Physics 7B Problems
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1 Problems

See accompanying PDFs for figures to show setup for each problem.

1.1 Continuous Charge

Problem 1

A straight wire of length L carries a uniform linear charge density λ. One end of the wire is a distance d away from a point charge Q. What is the net force of the wire on the point charge?

Solution

To set this up, we first set a coordinate system. Let x_1 denote a position along the wire and $x_1 = 0$ denote the left side of the wire. Next, we identify the force on the point charge Q due to an infinitesimal chunk of charge dq at a location x_1:

$$dF = k\frac{Qdq}{(L - x_1 + d)^2} \tag{1}$$

To do this integral, we need to re-express dq as λdx_1:

$$F = \int dF = \int_0^L k\frac{Q\lambda}{(L - x_1 + d)^2}dx_1 = kQ\lambda \frac{1}{L - x_1 + d}\Bigg|_{x_1=0}^{x_1=L} = kQ\lambda\left(\frac{1}{d} - \frac{1}{L + d}\right) \tag{2}$$

Let's check our units: λ has units of charge/length, and it is multiplied by a charge and divided by another length, so we get k times a charge^2/length^2 as expected for electric force. Let us also introduce the appropriate vector notation and check that the sign of our answer makes sense when $Q > 0$ and $\lambda > 0$. We are expecting that in this case, the force should be some positive number times $\hat{\mathbf{i}}$ (since that means the line and the point charges have the same sign, they should repel).

$$\boxed{\vec{\mathbf{F}} = kQ\lambda\left(\frac{1}{d} - \frac{1}{L + d}\right)\hat{\mathbf{i}}} \tag{3}$$

Since $1/d > 1/(d + L)$, we indeed get the correct sign for the force.

Problem 2

Now imagine that *two* wires of length L each carry a uniform linear charge density λ. The two wires are laid as in the accompanying figure. What is the net electric force of wire 1 on wire 2?

Solution

This time we will need to first find the force on an infinitesimal chunk of charge dq_2 at a position x_2 on wire 2 due to an infinitesimal chunk of charge dq_1 at x_1. This will be

$$dF = k\frac{dq_1 dq_2}{(x_2 - x_1)^2} \tag{4}$$

The total force of wire 1 on the chunk of charge dq_2 will follow from the first problem, but with $d \to x_2 - L$:

$$dF = kdq_2\lambda\left(\frac{1}{x_2 - L} - \frac{1}{x_2}\right) \tag{5}$$

Next, $dq_2 = \lambda dx_2$ as before, and the integration limits of x_2 correspond to the endpoints of wire 2, $(L + d, 2L + d)$:

$$F = \int_{L+d}^{2L+d} k\lambda^2 \left(\frac{1}{x_2 - L} - \frac{1}{x_2}\right) dx_2 \tag{6}$$

$$= k\lambda^2 \ln(x_2 - L)\Big|_{x_2=L+d}^{x_2=2L+d} - \ln(x_2)\Big|_{x_2=L+d}^{x_2=2L+d} \tag{7}$$

$$= k\lambda^2 \ln\left(\frac{(L+d)^2}{d(2L+d)}\right) \tag{8}$$

Let us again check the sign of the answer. It should be positive since both wires carry the same charge density in this problem. For this to be true, we need to check that the argument of ln is larger than 1:

$$\frac{(L+d)^2}{d(2L+d)} = \frac{L^2 + 2Ld + d^2}{2Ld + d^2} = \frac{L^2}{2Ld + d^2} + 1 > 1 \tag{9}$$

So the answer has the correct sign:

$$\boxed{\vec{\mathbf{F}} = k\lambda^2 \ln\left(\frac{(L+d)^2}{d(2L+d)}\right)\hat{\mathbf{i}}} \tag{10}$$

Problem 3

A straight wire carrying uniform linear charge density λ lies along the x-axis. A point charge Q is placed somewhere at a distance d away from the end of the wire along the x-axis and at a height y above the x-axis. Find the net electric force of wire 1 on wire 2.

