyze why the initial velocity is even important. When the charge enters the field from the left, which way does the force due to the electric field point? Since it points to the left, it will begin to accelerate *opposite* the direction of its initial velocity, so it will slow down as it moves through the field. Eventually, it will turn back around. A good analogy here is to think about trying to toss a ball up into the air to land it on the roof of a house. Once the ball is tossed, gravity acts on it downward–opposite its initial upward velocity– and slows it down. Eventually, it slows down, stops, and then turns back around. So, if you do not throw it with enough initial velocity, it will not get high enough to land on the roof. If this is clear, then we are prepared to set up the problem.

Recall from classical mechanics that if a particle begins with initial velocity v_0 and experiences constant acceleration a over a distance d, then its velocity after traveling d is given by

$$v_f^2 = v_0^2 + 2ad \tag{14}$$

First, is the acceleration here constant? The acceleration is found by Newton's second law:

$$\mathbf{a} = \frac{\mathbf{F}}{m} = \frac{q\mathbf{E}}{m} = \frac{-qE}{m}\hat{\mathbf{i}} \tag{15}$$

Next, what is the final velocity we are expecting? We are looking for the minimum initial velocity required to get the particle across L. If we were to add extra velocity beyond that, it would still exit, but it would exit with a higher final velocity. So the minimum velocity is the one where the final velocity is zero–the particle just barely makes it to L:

$$0 = v_0^2 + 2\left(\frac{-qE}{m}\right)L \tag{16}$$

$$v_0 = \sqrt{\frac{2qEL}{m}} \tag{17}$$

Example 3.10 (Electric field of dipole on perpendicular axis (HRW 22.18).)**.** There is a dipole composed of a charge $+q$ located at $z = d/2$ and a charge $-q$ located at $z = -d/2$. What is $\mathbf{E}$ on the x-axis or y-axis at a distance from the origin $r \gg d$?

First, we explicitly evaluate the electric fields due to the positive charge and the negative charge. Note that in this situation, the $\hat{\mathbf{i}}$ components of the two contributions will cancel each other while the $\hat{\mathbf{k}}$ components will add.

$$\mathbf{E}(r) = 2 \times k \frac{q}{(d/2)^2 + r^2} \frac{-d/2}{((d/2)^2 + r^2)^{1/2}} \hat{\mathbf{k}} \tag{18}$$

We want to find a simpler but still good approximate expression for the regime $r \gg d$, when we are much farther away from the center of the dipole than the separation of the charges in the dipole. To do this, we want to manipulate this expression to be cast in terms of a variable I will define as $\xi = d/r$, and then Taylor expand the expression around $\xi = d/r = 0$:

$$\mathbf{E}(r) = k \frac{-dq}{((d/2)^2 + r^2)^{3/2}} \hat{\mathbf{k}} = k \frac{-dq}{r^3 \left(\left(\frac{d}{2r} \right)^2 + 1 \right)^{3/2}} = \frac{-kq}{r^2} \frac{\xi}{(\xi^2/4 + 1)^{3/2}} \hat{\mathbf{k}} \tag{19}$$

Now, we want to do the Taylor expansion on the right-side part of the expression in terms of ξ:

$$\frac{\xi}{(\xi^2/4 + 1)^{3/2}} \approx 0 + \xi + O(\xi^2) \tag{20}$$

Substituting this back in,

$$\mathbf{E}(r) \approx -k \frac{q}{r^2} \xi \hat{\mathbf{k}} = -k \frac{qd}{r^3} \hat{\mathbf{k}} \tag{21}$$

Conventionally, the "dipole moment" is defined as the vector $\mathbf{p} = q\mathbf{d}$ where $\mathbf{d}$ is the vector from the negative charge to the positive charge. So in terms of $\mathbf{p}$,

$$\mathbf{E}(r) \approx -k \frac{|\mathbf{p}|}{r^3} \hat{\mathbf{k}} \tag{22}$$

3.3.1 Review of Taylor Expansion

In Ex. 3.10, we found an approximate expression which was much simpler than the exact expression for the exact electric field at a point due to two charges. The simplification used is known as a Taylor expansion–it is a way of expressing a complicated function in terms of the sum of a (potentially infinite) set of polynomials. If I have a well-behaved (no cusps or discontinuities) function $f(x)$ which is complicated–such as $x/(x^2/2+1)^{3/2}$–then I can find an equivalent way of expressing it in terms of powers of x:

$$f(x) = a_0 + a_1 x + a_2 + x^2 + a_3 x^3 + \cdots + a_N x^N + \ldots \tag{23}$$

where the coefficients a_i are to be determined using a known $f(x)$. To find the a_i, we notice that if we differentiate both sides with respect to x, a_0 will disappear:

$$\frac{df(x)}{dx} = 0 + a_1 + 2a_2 x + 3a_3 x^2 + \cdots + (N-1) a_N x^{N-1} + \ldots \tag{24}$$

and if we differentiate again, a_1 also disappears:

$$\frac{d^2 f(x)}{dx^2} = 0 + 0 + 2a_2 + 6a_3 x + \cdots + (N-1)(N-2) a_N x^{N-2} + \ldots \tag{25}$$

so this suggests that if we were to evaluate each derivative at $x = 0$, then we could calculate the coefficients as follows:

