Physics 7B Problems
Week of February 14, 2022

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1 Problems

1.1 Entropy and Efficiency

Problem

N molecules of an ideal gas with adiabatic coefficient γ undergo the following cycle: 1. Isothermal compression at from volume V_1 to $V_2 < V_1$ at temperature T_1 2. Isochoric heating 3. Adiabatic expansion Draw the PV-diagram. Calculate the efficiency. Calculate the change in entropy ΔS_3 for step 3. Calculate the heat added in step 2. Calculate the change in entropy ΔS_2 for step 2.

Solution

First we draw the PV diagram. Process (1) is the isothermal compression, process (2) is the isochoric heating, process (3) is the adiabatic expansion. We also go ahead and label all the information availble at each vertex: The key observation is that we have enough information to the find the pressures at the bottom

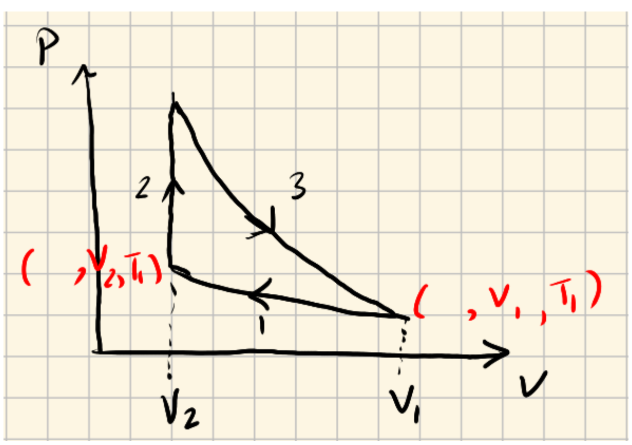

Figure 1: PV diagram for Problem 1

two vertices as well. First, to calculate the efficiency, To find the effiency η, we need to calculate W_{net} and Q_{added}:

$$\eta = \frac{W_{\text{net}}}{Q_{\text{added}}} \tag{1}$$

First, we calculate $W_{\text{net}} = W_1 + W_2 + W_3$. W_2 is zero because it is an isochoric process.

$$W_1 = \int PdV = Nk_BT \int_{V_1} V_2 \frac{1}{V} dV = Nk_BT \ln\left(\frac{V_2}{V_1}\right) \tag{2}$$

Observe that $W_1 < 0$ because the gas is compressed (mathematically, $V_2 < V_1$, so $\ln(V_2/V_1) < 0$. The work in the adiabatic leg is found using the adiabatic relationship and the ideal gas law. The pressure at the beginning of step 1 is:

$$P_{1,i} = \frac{Nk_BT_1}{V_1} \tag{3}$$

In adiabatic process, $PV^\gamma =$ constant, so we can evaluate the constant at this point in the cycle:

$$PV^\gamma = P_{1,i}V_1^\gamma = Nk_BT_1V_1^{\gamma-1} \tag{4}$$

$$W_3 = \int_{V_2}^{V_1} P_{1,i}\frac{V_1^\gamma}{V^\gamma} dV = Nk_BT_1V_1^{\gamma-1}\int_{V_2}^{V_1} V^{-\gamma} dV \tag{5}$$

$$= \frac{1}{1-\gamma} Nk_BT_1V_1^{\gamma-1}V^{1-\gamma}\Big|_{V_2}^{V_1} \tag{6}$$

$$= \frac{1}{\gamma-1}\left(\left(\frac{V_1}{V_2}\right)^{\gamma-1} - 1\right) Nk_BT_1 \tag{7}$$

which is positive as expected because $\gamma > 1$ and $V_1 > V_2$.

$$W_{\text{net}} = W_1 + W_3 = \left(\ln\left(\frac{V_2}{V_1}\right) + \frac{1}{\gamma-1}\left(\left(\frac{V_1}{V_2}\right)^{\gamma-1} - 1\right)\right) Nk_BT_1 \tag{8}$$

For the total heat added, we note that process 3 is adiabatic, so $Q = 0$, and process 1 has $\Delta U = Q_1 - W_1 = 0$, so $Q_1 = W_1$. But we noted earlier that $W_1 < 0$. So $Q_1 < 0$. We only need to calculate Q_2. But actually we don't need to do any more calculation! To see this, look at the total change in internal energy around a cycle:

$$\Delta U = Q_1 - W_1 + Q_2 - W_2 + Q_3 - W_3 = 0 \tag{9}$$

Now, $Q_1 = W_1$ (isothermal compression), $W_2 = 0$ (isochoric) and $Q_3 = 0$ (adiabatic). This makes conservation of energy give us

$$\Delta U = Q_2 - W_3 = 0 \tag{10}$$

So

$$\boxed{Q_2 = W_3} \tag{11}$$

and the efficiency η:

$$\boxed{\eta = \frac{W_{\text{net}}}{Q_{\text{added}}} = 1 + \frac{W_1}{W_3} = 1 - \frac{(\gamma-1)\ln(V_1/V_2)}{(V_1/V_2)^{\gamma-1} - 1}} \tag{12}$$