Solution

This problem is a generalization of the first. Now the force of a chunk of charge on wire 1 on a chunk of charge on wire 2 will have separate $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ components. **It is very important that we solve these two components separately. The magnitudes cannot simply be added up.**Recall that for a vector in a 2D plane, $\vec{\mathbf{v}} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}}$ and $v_x = |\vec{v}|\cos(\theta)$ and $v_y = |\vec{v}|\sin(\theta)$. First, we do the x-component

$$dF_x = k\frac{Qdq_1}{(L - x_1 + d)^2 + y^2}\frac{(L - x_1 + d)}{\sqrt{(L - x_1 + d)^2 + y^2}} \tag{11}$$

Substituting $dq_1 = \lambda dx_1$:

$$F_x = kQ\lambda \int_0^L \frac{(L - x_1 + d)}{((L - x_1 + d)^2 + y^2)^{3/2}} dx_1 \tag{12}$$

$$= kQ\lambda \frac{1}{((L - x_1 + d)^2 + y^2)^{1/2}}\Big|_0^d = kQ\lambda\left(\frac{1}{(d^2 + y^2)^{1/2}} - \frac{1}{((L+d)^2 + y^2)^{1/2}}\right) \tag{13}$$

Note that if $y = 0$, this reduces to the answer we obtained in the first problem. Next, we solve for the y-component:

$$dF_y = k\frac{Qdq_1}{(L-x_1+d)^2+y^2}\frac{y}{\sqrt{(L-x_1+d)^2+y^2}} \tag{14}$$

Substituting $dq_1 = \lambda dx_1$:

$$F_y = kQ\lambda\int_0^L \frac{y}{((L-x_1+d)^2+y^2)^{3/2}}dx_1 \tag{15}$$

$$= kQ\lambda\frac{1}{((L-x_1+d)^2+y^2)^{1/2}}\Bigg|_0^d = kQ\lambda\left(\frac{1}{(d^2+y^2)^{1/2}} - \frac{1}{((L+d)^2+y^2)^{1/2}}\right) \tag{16}$$

We make the substitution $u = L - x_1 + d$ so $du = -dx_1$ and the integration limits become $u = L + d$ and $u = d$:

$$F_y = -kQ\lambda y\int_{L+d}^d \frac{1}{(u^2+y^2)^{3/2}}du \tag{17}$$

$$= -kQ\lambda\frac{1}{y^2}\int_{L+d}^d \frac{1}{((u/y)^2+1)^{3/2}}du \tag{18}$$

Making another substitution $v = u/y$:

$$F_y = -kQ\lambda\frac{1}{y}\int_{(L+d)/y}^{d/y} \frac{1}{(v^2+1)^{3/2}}dv \tag{19}$$

$$= -kQ\lambda\frac{1}{y}\frac{v}{\sqrt{v^2+1}}\Bigg|_{v=(L+d)/y}^{v=d/y} \tag{20}$$

$$= -kQ\lambda\left(\frac{d}{y^2\sqrt{d^2/y^2+1}} - \frac{L+d}{y^2\sqrt{(L+d)^2/y^2+1}}\right) \tag{21}$$

$$= -kQ\lambda\left(\frac{d}{y\sqrt{d^2+y^2}} - \frac{L+d}{y\sqrt{(L+d)^2+y^2}}\right) \tag{22}$$

Note that $\lim_{y\to 0} F_y = 0$ as expected. So the total force on charge Q is:

$$\vec{\mathbf{F}} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}} \tag{23}$$

$$= \boxed{kQ\lambda\left(\frac{1}{(d^2+y^2)^{1/2}} - \frac{1}{((L+d)^2+y^2)^{1/2}}\right)\hat{\mathbf{i}} - kQ\lambda\left(\frac{d}{y\sqrt{d^2+y^2}} - \frac{L+d}{y\sqrt{(L+d)^2+y^2}}\right)\hat{\mathbf{j}}} \tag{24}$$

1.2 Electric Flux and Gauss's Law

Problem 4

A solid sphere of radius R carries uniform charge density (charge per unit volume) ρ. Find the electric field as a function of the distance r from the center of the sphere.