$$a_0 = f(0) \tag{26}$$

$$a_1 = \left.\frac{df}{dx}\right|_{x=0} \tag{27}$$

$$a_2 = \left.\frac{1}{2}\frac{d^2 f}{dx^2}\right|_{x=0} \tag{28}$$

$$a_N = \left.\frac{1}{N!}\frac{d^N f}{dx^N}\right|_{x=0} \tag{29}$$

In principle, as long as the derivative is always finite, you can keep calculating a_i's, and you would need to calculate an infinite number of them to obtain an expression that is *exactly* equivalent to $f(x)$. However, there is one regime where terms that have *lower* exponents will dominate those that have *higher* exponents–that is when $0 < x < 1$. Think about what happens to higher powers in this case: $x^N \ll x^3 < x^2 < x < 1$. So a good approximation of $f(x)$ in the area where $0 < x < 1$ can be given by truncating the expansion to just the first few powers of x:

$$f(x) \approx a_0 + a_1 x + a_2 x^2 \tag{30}$$

Typically, in such an approximation, one stops calculating a_i the first time it is nonzero *and* $i > 0$. So how is this idea used above? The problem specified that $r \gg d$. This meant that $d/r \ll 1$, and, since d and r are positive, $0 < d/r$. So we the conditions were ripe to simplify the complicated expression we obtained initially for $\mathbf{E}$ by expanding it in powers of (d/r). The first nonzero coefficient that appeared above was a_1, so we stopped calculating further coefficients.

3.4 Continuous Charge Distributions

Earlier, we said that the total electric field at a point $\mathbf{r}$ due to multiple source charges at locations $\mathbf{r}'_i$ was just the sum (over i) of the individual electric fields due to each source charge. One question is if I take a fixed amount of charge and plaster it (say, uniformly) over a surface, what is the resulting electric field at a point $\mathbf{r}$? When we put a large enough amount of charge on a small enough surface, the charges are very close together, and although they are discrete little pieces, we can treat them as effectively behaving like a continuous distribution. This is not any different than when you say that fluids have density that varies in space–although molecules are discrete, we can approximate their distribution by some smooth function of position.

To find $\mathbf{E}(\mathbf{r})$ due to a continuous distribution of charges ρ, we simply turn the sum in Eq. 13 into an integral:

$$\mathbf{E}(\mathbf{r}) = \int d\mathbf{r}' \frac{\rho(\mathbf{r}')}{|\boldsymbol{\mathscr{r}}|^2}\hat{\boldsymbol{\mathscr{r}}} = \int d\mathbf{r}' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \tag{31}$$

Example 3.11 (Electric field due to a ring of charge.)**.** Suppose we bend a wire with constant charge density λ (charge per unit length) into a circle with radius R. What is the electric field at a point located at a height z above the center of the circle? As always, we begin by sketching what the field at a point $\mathbf{r} = z\hat{\mathbf{k}}$ looks like due to a small piece of charge dq along the circle, where the d in dq refers to a differential element (infinitesimal piece of charge). By observing that the field along the $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ directions due to a piece of charge on one side of the circle gets canceled by the field along the $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ directions due to a piece of charge on the opposite side of the circle, it becomes clear that all of the $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ components will cancel and only the $\hat{\mathbf{k}}$ components will add. So now we proceed to calculate *just the* $\hat{\mathbf{k}}$ *components* of the field due to an infinitesimal piece of charge:

$$d\mathbf{E}\cdot\hat{\mathbf{k}} = k\frac{dq}{R^2+z^2}\frac{z}{(R^2+z^2)^{1/2}} \tag{32}$$

Now we want to integrate both sides to find the total electric field along the $\hat{\mathbf{k}}$ direction:

$$\mathbf{E}\cdot\hat{\mathbf{k}} = \int k\frac{z}{(R^2+z^2)^{3/2}}dq \tag{33}$$

Notice that every piece of charge on the wire has the same magnitude of contribution along the $\hat{\mathbf{k}}$ direction to the electric field at $z\hat{\mathbf{k}}$. This is manifested in the RHS of the equation above by the fact that the integrand doesn't depend on q. So we can pull it outside the integral:

$$\mathbf{E} \cdot \hat{\mathbf{k}} = k\frac{z}{(R^2+z^2)^{3/2}} \int dq = k\frac{z}{(R^2+z^2)^{3/2}} Q_{\text{total}} = k\frac{z}{(R^2+z^2)^{3/2}} \lambda(2\pi R) \tag{34}$$

So

$$\mathbf{E} = k\frac{2\pi\lambda R z}{(R^2+z^2)^{3/2}} \hat{\mathbf{k}} \tag{35}$$

Example 3.12 (Electric field due to a disk of charge.)**.** Suppose we have a circular disk of radius R carrying a uniform charge density σ (charge per unit area). What is the electric field at any point along the axis perpendicular the disk through its center?

First we observe what the field contribution will be from a little piece of charge at a point (r, ϕ) on the disk. It points along the vector from that location to the point $\mathbf{r} = z\hat{\mathbf{k}}$ where we are trying to evaluate $\mathbf{E}$. Now consider another little piece of charge at a point $(r, \phi + \pi)$. This little piece contributes the same magnitude of field as the previous piece, since they are equidistant form $\mathbf{r}$, but the $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ contributions to the field have the opposite sign. So pieces of charge that are at the same distance from the center but opposite from each other on the disk cancel each others' $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ contributions, leaving only the $\hat{\mathbf{k}}$ contribution. Now, we are ready to set up the problem.

