AC Circuits

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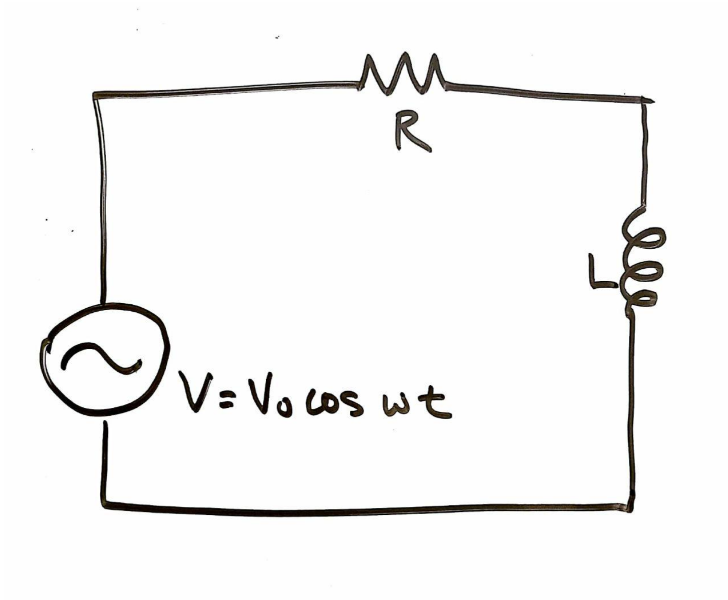
1 AC Circuits - An Introduction

So far, we have only dealt with voltage sources which do not vary in time, such as batteries. Circuits containing only these elements are referred to as "DC" circuits, meaning "direct current". However, most interesting applications of circuit design and power transmission rely on alternating currents, which are driven by voltages which vary sinusoidally in time. Generally, these are driven by voltage sources of the form

$$V(t) = V_0 \cos(\omega t) \tag{1}$$

Let us understand how this works by setting up a simple RL circuit. This circuit alone will prove to be a useful exercise in creating a physically useful device.



Kirchoff's loop rule for this circuit reads:

$$V(t) - IR - L\frac{dI}{dt} = V_0 \cos(\omega t) - IR - L\frac{dI}{dt} = 0 \tag{2}$$

where $I(t)$ is the function we are trying to find. This is a linear, inhomogeneous differential equation–generally, these are challenging to solve. Luckily, we can make such circuits quite easy. To do this, we will make some simple transformations and use a useful fact about oscillatory functions. The main trick is to represent all of the observable quantities (voltages, currents, charges) as *complex*-valued quantities. Euler's identity says

$$e^{ix} = \cos(x) + i\sin(x) \tag{3}$$

where

$$i = \sqrt{-1} \tag{4}$$

From now on, when we replace an observable quantity (voltage, current, charge) with a complex-valued counterpart, what we really mean is that the value you would measure in lab is the *real* part of it. For example, we will write the voltage source above as

$$\tilde{V}(t) = V_0 e^{i\omega t} \tag{5}$$

which makes sense because if we plug in Euler's identity,

$$\tilde{V}(t) = V_0 \cos(\omega t) + iV_0 \sin(\omega t) \tag{6}$$

If we take the real part of this, we recover the original $V(t)$. Similarly, we will substitue $I \to \tilde{I}$, where the tildes remind us that these are the complex-valued versions of the quantities. Kirchoff's law for this circuit becomes

$$V_0 e^{i\omega t} - \tilde{I}R - L\frac{d\tilde{I}}{dt} = 0 \tag{7}$$

Now, we need to use another fact. It turns out that the only possible solutions to such an equation are given by functions which are also proportional to $e^{i\omega t}$. To get some intuition for why this is true, see the note at the end. This means that we can use an *ansatz*, a guess for what the function $\tilde{I}(t)$ looks like:

$$\tilde{I}(t) = \tilde{I}_0 e^{i\omega t} \tag{8}$$

where $\tilde{I}_0$ is a complex-valued, unknown amplitude which we will now try to determine by solving the differential equation. While we are at it, let us go ahead and calculate $d\tilde{I}/dt$:

$$\frac{d\tilde{I}}{dt} = i\omega \tilde{I}_0 e^{i\omega t} = i\omega \tilde{I}(t) \tag{9}$$

