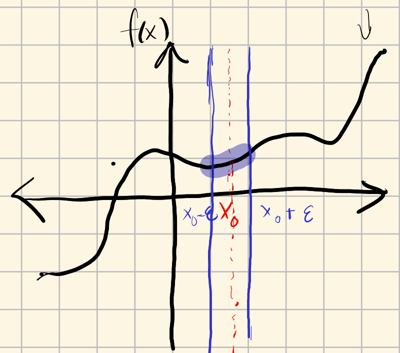
Last Time

- Scalars
- Vectors
- approximating functions by Taylor Series

This time

- practice with Taylor series
- thermal expansion
- ideal gas law

Recap



Goal: approximate $f(x)$ by a polynomial in a small neighborhood about some point x_0, denoted $x_0-\varepsilon$ to $x_0+\varepsilon$, using derivatives of $f(x)$ evaluated at x_0.

$$\boxed{f(x) \approx \sum_{n=0}^{\infty} c_n (x-x_0)^n \qquad c_n = \frac{1}{n!} \frac{d^n f}{dx^n}\bigg|_{x=x_0}}$$

Example: first few terms of Taylor Series of $f(x) = e^x$ around $x_0 = 0$ is $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots \frac{x^n}{n!}$

PROBLEM 1 (Warmup)

A) Let $f(x) = (1+x)^\alpha$ where α is some real number. Find the Taylor series around $x_0 = 0$ up to and including the term linear in x.

Note that this series converges in the interval $|x| < 1$ and $|x| < 1/|\alpha|$

B) Suppose UC Berkeley has 10000 undergrads, and every day one is randomly chosen to get a homework pass. The choosing is done with replacement. Without using a calculator, what is the probability of getting it at least once after 100 days?

1A) $f(x) = (1+x)^{\alpha}$ $\quad x_0 = 0$

linear term is $n=1$, $\quad \sum_{n=0}^{\infty} c_n (x-x_0)^n$

So we compute c_0, c_1

$$c_0 = f(x_0) = 1$$

$$c_1 = \frac{1}{1!} \left.\frac{df}{dx}\right|_{x=x_0} = \left.\alpha(1+x)^{\alpha-1}\right|_{x=0} = \alpha$$

$$f(x) \approx \underline{1 + \alpha x}$$

1B) P(getting the pass at least once in 100 days)

$$= 1 - \underbrace{P(\text{not getting at all in 100 days})}$$

$$1 - \boxed{\left(1 - \frac{1}{10^4}\right)^{100}}$$

$\hookrightarrow$ result of 1A)

$$\left(1-\frac{1}{10^4}\right)^{100} \approx 1 - \frac{100}{10^4}$$

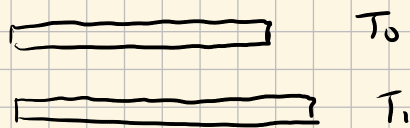
$$= 1 - \frac{1}{100}$$

$$1 - \left(1 - \frac{1}{100}\right) = \frac{1}{100}.$$

Thermal Expansion

1) Why do materials expand or contract?

2) Linear Thermal Expansion



empirical — temperature

$$\underbrace{\frac{\Delta L}{L_0}}\ = \underbrace{\alpha_L \ \Delta T}$$

only valid to LINEAR order in $(T-T_0)$

T_0 $\quad c_0 = f(x_0)$

$f(x) \approx c_0 + c_1(x - x_0)$

$$L(T) = L(T_0) + C_1(T-T_0)$$

$$C_1 = \left.\frac{dL(T)}{dT}\right|_{T=T_0} \qquad C_1 = L(T_0)\cdot\alpha_L$$

$$L(T) = L(T_0) + L(T_0)\,\alpha_L(T-T_0)$$

$$\boxed{\frac{L(T)-L(T_0)}{L(T_0)} = \alpha_L(T-T_0)}$$

empirical

$$V(T) \rightarrow \quad \frac{\Delta V}{V} = \alpha_V \Delta T$$

$$L(T) \rightarrow \quad \frac{\Delta L}{L} = \alpha_L \Delta T$$

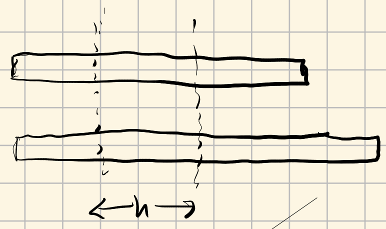
units of α_L, α_V : $\frac{1}{\text{temp.}}$

PROBLEM 3

α_L

I have two rods, A and B, each with coefficient of linear thermal expansion α_A and α_B, respectively. Both rods are initially at temperature T_0. Each rod has two ticks marked on it, separated by a distance h. After heating up both rods to temperature T_1, what is the ratio of the distances between the ticks on the two rods?