$$d\mathbf{E} \cdot \hat{\mathbf{k}} = k\frac{\sigma r' dr' d\phi'}{(r'^2+z^2)^{3/2}} z \tag{36}$$

$$\begin{aligned}
\mathbf{E(z)} \cdot \hat{\mathbf{k}} &= \int_0^R \int_0^{2\pi} k\frac{\sigma r' dr' d\phi'}{(r'^2+z^2)^{3/2}} z && (37)\\
&= 2\pi\sigma k z \int_0^R \frac{r' dr'}{(r'^2+z^2)^{3/2}} dr' && (38)\\
&= -\frac{2\pi\sigma k z}{(r'^2+z^2)^{1/2}}\Bigg|_{r'=0}^{r'=R} && (39)\\
&= 2\pi\sigma k \left(1 - \frac{z}{(R^2+z^2)^{1/2}}\right) && (40)\\
&= \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{(R^2+z^2)^{1/2}}\right) && (41)\\
&= \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z/R}{(1+(z/R)^2)^{1/2}}\right) && (42)
\end{aligned}$$

The reason for writing the field this way will be made clear in Ex. 3.16

Example 3.13 (Electric field due to a donut of charge.)**.** In this setup, we have a disk of radius R carrying a uniform charge density σ, but we then punch out a hole of radius $r < R$ in the center of the disk. What is $\mathbf{E}$ anywhere on the axis perpendicular to the donut through its center?

We can solve this problem in two ways. One would be to repeat the analysis and integral from Ex.3.12 using a lower limit of r instead of 0 on the integral over the disk area. Exactly equivalent to that is to conceptualize the donut as a full disk of charge density σ and radius R with another full disk of charge density $-\sigma$ and radius r superimposed at its center. If we were to really do this, the net charge in the center, small disk would be zero as desired. So using the principle of superposition all we need to do is plug in the results:

$$\mathbf{E}(z) = \frac{\sigma}{2\varepsilon_0}\left(\frac{z/R}{(1+(z/R)^2)^{1/2}} - \frac{z/r}{(1+(z/r)^2)^{1/2}}\right)\hat{\mathbf{k}} \tag{43}$$

3.5 Electric Flux

One useful quantity in electrostatics is the electric flux through some region of space. The flux over a surface S can be thought of as the amount of a vector field E that flows into or out of S. The conceptual picture is to think of a surface, such as a plane and ask how

3.6 Gauss's Law and Symmetries

There are a few (equivalent) versions of Gauss's law that are useful in different scenarios. The first (and potentially only one for this course) that we discuss uses the total flux through a closed surface S:

$$\Phi = \int \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{enclosed}}}{\varepsilon_0} \tag{44}$$

Let's break this down. The left hand side contains the total electric flux for a given closed surface. This is obtained by summing the dot product of the electric field and the area vector at every point on the surface. When the surface and electric field are simple, the integral can be simplified to just sums and multiplications, but for an arbitrary surface, you would have to do an integral. The right hand side says that the total flux is proportional to all of the charge enclosed in the region of space inside the surface.

If we consider the total flux through the surface of a sphere containing a point charge $+q$, Gauss's law says that is equal to q/ε_0. But interestingly *it does not matter where you put the charge inside the sphere.* If you put it at the center of the sphere, the magnitude of the field is the same everywhere on its surface and its direction points outward, perpendicular to the surface. But if you move the charge to another location inside the sphere, the field is no longer constant everywhere on the surface. *But the flux is the same in both situations.*

Example 3.14 (Flux through a cube with point charge at vertex. (Griffiths 3.?))**.** Suppose we have a cube with side length L and a point charge of magnitude $+q$ sits at one of the vertices. What is the total flux through the cube?

The most important use case for Gauss's law is to find $\mathbf{E}$ along some surface when it is easier to use Gauss's law than to add the fields manually. Sometimes it is even necessary to use Gauss's law rather than sum individual fields because the sum diverges even though the field is finite. This use is only possible when there is some symmetry in the scenario that allows $\mathbf{E}$ to be constant across some surface. In that case, the integral for the flux simplifies greatly.

Example 3.15 (Field of an infinite line of charge.)**.** Suppose we have an infinitely long line of charge with constant charge density λ (again, charge per unit length). What is the electric field at a point which is a distance R away from the axis of the line? You can do this problem in two ways. In one, you can integrate all of the contributions across the whole wire. In the other, you can consider a surface enclosing part of the wire and analyze how $\mathbf{E}$ points on that surface.

Example 3.16 (Field of an infinite plane of charge.)**.** Suppose now we have an infinite plane of charge with constant charge density σ (charge per unit area). What is the electric field at a point which is a distance R away from the plane? Try doing this problem by integrating Eq. 31. Or try doing the problem using the result of Ex. 3.12. Finally, try doing it by arguing about the magnitude and direction of the field everywhere and then finding a surface to enclose part of the plane and use Gauss's law to find $\mathbf{E}$. If you did not already have the result of the Ex. 3.12 on hand, which method would be fastest?

Example 3.17 (Charged hollow sphere.)**.** Suppose we have a spherical shell of radius R carrying a total charge q distributed uniformly on its surface. What is $\mathbf{E}$ inside the cavity? What is $\mathbf{E}$ outside the cavity?

Example 3.18 (Charged sphere with a cavity.). Suppose we have a sphere of radius R carrying a uniform charge density ρ (charge per unit volume). We then carve out two small spherical cavities each of radius $R/2$ such that their centers are R apart from each other, and each cavity's center is a distance $R/2$ from the center of the whole sphere. What is $\mathbf{E}$ at any point along the axis perpendicular to the axis formed by the centers of the two cavities?