As expected, the efficiency is less than 1. The change in entropy in step 3 is zero because it is an adiabatic process.

$$\boxed{\Delta S_3 = 0} \tag{13}$$

To find the change in entropy in step 2, note that because S is a state function

$$\Delta S_1 + \Delta S_2 + \Delta S_3 = 0 \tag{14}$$

but we just said that the entropy change in step 3 is zero, so

$$\Delta S_2 = -\Delta S_1 \tag{15}$$

In an isothermal process, $dU = dQ - dW = 0$ so $dQ = dW = PdV$.

$$\Delta S_1 = \int \frac{dQ}{T} = \int_{V_1}^{V_2} \frac{P}{T} dV = \int_{V_1}^{V_2} \frac{Nk_B}{V} dV = Nk_B \ln\left(\frac{V_2}{V_1}\right) \tag{16}$$

Therefore,

$$\boxed{\Delta S_2 = Nk_B \ln\left(\frac{V_1}{V_2}\right)} \tag{17}$$

1.2 Thermal Conduction

Problem

(Taken from Bordel's Sp 18 MT1) Consider a cylindrical chimney of height h, with inner radius R_1 and outer radius R_2. The chimney is made of a homogeneous material whose thermal conductivity is k. The inner and outer surfaces are maintained at constant temperatures T_1 and T_2 ($T_1 > T_2$) respectively, and you may assume that a steady state of heat transfer has been reached within the wall of the chimney. You may ignore any heat loss through the top and bottom caps. Determine the rate of conductive heat flow through the lateral wall and justify its sign. Determine the temperature profile $T(r)$ through the wall.

Solution

First, let us visualize the setup: Since $T_1 > T_2$, heat will flow from the central surface to the outside. We are

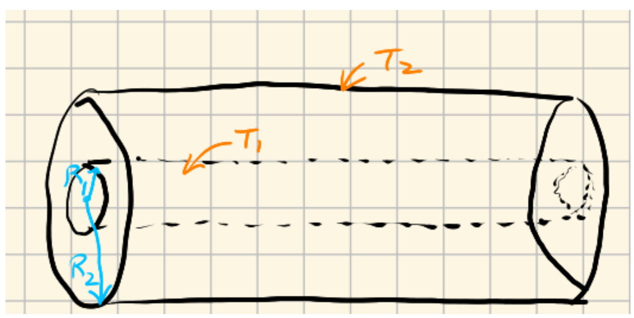

Figure 2: Setup for Problem 1.2

told to ignore heat loss through the end caps. This tells us that the direction of the heat flow should be taken to be purely radial. In that case, we will take our integral over this direction. Note that the cross-sectional area through which the heat flows is no longer constant. It depends on how far we are from the center. Let us parameterize this using the variable r'. (It does not matter what we call it, we are going to integrate over it.). The thermal conduction equation is

$$\frac{dQ}{dt} = -kA\frac{dT}{dx} \tag{18}$$

The variable x here refers to the variable which parameterizes our heat flow path. For us, that is r'. A is the cross-sectional area of the heat flow. That is not constant here, it is $A = 2\pi r' h$. k is the thermal conductivity which is constant here. Now, the goal is to integrate away dT and dr':

$$\frac{dQ}{dt} \int_{R_1}^{R_2} \frac{1}{2\pi r' h} dr' = -k \int_{T_1}^{T_2} dT \tag{19}$$

$$\frac{dQ}{dt} \frac{1}{2\pi h} \ln\frac{R_2}{R_1} = -k(T_2 - T_1) \tag{20}$$

$$\frac{dQ}{dt} = -\frac{2\pi h k(T_2 - T_1)}{\ln(R_2/R_1)} \tag{21}$$

Let's check that the answer makes sense. When $T_1 > T_2$, dQ/dt is positive. This makes sense because we defined our direction of heat flow and integration to be radially outward (instead of radially inward). At steady state, there is a temperature gradient through the wall, $T(r)$. To get this, we want to substitute our expression for dQ/dt into the conduction equation and then integrate r' from R_1 to r and T from T_1 to $T(r)$ similar to before:

$$-\frac{2\pi hk(T_2 - T_1)}{\ln(R_2/R_1)} = -k(2\pi hr')\frac{dT}{dr'} \tag{22}$$