3)



A

B

both at T_0

$h_{f,A}$, $h_{f,B}$

let's compute $\frac{h_{f,A}}{h_{f,B}}$

$$h_{f,A} = h\,(1 + \alpha_A(T_1 - T_0))$$

$$h_{f,B} = h\,(1 + \alpha_B(T_1 - T_0))$$

$$\frac{h_{f,A}}{h_{f,B}} = \frac{\cancel{h}\,(1 + \alpha_A(T_1 - T_0))}{\cancel{h}\,(1 + \alpha_B(T_1 - T_0))} = \boxed{\frac{1 + \alpha_A(T_1 - T_0)}{1 + \alpha_B(T_1 - T_0)}}$$

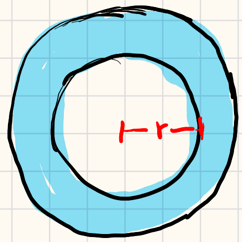
PROBLEM 2

Suppose a material is "isotropic" meaning it is the same in every direction. Show that the coefficient of volume thermal expansion α_V and the coefficient of linear thermal expansion α_L are related by the following equation:

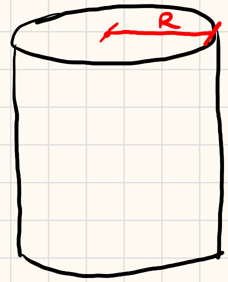
$$\alpha_V = 3\alpha_L$$

Hint: start with $V = L^3$

PROBLEM 4



(a) ring



(b) rod

At temperature T_0, the inner radius of the ring r is less than the radius of the rod R. The ring has coefficient of linear thermal expansion α_{ring} and the rod has coefficient of linear thermal expansion α_{rod}.

A) To fit the ring onto the rod, should you heat the ring, cool the ring, heat the rod, or cool the rod?

B) Heat up the ring only. What temp should I heat up the ring to?

4A) heat up ring or cool the rod

4B) Inner radius of ring should equal radius of rod

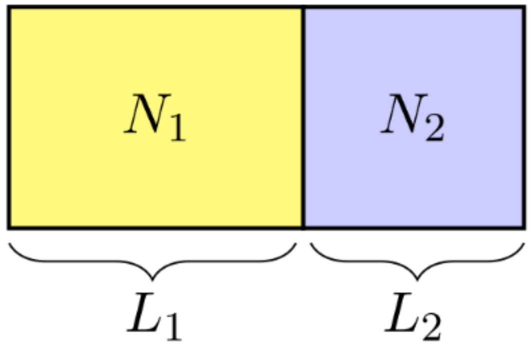
$$\frac{R-r}{r} = \alpha_{ring}\left(T_f - T_0\right)$$

$$T_f = T_0 + \frac{1}{\alpha_{ring}} \frac{R-r}{r} = \boxed{T_0 + \frac{1}{\alpha_{ring}}\left(\frac{R}{r} - 1\right)}$$

Ideal Gases

(01/25/22)

A) Suppose we have a box length L and cross-sectional area A, as depicted in Fig. 1.1. There is a movable partition inside the box, and an ideal gas is added to the containers on both sides of the partition. Specifically, there are N_1 molecules of gas in the left side and N_2 molecules in the right side. Assuming that the entire box is in thermal equilibrium at temperature T, what are the lengths L_1 and L_2 of the left and right containers, respectively, after the movable partition comes to a rest?



A) Main idea: partition stops moving when the forces on both sides are equal (remember from 7A)

$$F_{left} + F_{right} = 0 \qquad \left(P = \frac{F}{A}\right)$$

$$P_{left} A - P_{right} A = 0$$

$$P_{left} = P_{right} \qquad (PV = N k_B T)$$

$$\frac{N_1 k_B T}{V_{left}} = \frac{N_2 k_B T}{V_{right}} \qquad \left(V_{left} = L_1 A\right) \quad \left(V_{right} = L_2 A\right)$$

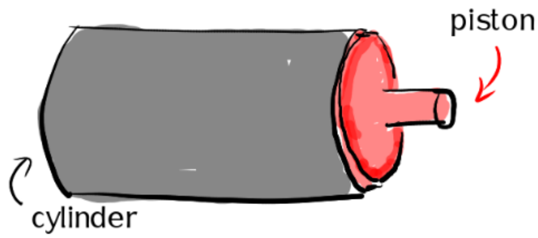
$$\frac{N_1}{L_1 \cancel{A}} = \frac{N_2}{L_2 \cancel{A}}$$

$$\frac{L_1}{L_2} = \frac{N_1}{N_2}$$

$$L_1 + L_2 = L$$

$$L_1 + \frac{N_2}{N_1} L_1 = L \rightarrow \boxed{\begin{cases} L_1 = \dfrac{N_1}{N_1 + N_2} L \\ L_2 = \dfrac{N_2}{N_1 + N_2} L \end{cases}}$$

B) A cylinder (shown in Fig. 1.1 of length L contains gas. One side of the cylinder is a movable wall–also known as a piston. It is suddenly moved inward to compress the gas into 1/8 of the initial volume. By monitoring the pressure gauge, you see that the pressure has gone up by a factor of 10. By what factor has the temperature changed?