3.7 Conductors

We now turn to talking about an important material in electrostatics–conductors. Our daily experience (and usage of the word 'conductor') is that materials like metals are conductors and materials like rubber are insulators. We want to make precise what this word means in terms of electric fields and charge distributions. To do this, we will idealize a conductor as an object in which charges are free to move apart from each other if there is an electric field acting on them, and they are free to move anywhere inside the body of the material but cannot leave the surface. If a conductor is said to be 'neutral' or 'no net charge', then it contains an equal number of positive and negative charges which can be separated from each other if an external electric field is applied. So the question is, how do the charges in a conductor align themselves if an external electric field is applied?

If we applied a field $\vec{E} = E_0\hat{\mathbf{i}}$ with $E_0 > 0$, then we anticipate that positive charges in the conductor will move in the $+\hat{\mathbf{i}}$ direction while negative charges will move in the $-\hat{\mathbf{i}}$ direction, and they will keep doing so until they reach the surface of the conductor. Now what? Well, if we looked at what field is produced inside the conductory *by* the separated charges in the conductor, we can guess that it points in the $-\hat{\mathbf{i}}$ direction. Now, once the charges reach equilibrium, which experimentally happens very fast, there is no net force acting on any of the charges, except potentially outward at the surface (since the charges are assumed not to leave the surface). So the field inside must *exactly cancel* the external field so that $\vec{E} = 0$ everywhere inside a conductor. At the surface of a conductor though, again, $\vec{E}$ may point perpendicular to the surface. But there can be no component of $\vec{E}$ tangential to the surface. Why?

Example 3.19 (Spherical conductor with spherical cavity containing point charge.). A neutral, solid spherical conductor of radius R has a spherical cavity of radius $r < R$ at its center. At the center is a point charge q. What is $\vec{E}$ everywhere? Identify where all charges are distributed and how much charge is distributed at each surface. How do the answers change if the spherical conductor has a net charge q' or $-q'$?

If a conductor has a net charge, you can first figure out where the charges would go if the conductor were neutral. Then, since you have constructed a neutral conductor with $\vec{E} = 0$ inside, the remaining net charge, whether it is positive or negative, must be plastered on the surface.

Example 3.20 (Spherical conductor with cavity of arbitrary shape containing point charge.). A neutral, solid spherical conductor of radius R has a cavity of arbitrary shape, off-center of the conductor. In the cavity, there is a point charge q. What is $\vec{E}$ outside the conductor?

If you carefully think through the Ex. 3.20, you find that $\vec{E}$ outside the conductor is the same as though you had a hollow shell of with total charge q plastered on it, the same as though you had a solid sphere of total charge q spread throughout it, the same as though you had just a point charge of value q at the origin. So in other words, outside the conductor, the only information you can ever know is the total charge in the conductor – that is, you can only know the sum of the net charge of the conductor and any point charges that may reside in cavities in the conductor. But you cannot know how much of each is present. Weird, right?

3.8 Energy

So far we have discussed forces, electric fields, and charge distributions. But we know that there is another concept related to forces–work and energy. If electric fields exert forces, then it is possible that they do work on a charge. So first we recall what work is:

$$W = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = \Delta\text{KE} \tag{45}$$

The work done *on* an object is equal to its change in kinetic energy $(1/2)mv^2$. This means that if the object has negative work, it loses kinetic energy and transfers it to its surroundings.

Example 3.21 (Strength of the electric field.)**.** A uniform electric field is applied in some direction. A charge of value q is launched at velocity v_0 in that direction and travels a distance d, at which point it has a velocity $v_1 < v_0$. What must the direction of the force on q be relative to the direction of the electric field? Find E_0.

If we consider the electric field as part of the entire system, and we know that closed systems must conserve energy, then we realize that if a particle gains or loses kinetic energy, it must be converted to some other kind of energy. You might recall that this is called potential energy (U).

Example 3.22 (Tennis ball falls from roof.)**.** A tennis ball falls off a roof at height h. What is its speed as it its the ground?

So we recognize that when work is done on a particle, the potential energy must change by the opposite amount so that $\Delta\text{KE} + \Delta U = 0$. So,

$$\Delta U = -\Delta\text{KE} = -W \tag{46}$$

However, we see that since W depends on the *force* applied on a test particle over a distance, and the electric force $\vec{F} = q\vec{E}$ is proportional to the charge of the test particle, then W and ΔU must both depend on the charge of the test particle as well. So, just as we had introduced $\vec{E}$ as the *force per unit test charge* we want to introduce an electric potential function ΔV as the potential energy per unit test charge:

$$\Delta V = \frac{\Delta U}{q} \tag{47}$$

(recall the similar form for $\vec{E}$:

$$\vec{E} = \frac{\vec{F}}{q}$$

Now let's see if we can write down ΔV in terms of $\mathbf{E}$. Recall that the work done on a particle over some path P_{12} from $\mathbf{r}_1$ to $\mathbf{r}_2$ by a force $\mathbf{F}$ is given by

$$W = \int_{P_{12}} \mathbf{F} \cdot d\vec{\ell} \tag{48}$$

where $d\vec{ell}$ is an infinitesimal vector along the path P_{12}. This is just the generalized, integral version of the dot product $\mathbf{F} \cdot \mathbf{d}$ for cases where $\mathbf{F}$ is not necessarily constant over the path. Since $\Delta U = -W$:

$$\Delta U = -\int_{P_{12}} \mathbf{F} \cdot d\vec{\ell} \tag{49}$$

and for electric fields, the change in potential energy due to the field acting on a particle of charge q is