This makes the back EMF of the inductor

$$-L\frac{d\tilde{I}}{dt} = -i\omega L\tilde{I}(t) \tag{10}$$

Substituting these into Kirchoff's rule,

$$V_0 e^{i\omega t} - \tilde{I}_0 e^{i\omega t} R - Li\omega \tilde{I}_0 e^{i\omega t} = 0 \tag{11}$$

Now, **this is the important point**. We can divide out all of the $e^{i\omega t}$'s:

$$V_0 - \tilde{I}_0 R - Li\omega \tilde{I}_0 = 0 \tag{12}$$

and what we are left with is no longer a differential equation at all! In fact, this just looks like a regular Kirchoff's rule with a resistor with a resistance R and some weird object with imaginary resistance $i\omega L$. This quantity is referred to as the *impedance* of an inductor. Impedance is just a generalization of resistance to inductors and capacitors so that you can write the voltage across each of those objects as a quantity proportional to the current in the same way that Ohm's law says that the voltage drop across a resistor is proportional to the current through it. The impedance of a component is usually denoted Z, and in this form, Kirchoff's law above takes the form

$$V_0 - \tilde{I}_0 Z_R - \tilde{I}_0 Z_L = 0 \tag{13}$$

where $Z_R = R$ is the impedance (resistance) of a resistor and $Z_L = i\omega L$ is the impedance of an inductor. Back to our goal: we want to solve for $\tilde{I}_0$. Here's the beauty of this approach: having written everything in terms of impedance, we can take our usual approach to circuits that only contain resistors. That is, we find an equivalent impedance for the circuit by combining them together in the right ways. The substitution above tells us that when components are wired in series, as the resistor and inductor are in this circuit, the equivalent impedance is simply the sum of the impedances:

$$Z_{\text{eq}} = \sum_i Z_i \tag{14}$$

So here,

$$Z_{\text{eq}} = Z_R + Z_L = R + i\omega L \tag{15}$$

and we write Kirchoff's rule for the equivalent circuit as

$$V_0 - \tilde{I}_0 Z_{\text{eq}} = 0 \tag{16}$$

$$\tilde{I}_0 = \frac{V_0}{Z_{\text{eq}}} = \frac{V_0}{R + i\omega L} \tag{17}$$

Now comes the algebra. We want to get this into a more useful form. One good way to write complex numbers is as a real magnitude times a phase (henceforth referred to as polar form):

$$\tilde{I}_0 = |\tilde{I}_0| e^{i\phi} \tag{18}$$

To find $|\tilde{I}_0|$,

$$\tilde{I}_0 = \frac{V_0}{R + i\omega L} \times \frac{R - i\omega L}{R - i\omega L} = \frac{V_0(R - i\omega L)}{R^2 + \omega^2 L^2} \tag{19}$$

So

$$|\tilde{I}_0| = \frac{V_0}{R^2 + \omega^2 L^2} \sqrt{R^2 + \omega^2 L^2} = \frac{V_0}{\sqrt{R^2 + \omega^2 L^2}} \tag{20}$$

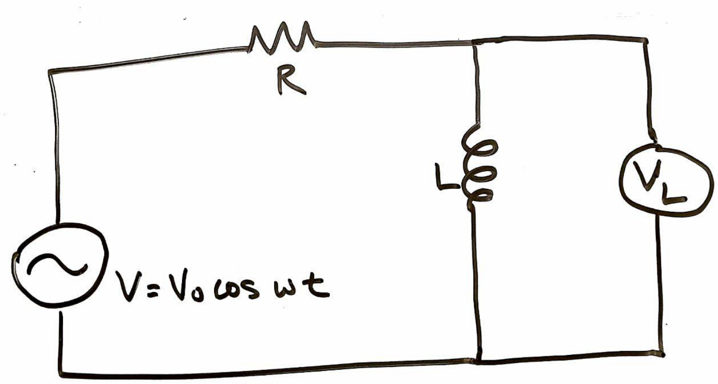
(or just remember that the magnitude of a fraction of two complex numbers $|a/b|$ is the fraction of the magnitudes of those complex numbers $|a|/|b|$)and the phase ϕ here is

$$\phi = \tan^{-1}\left(-\frac{\omega L}{R}\right) \tag{21}$$

Putting it all together,

$$\tilde{I}(t) = \frac{V_0}{\sqrt{R^2 + \omega^2 L^2}} e^{i(\omega t + \phi)} \tag{22}$$

where ϕ is the value found above. Here's a question. If we connected a voltmeter across the inductor as shown below, what would we read?