Solution

Remember that whenever the charge distribution changes discontinously across a region, such as at the boundary of the sphere, you need to stop and solve the problem in the two regions separately, i.e. one solution for $r < R$ and one for $r > R$. First we do $r < R$:

First, we determine the direction of the field. By symmetry, the field must point radially outward. Next we determine along what surfaces the magnitude of the field would be constant. Those would be spherical surfaces centered on the center of the sphere. Remember, the goal is to pick an imaginary (Gaussian) surface such that the field and the normal vector to the imaginary surface are either aligned or perpendicular along the surface. In this case, we can choose our imaginary surface S to be a sphere of radius r centered on the center of the sphere. This means that the electric flux simplifies as follows:

$$\Phi = \int_S \vec{E} \cdot d\vec{A} = \int_S |E| d|A| = |E| \int_S d|A| = |E|(4\pi r^2) \tag{25}$$

Next, we find another expression for Φ using Gauss's law. How much charge does our surface S enclose? $q_{\text{enc}} = \rho 4\pi r^3/3$. So

$$\Phi = \frac{1}{\varepsilon_0}\frac{4}{3}\rho 4\pi r^3 \tag{26}$$

Setting these two values for the flux equal to each other:

$$|E|(4\pi r^2) = \frac{1}{\varepsilon_0}\frac{4}{3}\rho 4\pi r^3 \tag{27}$$

We get

$$\boxed{\vec{\mathbf{E}} = \frac{\rho r}{3\varepsilon_0}\hat{\mathbf{r}} \quad (\text{if } r < R)} \tag{28}$$

If we repeat this exercise for $r > R$, all the geometric steps and arguments remain the same. But the charge enclosed is constant: $q_{\text{enc}} = \rho(4\pi R^3/3)$, so the last step becomes

$$|E|(4\pi r^2) = \frac{1}{\varepsilon_0}\frac{4}{3}\rho 4\pi R^3 \tag{29}$$

$$\boxed{\vec{\mathbf{E}} = \frac{\rho R^3}{3r^2\varepsilon_0}\hat{\mathbf{r}} \quad (\text{if } r > R)} \tag{30}$$

Problem 5

Now imagine a spherical shell of radius R carrying total charge Q spread uniformly on its surface. There is no charge inside the shell. What is the electric field inside and outside?

Solution

Following the same symmetry arguments as before, all of the geometric steps remain the same. But the charge enclosed for $r < R$ will be zero. So we will have

$$|E|(4\pi r^2) = 0 \tag{31}$$

and so

$$\boxed{\vec{\mathbf{E}} = 0 \quad (\text{if } r < R)} \tag{32}$$

while for points outside the sphere ($r > R$), the total charge enclosed will be the total charge of the spherical shell, Q:

$$|E|(4\pi r^2) = \frac{1}{\varepsilon_0}Q \tag{33}$$

So

$$\boxed{\vec{\mathbf{E}} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} \quad (\text{if } r > R)} \tag{34}$$

Problem 6

A solid sphere of radius R carries uniform charge density (charge per unit volume) ρ. However, now a spherical cavity of radius $R/2$ is carved out as shown below. What is the electric field at all points on the positive x axis?

Solution

Before you can jump into applying Gauss's law, you must always analyze the symmetry and determine what the direction of the field is and what imaginary surface would have a constant magnitude of field and a constant angle between the field and the normal vector of the surface. The only symmetry we have here is rotation about the x axis. It is very hard to imagine what the direction of the field would be and what the constant magnitude surfaces would be. There is an easier way to deal with this problem. We can view the**absence** of charge as the **superposition** of positive and negative charge densities. In other words, imagine that the "cavity" is replaced by a superposition of solid spheres of radius $R/2$ carrying charge ρ and $-\rho$. The reason this is useful is because if we take into account the rest of the solid sphere of radius R carrying charge density ρ, then together with our first imaginary solid sphere of radius $R/2$, it now becomes a complete solid sphere of radius R, carrying charge ρ with no cavity. This problem has complete spherical symmetry and indeed we solved it above. Next, we have to deal with the imaginary solid sphere of charge density $-\rho$. But again this has complete spherical symmetry and we know how to use Gauss's law to find its field. So we just need to add the contribution due to both spheres in the regions $x < R$ and $x > R$. There's no real Gauss's law problem to solve: we already did it above.