$$\Delta U = -\int_{P_{12}} q\mathbf{E} \cdot d\vec{\ell} \tag{50}$$

so this means

$$\Delta V = -\int_{P_{12}} \mathbf{E} \cdot d\vec{\ell} \tag{51}$$

Now, it turns out that $\mathbf{E}$ is special. If I want to know what the potential difference ΔV is between points $\mathbf{r}_1$ and $\mathbf{r}_2$, *it does not matter what path I take between them.* So a convenient way to find ΔV's between points $\mathbf{r}_1$ and $\mathbf{r}_2$ is to define a function $f(\mathbf{r})$ which is zero at a special point $\mathbf{r}_0$ so that ΔV between r_0 and r_1 is $f(\mathbf{r}_1)$ and ΔV between r_0 and r_2 is $f(\mathbf{r}_2)$. Now if we want to find ΔV from r_1 to r_2, we can take any path between them, so we could add r_0 in the middle of the path. So ΔV from $\mathbf{r}_1$ to $\mathbf{r}_2$ is ΔV from $\mathbf{r}_1$ to $\mathbf{r}_0$ + ΔV from $\mathbf{r}_0$ to $\mathbf{r}_2$, which in terms of $f(\mathbf{r})$ is $f(\mathbf{r}_2) - f(\mathbf{r}_1)$. We can usually choose any point in space to be $\mathbf{r}_0$ by appropriately adjusting the form of f. When we speak of the electric potential function V from now on, we really mean ΔV relative to some point $\mathbf{r}_0$, which is precisely what $f(\mathbf{r})$ is.

Example 3.23 (Potential of a point charge.)**.** Find a potential function of a point charge of value q at the origin.

The last thing we need to do to understand how to use $V(\mathbf{r})$ is to figure out how we can get $\mathbf{E}$ if we already know $V(\mathbf{r})$. To get the components of $\mathbf{E}$, we need to invert Eq. 51:

$$E_x = -\frac{\partial}{\partial x}V(x,y,z) \tag{52}$$

$$E_y = -\frac{\partial}{\partial y}V(x,y,z)$$

$$E_z = -\frac{\partial}{\partial z}V(x,y,z)$$

Now we make a very important observation. Suppose we have a potential function $V(x,y,z)$ whose derivatives give us an electric field $\mathbf{E}$ by Eqs. 52. If we now choose a different potential $V'(x,y,z) = V(x,y,z) + C$ where C is an arbitrary constant, does $\mathbf{E}$ change?

The fact that it does not change is significant. There are an infinite number of equally good potential functions which all produce the same electric field, and they all differ from each other by just a constant. This is OK because the *electric field* and ΔV can be measured, but $V(x,y,z)$ *cannot*:

Quantity	Measurable
Potential ($V(x,y,z)$)	No
Differences in potential ($\Delta V = V(x_1,y_1,z_1) - V(x_0,y_0,z_0)$)	Yes
Derivative of potential ($\mathbf{E}$)	Yes

Convince yourself that ΔV between two given points will not change if the value of V at each point is shifted by the same, arbitrary constant.

If we have a collection of sources charges and want to find $V(x,y,z)$ for all of them, then we can use the superposition principle to find it by finding potentials for each source independently and adding them. But this only works if they are all zero at the same point (that is, the reference point of the potential is the same for all of them).

Example 3.24 (Potential of four charges.)**.** Four charges, each of value q, are placed on the vertices of a square of side length L. What is V at the center? Four charges are now placed on the circumference of a circle of radius $R = L/\sqrt{2}$ separated by equal distances. What is V at the center? The four charges are now pushed all to one side of the circle. What is V at the center?

When you have a finite amount of source charge that creates the electric field and electric potential, then you can safely take $\mathbf{r} = \infty$ to be the reference point of your potential function V (the point at which your function will be zero). This is because you know that at $\mathbf{r} = \infty$, $\mathbf{E} = 0$. But we have already seen some examples where an infinite amount of charge creates a nonzero amount of field even at infinite distance from it (see the infinitely long, uniformly charged wire and the infinitely wide/long, uniformly charged plane). In these cases, you cannot take V to go to zero at infinite distance. To see this, think about the line integral in Eq. 51 from any finite point $\mathbf{r}_1$ out to some point at ∞. Since $\mathbf{E}$ does not go to zero, the line integral must also go to ∞. You cannot even correct for this by adding an arbitrary constant because your arbitrary constant would have to be $\pm\infty$. We now turn to do a fairly challenging problem which illustrates all of these points.

Example 3.25 (Infinite conducting slab in uniform electric field.)**.** A conducting slab of thickness d and infinite length and width is placed perpendicular to a uniform electric field (say, $\mathbf{E} = E_0\hat{\mathbf{i}}$). Find $\mathbf{E}$ everywhere at equilibrium, find the surface charge density on the surfaces of the slab, and find a continuous potential function V for this electric field.