$$\frac{(T_2 - T_1)}{\ln(R_2/R_1)}\int_{R_1}^{r}\frac{dr'}{r'} = \int_{T_1}^{T(r)} dT \tag{23}$$

$$\frac{(T_2 - T_1)\ln(r/R_1)}{\ln(R_2/R_1)} = T(r) - T_1 \tag{24}$$

$$T(r) = T_1 + \frac{(T_2 - T_1)\ln(r/R_1)}{\ln(R_2/R_1)} \tag{25}$$

Let's check that the answer makes sense: at $r = R_2$, we should get $T(r) = T_2$, and we do. At $r = R_1$, we should get $T(r) = T_1$, and we do. If $T_2 = T_1$, we should get $T(r) = T_1 = T_2$, and we do.

1.3 Equilibrium Point between Two Charges

Problem

Two point charges of value Q and $3Q$ are placed a distance d away from each other as in fig. 3. If a third test charge of value Q' is to be placed such that it experiences no net force, where should it be placed? Does the sign or magnitude of the test charge matter?

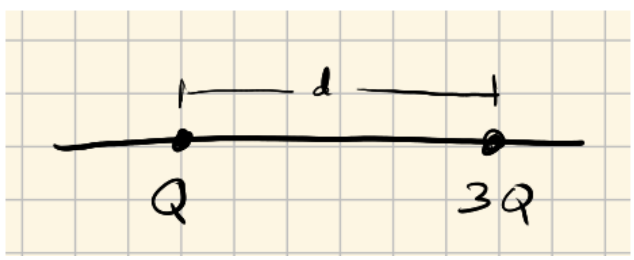

Figure 3: Setup for Problem 1.3

Solution

Let us think about the net force on a charge Q' placed on the axis of Fig. 3 at a distance x to the right of charge Q (and so a distance $d - x$ to the left of charge $3Q$). The force due to each of two charges will be:

$$\vec{F}_Q = k\frac{QQ'}{x^2}\hat{\mathbf{i}} \tag{26}$$

$$\vec{F}_{3Q} = -k\frac{3QQ'}{(d-x)^2}\hat{\mathbf{i}} \tag{27}$$

The net force F_{net} is then:

$$\vec{F}_{\text{net}} = \vec{F}_Q + \vec{F}_{3Q} = k\frac{QQ'}{x^2}\hat{\mathbf{i}} - k\frac{3QQ'}{(d-x)^2}\hat{\mathbf{i}} \tag{28}$$

The condition we are looking for is $\vec{F}_{\text{net}} = 0$. Solving for x:

$$k\frac{QQ'}{x^2}\hat{\mathbf{i}} - k\frac{3QQ'}{(d-x)^2}\hat{\mathbf{i}} = 0 \tag{29}$$

$$\frac{1}{x^2} = \frac{3}{(d-x)^2} \tag{30}$$

$$\pm\sqrt{3}x = d - x \tag{31}$$

$$x = \frac{d}{1 \pm \sqrt{3}} \tag{32}$$

The "-" solution cannot be physical as it places point charge to the left of charge Q. The reason this solution is not valid is that the net force on the test charge Q' is different than Eq. 28 above when Q' is to the left of charge Q. So our equation above does not apply. Instead, the true solution for zero net force is

$$\boxed{x = \frac{d}{1 + \sqrt{3}}} \tag{33}$$

Note that the answer does not depend on the sign or magnitude of Q'. This is an indicator of the utility of thinking in terms of electric fields instead of forces and test charges: we could have ignored the value of the test charge and calculated where the electric field created by the two charges Q and $3Q$ is zero.

1.4 Electric Dipole

Problem

Two point charges with values Q and $-Q$ are separated by a distance d as shown below. Calculate the electric field at any point P along the axis halfway between them as a function of z. Sktech the field lines assuming $Q > 0$.