If this were a resistor, we would use Ohm's law ($V = IR$) to find the voltage drop across the resistor since we now know the current through it. The equivalent statement using impedances is

$$\tilde{V} = \tilde{I}Z \tag{23}$$

So for an inductor,

$$\tilde{V}_L = \tilde{I}(t)Z_L = \frac{V_0}{\sqrt{R^2 + \omega^2 L^2}} e^{i(\omega t + \phi)}(i\omega L) \tag{24}$$

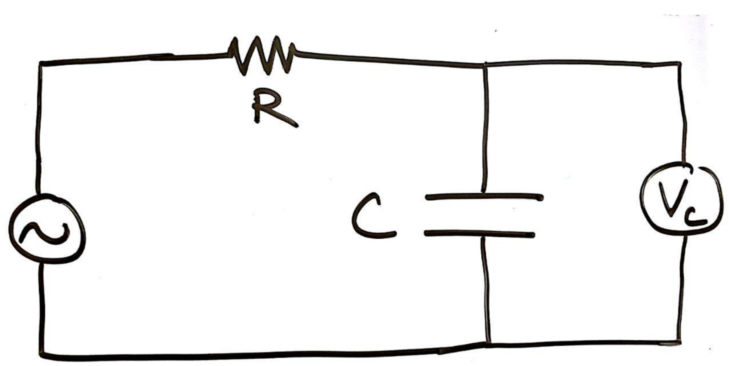
What is the magnitude of this voltage?

$$|\tilde{V}_L| = \frac{V_0 \omega L}{\sqrt{R^2 + \omega^2 L^2}} = \frac{V_0}{\sqrt{\frac{R^2}{\omega^2 L^2} + 1}} \tag{25}$$

Let us now stop and study the *physics* of this. What happens at low frequencies, as $\omega \to 0$? Study this carefully and you will see that $|\tilde{V}_L| \to 0$. What about at high frequencies, as $\omega \to \infty$? $|\tilde{V}_L| \to V_0$. So at low frequencies, none of the input voltage V_0 is seen across the inductor, and at high frequencies all of it is seen. This object is called a **high-pass filter** because it allows high frequencies to pass through. This has *many* engineering and experimental research applications. For example, if you want to isolate high frequencies for a speaker which is designed to reproduce high frequency sounds with high fidelity, you would want to pass the signal through such a filter first before passing it on to the speaker hardware. Or, suppose you are doing an experiment where there is electronic noise but most of it is at low frequencies. To purify your signal, you might pass it through a high-pass filter first before recording. Try to think of some more applications of this on your own.

2 AC Circuits - The next example

If an *RL* circuit acts as a high-pass filter, you can only imagine what behaves as a low-pass filter. Yes, that's right, it's an RC circuit:



First we need to figure out what the charge on the capacitor looks like as a function of time and determine the impedance of a capacitor Z_C. Well, if the current is

$$\tilde{I}(t) = \tilde{I}_0 e^{i\omega t} = \frac{d\tilde{Q}}{dt} \tag{26}$$

then we guess

$$\tilde{Q}(t) = \frac{1}{i\omega}\tilde{I}_0 e^{i\omega t} = \frac{1}{i\omega}\tilde{I}(t) \tag{27}$$