If the field points along the x-axis, then per this description, the slab is parallel ot the yz-plane. Let us take the left-side plane of the slab to be situated at $x = 0$ and the right-side plane to be situated at $x = d$. We know that since the slab is a conductor, $\mathbf{E} = 0$ inside it. This can only be achieved if the charges in the slab separate to create a field of equal magnitude and opposite direction of the applied field $E_0\hat{\mathbf{i}}$. From our solutions to the charged infinite plane (see Ex. 3.16), we know that if the leftside plane acquires a surface charge density of σ, then it will create a field $(\sigma/2\varepsilon_0)\hat{\mathbf{i}}$ in the region $0 < x < d$. If σ appears on the leftside plane, then $-\sigma$ must appear on the rightside plane to keep the conductor neutral. The rightside plane then generates a field $(\sigma/2\varepsilon_0)\hat{\mathbf{i}}$ in the region $0 < x < d$ as well. So the total field in the body of the slab is

$$\mathbf{E}_{\text{slab}} = E_0\hat{\mathbf{i}} + \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{i}} + \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{i}} = 0 \tag{53}$$

$$\sigma = -\varepsilon_0 E_0 \tag{54}$$

Now let us find the total field left of the slab ($x < 0$). Again, using our previous solutions to the infinite charged plane (see Ex. 3.16), the leftside plane contributes $-(\sigma/2\varepsilon_0)\hat{\mathbf{i}}$ while the rightside plane contributes $(\sigma/2\varepsilon_0)\hat{\mathbf{i}}$. So in $x < 0$:

$$\mathbf{E} = E_0\hat{\mathbf{i}} - \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{i}} + \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{i}} = E_0\hat{\mathbf{i}} \tag{55}$$

If you reason through the argument for $x > d$, you find that the contributions from the two planes again cancel, so for $x > d$:

$$\mathbf{E} = E_0\hat{\mathbf{i}} \tag{56}$$

Now we need to find a potential function. We need to be careful because $\mathbf{E}$ does not go to zero as we go out to infinity. Instead, it stays at a constant $\mathbf{E} = E_0\hat{\mathbf{i}}$. So we must try to be a bit more clever. Let us use the Eqs. 52. Let's start with the region inside the slab. We know that $\mathbf{E} = 0$ here, so Eqs. 52 tell us that the partial derivatives of V must be zero here. What function's derivative with respect to all three variables is zero? A constant. So we know

$$V_{\text{slab}} = c_{0,\text{slab}} \tag{57}$$

for some constant c_1 here. Next let us do the region $x < 0$. The field here is $\mathbf{E} = E_0\hat{\mathbf{i}} = -(\partial/\partial x)V\hat{\mathbf{i}}$. So we want a function whose partial derivative with respect to x is a nonzero constant E_0–something like

$$V_{\text{left}} = E_0 x + c_{0,\text{left}} \tag{58}$$

Since $E_y = E_z = 0$, this just says that V must not depend on y or z. Lastly, let's write down the form of V for $x > d$. Here $\mathbf{E} = E_0\hat{\mathbf{i}}$ again. So the solution should look the same,

$$V_{\text{right}}(x) = E_0 x + c_{0,\text{right}} \tag{59}$$

Now, $V(x, y, z)$ needs to be a continuous function. What does that condition mean? It means that $V_{\text{left}}(x = 0) = V_{\text{slab}}(x = 0)$ and $V_{\text{slab}}(x = d) = V_{\text{right}}(x = d)$. We need to solve for the three remaining constants using these two conditions. However, this means we have more variables than constraints–so we can choose to set one of them to zero (or any other value) if we want. However, we cannot set all of them to zero, because otherwise $V_{\text{slab}}(x = d) = 0$ while $V_{\text{right}}(x = d) = E_0 d$. Let us choose $c_{0,\text{slab}} = 0$. Now let us solve for $c_{0,\text{left}}$:

$$V_{\text{left}}(x = 0) = E_0 \cdot 0 + c_{0,\text{left}} = 0$$

$$V_{\text{slab}}(x=0) = 0$$

so

$$c_{0,\text{left}} = 0$$

and

$$V_{\text{left}}(x) = E_0 x \tag{60}$$

Now let us solve for $c_{0,\text{right}}$. Again, evaluating at $x = d$:

$$V_{\text{slab}}(x=d) = 0$$

$$V_{\text{right}}(x=d) = E_0 \cdot d + c_{0,\text{right}}$$

$$E_0 d + c_{0,\text{right}} = 0$$

so

$$c_{0,\text{right}} = -E_0 d$$

and

$$V_{\text{right}} = E_0 x - E_0 d \tag{61}$$

This completes our solution. The zero of potential is everywhere inside the slab, rather than at $x = \infty$. However, we can move it to any other finite point. Suppose we added another arbitrary, finite constant c_0' to all three parts of the potential. Then the zeros of potential are now at $x = -c_0'/E_0$ and $x = (E_0 d - c_0')/E_0$ (note that there are two zeros here because the original setup has a symmetry about $x = d/2$. Since we added the same constant to all three parts, V is still continuous, and since we added a constant, $\mathbf{E}$ is unchanged and Eqs. 52 are still satisfied.

3.9 Tips for Solving Problems involving E and V

When you have a system of multiple/continuous charge distributions, you can, in principle, calculate the electric field using Eq. 31. This method always works when there is a localized amount of charge, such as a finite number of point charges or a finite region of continuous charge. However, in many cases, if there is a symmetry to the problem, it is often much more efficient to exploit Gauss's law. If symmetry is not present, it is generally difficult if not impossible to derive the electric field at all points in space using Gauss's law. On the other hand, if the charge is not localized, such as an infinitely long line of uniform charge, then it is expedient to use Gauss's law or some other clever trick that exploits a translational, spherical, or cylindrical symmetry (these are simply the most common types). When it comes to finding a suitable potential function V, remember that V must be continuous everywhere (though $\mathbf{E}$ need not be) and when an infinite amount of charge is present, it is not possible to set the reference point or zero point of your potential to be $\mathbf{r} = \infty$.