B) $PV = N k_B T$

N is constant

$$T_i = \frac{P_i V_i}{N k_B} \qquad T_f = \frac{P_f V_f}{N k_B}$$

$$P_f = 10 P_i$$

$$V_f = \frac{1}{8} V_i$$

$$T_f = \frac{10}{8} \frac{P_i V_i}{N k_B} = \frac{5}{4} \frac{P_i V_i}{N k_B} = \frac{5}{4} T_i$$

$$\boxed{\frac{T_f}{T_i} = \frac{5}{4}}$$

C) A cylinder of N molecules of gas as shown in Fig. 1 has one movable wall which has mass m and area A. The atmospheric pressure is P_a. By monitoring the height h at which the movable wall sits at equilibrium, how can you determine the temperature of the gas? What is the temperature as a function of h (and the other variables given)?

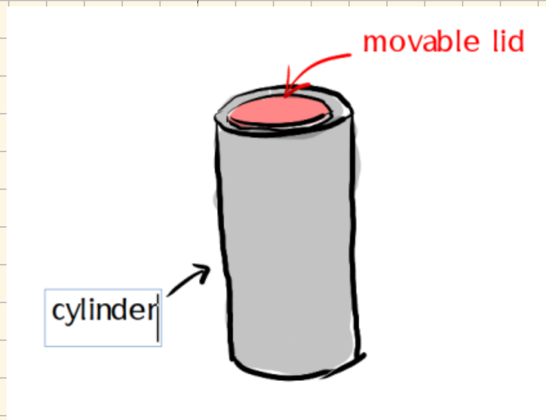
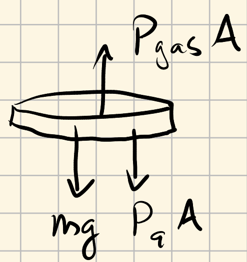


Figure 1: Caption

C) Again, when the lid stops moving, the forces on it sum to zero. But now the forces are

1) the pressure from the gas, P_{gas}
2) the atmospheric pressure, P_a
3) the force due to gravity, mg



$$P_{gas} A - (mg + P_a A) = 0$$

$$\frac{N k_B T}{V} A - (mg + P_a A) = 0 \qquad (PV = N k_B T)$$

units of k_B are $\frac{\text{energy}}{\text{temp}}$ (SI: $\frac{J}{K}$)

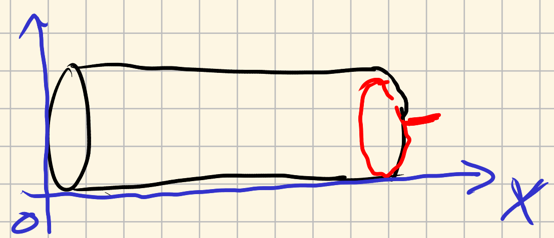
$$\frac{N k_B T}{Ah} A - (mg + P_a A) = 0 \qquad (V = Ah)$$

$$T = \frac{\overset{\text{length}}{h}}{N \underset{\text{energy/temp}}{k_B}} \underbrace{(mg + P_a A)}_{\text{force}}$$

$\frac{\text{temp}}{\text{energy}} \cdot \text{length} \cdot \text{force} = \text{temp}$ ✓

D) Consider again the cylinder with the piston. The cylinder initially contains N molecules of gas at temperature T_0. The piston is pushed to compress the gas to $1/8$ of its initial volume, but this time the temperature of the gas is maintained at T_0 throughout the process (for example by submerging it in a larger bath at temperature T_0). How much energy is required to compress the gas this way?

D)



force = pressure × area

Key idea: $W = \int \vec{F} \cdot d\vec{x}$

but pressure is not constant!

$P = \frac{N k_B T}{V}$ ← changing!

Let x denote the position of the piston.
It is initially at $x = L$, and stops at $x = L/8$.
The volume of the gas then is $V(x) = Ax$.

$\hookrightarrow P = \frac{N k_B T}{V(x)} = \frac{N k_B T}{Ax}$

$\vec{F} = -\frac{N k_B T}{\cancel{A} x} \cdot \cancel{A}\, \hat{x}$ $\quad (F = PA)$

$$W = \int \vec{F} \cdot d\vec{x} = -\int_L^{L/8} \frac{N k_B T}{x}\, dx = -N k_B T \ln(x) \Big|_L^{L/8}$$

$$= N k_B T \left[\ln(L) - \ln(L/8) \right]$$

$$= \boxed{N k_B T \ln(8)}$$

check units: $\frac{\text{energy}}{\text{temp}} \cdot \text{temp} = \text{energy}$ ✓