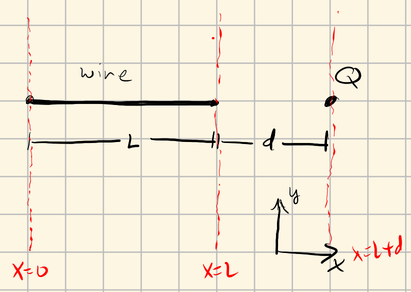
Case $x < R$:

$$\boxed{\vec{\mathbf{E}}(x) = \left(\frac{\rho x}{3\varepsilon_0} - \frac{\rho(x - R/2)}{3\varepsilon_0}\right)\hat{\mathbf{i}} = \frac{\rho R}{6\varepsilon_0}\hat{\mathbf{i}} \quad (\text{if } x < R)} \tag{35}$$

Case $x > R$:

$$\boxed{\vec{\mathbf{E}}(x) = \left(\frac{\rho R^3}{3x^2\varepsilon_0} - \frac{\rho(R/2)^3}{3(x - R/2)^2\varepsilon_0}\right)\hat{\mathbf{i}} = \left(\frac{\rho R^3}{3x^2\varepsilon_0} - \frac{\rho R^3}{24(x - R/2)^2\varepsilon_0}\right)\hat{\mathbf{i}} \quad (\text{if } x > R)} \tag{36}$$

#1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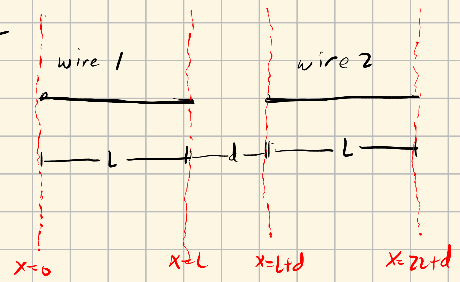
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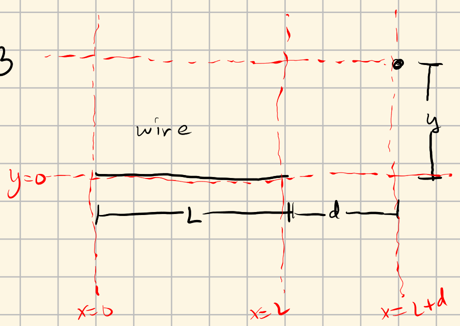
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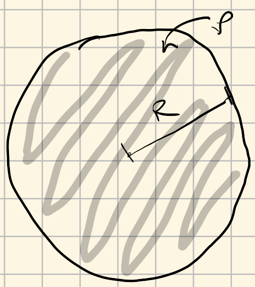

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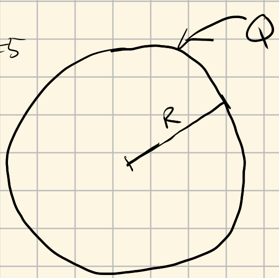

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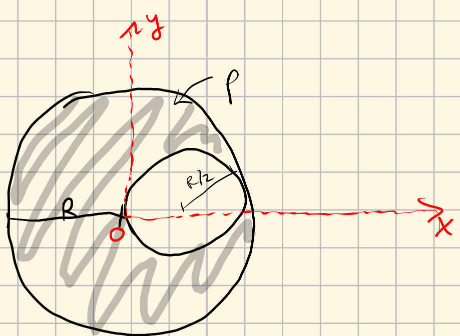

solid