3.10 Conductors and Equipotentials

In Ex. 3.25 we found that V was constant inside of the conducting slab no matter where we set the reference point of our potential (no matter what overall constants we added to V). We call such regions of space *equipotentials*. The fundamental (physical) reason that the conductor became an equipotential just goes back to the point that $\mathbf{E} = 0$ in its body, so the line integral for ΔV (Eq. 51) between any two points inside the conductor must be zero, meaning there is no change in V. The main message on that point, then, is that *conductors, by virtue of always cancelling the electric fields inside them, are always equipotentials.*

Thinking of potentials in terms of equipotentials has another intuitive meaning–it tells us which way test charges will move when placed in those potentials. Since $\mathbf{E}$ is zero along an equipotential, $\mathbf{F} = q_{\text{test}}\mathbf{E} = 0$, so the charge does not move along an equipotential. Instead, it moves either up or down the potential, depending on the sign of the test charge. Positive charges move from regions of high potential to low potential. To get the right intuition, picture a ball of mass m sitting atop a hill of height h. It has a potential energy

mgh, and if perturbed from the top, it will roll down to some place with lower potential energy. Positive charge behave similiarly. Negative charges behave oppositely. If masses could be negative, then objects with negative mass would move from regions of low potential to regions of high potential.

Following the hill analogy, equipotentials are then regions of the same height. Picture a topographic map. The lines drawn indicate points in space with the same elevation so that if the ball were placed at any of the points on the same line, it would have the same potential energy. This is the intuition behind the equipotential.

Example 3.26 (Two charged spheres.)**.** Two conducting spheres of radius R and radius $2R$ are held with their centers a distance d apart. The sphere of radius R has charge $2Q$ on it. The sphere of radius $2R$ has charge $-Q$ on it. The two spheres are now connected by a (conducting) wire. How much charge flows from one sphere to the other and in which direction?

3.11 Capacitors

So far we have assembled all of the fundamental components of the theory of electrostatics. One application is a type of deviced known as a capacitor. Broadly it can be thought of as a device which can store energy in it in the form of an electric field. It is usually composed of two pieces of material which are initially electrically neutral, but (by some process, described later) acquire extra positive charge on one surface and extra negative charge on the other surface. This creates a potential difference between the two surfaces. The ratio of this charge Q to the potential it creates V is called the capacitance C:

$$C = \frac{Q}{V} \tag{62}$$

Note that V implicitly depends on Q: the more charge I add, the bigger a potential I create usually. Capacitance is defined uniquely for each configuration of surfaces that are used to construct the capacitor. In lecture, you constructed a capacitor of two parallel plates of area A and separated a distance d, and by approximating them as being infinitely large, you found that that the electric field between them as

$$\mathbf{E} = \frac{Q}{2\varepsilon_0 A} + \frac{Q}{2\varepsilon_0 A}\hat{\mathbf{k}} \tag{63}$$

(one term from each plate) and the potential difference created for two parallel plates was

$$V = \frac{Q}{\varepsilon_0 A} d \tag{64}$$

so

$$C = \frac{Q}{V} = \frac{\varepsilon_0 A}{d} \tag{65}$$

This capacitance is true *only for the parallel plate* setup and depends on *only on its geometries–area, distance.* If I make any alterations to the setup, the capacitance will depend on the new setup's geometric parameters– so you must rederive the potential difference for a hypothetical charge Q placed on one of the new surfaces and $-Q$ placed on the other. Note that this potential difference depends on Q, which is why when we do $C = Q/V$, the Q's cancel and C does not depends on Q.

Example 3.27 (Spherical capacitor.)**.** A hollow sphere (spherical shell, if you would like) of radius a is surrounded by another hollow sphere of radius $b > a$ so that the centers of the two spheres coincide. What is the capacitance of this setup? What changes if the sphere of radius a is solid instead of hollow?

One thing that can be used to raise the capacitance of a device is to replace the vaccum (really air) between the two surfaces with some other substance, like a gel. These substances *decrease* the strength of the electric field created by placing Q on one of the surfaces and $-Q$ on the other, and *decrease* the potential difference V that is created. So since $C = Q/V$, the capacitance *increases.* The reason the $\mathbf{E}$ is weakened in these materials is that they are made up of atoms which are electrically neutral, but in the presence of an external field create a small field of their own in the opposite direction. Superimposing the two fields gives an electric field which is the same as the external field but smaller in magnitude. All of this behavior is encoded in an experimental constant specific to each material κ, the material's "dielectric constant" which is a multiplier on ϵ_0 so that when evaluating the field in a dielectric instead of in a vaccum, we just need to replace $\varepsilon_0 \to \kappa\varepsilon_0$ That's all that changes. To find the field in a dielectric, you can find the field as though you were in a vaccum and then finally swap ε_0 for $\kappa\varepsilon_0$.

Example 3.28 (Parallel plate with dielectric.)**.** The parallel-plate capacitor mentioned above is now filled with a dielectric material of dielectric constant κ. How does the capacitance change? If we know that the capacitance must increase, what must that mean about the value of κ?

Example 3.29 (Cylindrical capacitor partially filled with dielectric.)**.** A solid, cylindrical conducting rod of length L and radius a is surrounded by a cylindrical shell of length L and radius $b > a$ so that the axes of the rods coincide. The space in between them is filled from the surface of the inner rod out to a radius $(a+b)/2$ with a dielectric and filled with vaccum (air) from a radius $(a+b)/2$ to the outer shell. The dielectric has dielectric constant κ. What is the capacitance of this setup? Note that in order to solve this problem, you will want to find the electric field as though the cylinders are infinitely long (so that you can use Gauss's law) just as you did with the parallel plate, and you will want to be mindful of which regions of space have electric fields containing ε_0 vs those with fields containing $\kappa\varepsilon_0$.