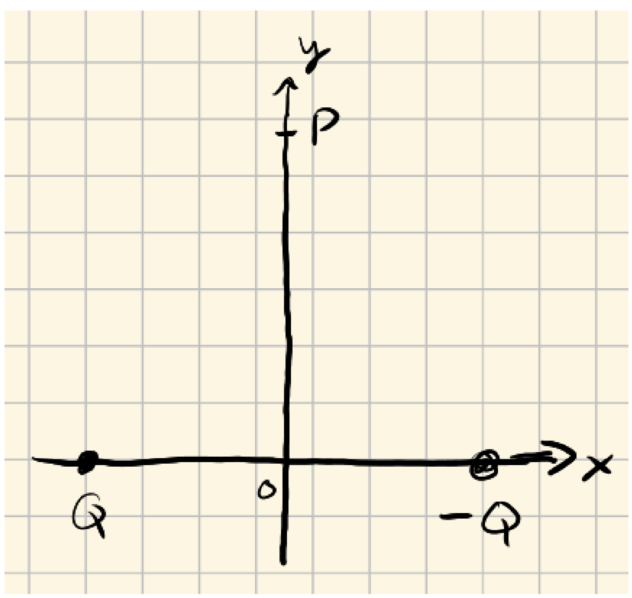

Figure 4: Setup for Problem 1.4

Solution

Since by the principle of superposition, the total electric field at a point P at a height z will be the sum of the electric fields due to the two charges individually, we can first try to visualize what each field will look like at that point. Without loss of generality, let's imagine $Q > 0$. Then the field due to charge Q at point P will point up and to the right. The field due to charge $-Q$ would point down and to the right. The two contributions will have equal magnitude, so we expect the vertical components to cancel and the horizontal components to add. To calculate the electric field, note that the two vectors are not co-linear, so we must

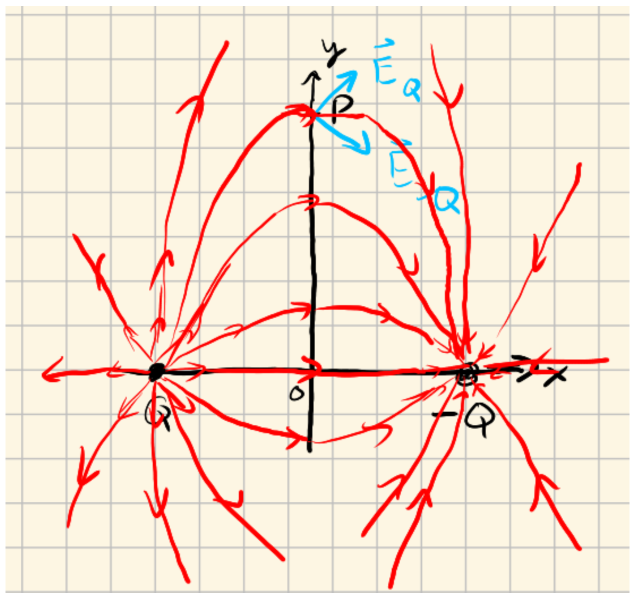

Figure 5: Sketch of electric field of dipole if $Q > 0$. Blue lines indicate the individual electric field contributions at a point P along the y-axis at a height z. Red lines indicate the net electric field.

decompose them into x and y components. Remember that for a vector $\vec{v}$ which makes an angle θ with respect to the x axis, we have

$$\vec{v} = |\vec{v}| \cos\theta \hat{\mathbf{i}} + |\vec{v}| \sin\theta \hat{\mathbf{j}} \tag{34}$$

Let us do this decomposition for each contribution. For the electric field due to charge Q, denoted E_Q, we have

$$|\vec{E}_Q| = k \frac{Q}{(d/2)^2 + z^2} \tag{35}$$

Next,

$$\cos(\theta) = \frac{d/2}{\sqrt{(d/2)^2 + z^2}} \tag{36}$$

$$\sin(\theta) = \frac{z}{\sqrt{(d/2)^2 + z^2}} \tag{37}$$

$$\tag{38}$$

So, decomposing the field due to Q, we get

$$\vec{E}_Q = k \frac{Q}{((d/2)^2 + z^2)^{3/2}} \left(\frac{d}{2} \hat{\mathbf{i}} + z \hat{\mathbf{j}} \right) \tag{39}$$

Now we carry out this analysis for the field due to charge $-Q$: For the electric field due to charge Q, denoted E_Q, we have

$$|\vec{E}_{-Q}| = k\frac{Q}{(d/2)^2 + z^2} \tag{40}$$

Next,

$$\cos(\theta) = \frac{d/2}{\sqrt{(d/2)^2 + z^2}} \tag{41}$$

$$\sin(\theta) = -\frac{z}{\sqrt{(d/2)^2 + z^2}} \tag{42}$$

$$\tag{43}$$

So, decomposing the field due to $-Q$, we get

$$\vec{E}_{-Q} = k\frac{Q}{((d/2)^2 + z^2)^{3/2}}\left(\frac{d}{2}\hat{\mathbf{i}} - z\hat{\mathbf{j}}\right) \tag{44}$$

Finally, adding the two contributions:

$$\vec{E}_{\text{net}} = \vec{E}_Q + \vec{E}_{-Q} = k\frac{Qd}{((d/2)^2 + z^2)^{3/2}}\hat{\mathbf{i}} \tag{45}$$