Later when we talk about circuits, we will establish rules for so-called "series" and "parallel" arrangements of capacitors in circuits.

4 Current and DC Circuits

5 Capacitors in Circuits

One setup of practical significance is to charge up and discharge a capacitor by placing it in a circuit with a battery of known voltage and a resistor. The result of applying the "loop rule" (the sum of the voltage changes around a closed loop must equal zero) is a first order differential equation in Q, the charge on the capacitor. To see this, consider a circuit with a battery, capacitor, and resistor all in series. Looking at the direction of the current, we can see that since positive charge will accumulate on the side of the capacitor closest to the battery, as it charges up a "force" *opposite* the direction of the battery's will be created. The magnitude will be given by $V_{\text{cap}} = Q/C$. Next, remember that current I is the charge flowing through a region per unit time, dQ/dt, and this Q must be the same as the Q on the capacitor because charge is conserved. So applying the loop rule:

$$V - \frac{1}{C}Q - \frac{dQ}{dt}R = 0 \tag{66}$$

whose solution is

$$Q(t) = CV(1 - e^{-t/RC}) \tag{67}$$

or more usefully, the voltage across the capacitor can be obtained from that equation (dividing through by C) as:

$$V_{\text{cap}}(t) = V(1 - e^{-t/RC}) \tag{68}$$

Now, if we look at $t = 0$, initially there is no voltage drop across the capacitor because it is uncharged. Since there is no voltage drop, we can regard it as a short-circuit. At this instant in time, the circuit behaves as though it has just a battery connected to a resistor. Now, look at what happens as $t \to \infty$. $V_{\text{cap}} \to V$, the capacitor's voltage increases until it reaches the battery's voltage. So this must mean that it pushes back against the battery with equal magnitude. The current therefore must be zero. Let us explicitly evaluate this by taking the time derivative of the charge equation:

$$I(t) = \frac{d}{dt}Q(t) = \frac{V}{R}e^{-t/RC} \tag{69}$$

Looking again at the $t = 0$ and $t \to \infty$ limits, this result confirms our intuition. In the beginning, the capacitor acts like a short circuit. At long times, it acts like an open circuit. This completes the analysis for charging up a capacitor. What about discharging? Suppose we now short out the battery so that circuit consists of the capacitor charged up to voltage V and the resistor. Our loop equation now yields:

$$\frac{1}{C}Q - \frac{dQ}{dt}R = 0 \tag{70}$$

Why did the sign on the first term change? Because the current now flows the *opposite* direction that it flowed before with the battery connected! This equation has solution

$$Q(t) = CVe^{-t/RC} \tag{71}$$

The quantity in RC in these equations is called the time constant and represents how long it takes to lose $1/e$ of the starting value (charge, voltage, current).

6 Midterm Exam 1 Practice

Example 6.1 (Charged hollow sphere.)**.** A hollow sphere of radius R has charge Q uniformly spread over it. A point charge of mass m and charge q such that q and Q have the same sign is released from infinitely far away with initial velocity v_0 toward the center of the hollow sphere. What is the minimum initial velocity v_{min} that the particle needs to reach the surface of the hollow sphere? If we now drill a tiny hole at the point where the particle would reach the sphere, and then drill a tiny hole on the exact opposite side of the sphere, what happens to the particle if it starts with initial velocity $v_0 > v_{\text{min}}$? Why? Is it possible to give the particle an initial velocity such that it stops at any given point inside the sphere? Sketch the potential energy of the system as a function of where the particle is along this trajectory. Find a potential function $V(r)$ for this hollow sphere. Finally, repeat this analysis for the case of Q spread uniformly through a solid sphere of radius R.

Example 6.2 (Block of metal.)**.** A piece of metal of dimensions LxLxW such that $L \gg W$ is placed in a uniform electric field $\mathbf{E} = E_0\hat{\mathbf{i}}$ such that the field points along the dimension of length W. The piece is then cut down the middle, along the LxL plane, perpendicular to the dimension of length W. How much charge is on each piece?

Example 6.3 (Capacitors charging each other.)**.** Two capacitors are connected to each other in a closed circuit. Capacitor 1 with capacitance C_1 has already been charged up while capacitor 2 with capacitance C_2 has no charge on it initially. What is the ratio of the charge on capacitor 2 after a long time to the charge on capacitor 1 initially (that is, what is $Q_2(t = \infty)/Q_1(t = 0)$)?

Example 6.4 (Removing dielectric from capacitor.)**.** A parallel plate capacitor with dielectric of dielectric constant $\kappa > 1$ and capacitance (with the dielectric) C is connected to a battery of voltage V. What happens to V_{cap}, Q_{cap}, and U_{cap} if the dielectric is removed *while the battery is connected*? What about if the dielectric is removed *while the battery is disconnected*? How much work is required to remove the dielectric? What do you feel if you are try to pull the dielectric out from inside the capacitor? What is happening microscopically inside the dielectric material?

Example 6.5 (Mercury Drops (Chien-I).)**.** Consider two mercury spherical drops with the same size, each carries potential $V1$ and $V2$ respectively. If now the two drops are merged, what it the electric potential of the new drop?