Check that if you take the time derivative of this, you indeed get $\tilde{I}_0 e^{i\omega t}$. The voltage V_C (see the diagram above) drop across a capacitor is, in its familiar form,

$$V_C = \frac{Q}{C} \tag{28}$$

so

$$\tilde{V}_C = \frac{\tilde{Q}}{C} = \frac{1}{i\omega C}\tilde{I}(t) = -\frac{i}{\omega C}\tilde{I}(t) \tag{29}$$

and now we know what the impedance of the capacitor is, because we have gotten this equation in the form of Eq. 23:

$$Z_C = -\frac{i}{\omega C} \tag{30}$$

Alright, so that means that the equivalent impedance here is

$$Z_{\text{eq}} = Z_R + Z_C = R - \frac{i}{\omega C} \tag{31}$$

and Kirchoff's rule for our equivalent circuit reads

$$V_0 - \tilde{I}_0 Z_{\text{eq}} = 0 \tag{32}$$

So

$$\tilde{I}_0 = \frac{V_0}{Z_{\text{eq}}} = \frac{V_0}{R - \frac{i}{\omega C}} \tag{33}$$

Work through the rest of the steps now, finding the magnitude and phase. You should get

$$\phi = \tan^{-1}\left(\frac{1}{R\omega C}\right) \tag{34}$$

and

$$|\tilde{I}_0| = \frac{V_0}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}}} \tag{35}$$

so that the total, time-dependent complex current is

$$\tilde{I}(t) = \frac{V_0}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}}} e^{i(\omega t + \phi)} \tag{36}$$

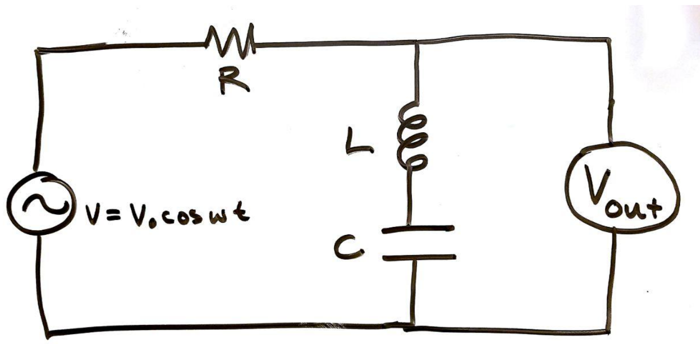
Now, find the voltage drop across the capacitor as a function of time and calculate its magnitude. You should get

$$|\tilde{V}_C| = \frac{V_0}{\omega C\sqrt{R^2 + \frac{1}{\omega^2 C^2}}} = \frac{V_0}{\sqrt{R^2\omega^2 C^2 + 1}} \tag{37}$$

Study this equation. What is its behavior at low frequencies? What is its behavior at high frequencies? What kind of a filter is this?

3 RLC Circuits

It's time to graduate. Put it all together to determine the magnitude of the voltage drop across the capacitor and inductor *together* (V_{out}) in the following circuit and (qualitatively sketch the magnitude of the voltage drop as a function of the AC frequency ω by analyzing what happens as $\omega \to 0$ and $\omega \to \infty$. To simplify your algebra a little, write $\sqrt{1/(LC)}$ as a constant ω_{LC}. Is this a high-pass filter? A low-pass filter? Or something *else*? You should find that it is something called a "band-stop" filter: a filter which allows low frequencies and high frequencies but attenuates frequencies in the middle range!



$$\tilde{I}_0(t) = \frac{V_0}{R + i\left(\omega L - \frac{1}{\omega C}\right)} \tag{38}$$

$$|\tilde{I}_0(t)| = \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{V_0}{\sqrt{R^2 + L^2\left(\omega - \frac{1}{\omega LC}\right)^2}} \tag{39}$$

$$|V_{\text{out}}| = \frac{V_0}{\sqrt{\frac{1}{\left(\frac{\omega}{R/L}\right)^2\left(1 - \frac{\omega_{LC}^2}{\omega^2}\right)^2} + 1}} \tag{40}$$

4 Parallel Impedances

Naturally, if we have multiple circuit components in parallel, like below, we can find an equivalent circuit and equivalent impedance

$$\frac{1}{Z_{\text{eq}}} = \sum_i \frac{1}{Z_i} \tag{41}$$

Try working out the current through the voltage source and the voltage drops across components of these circuits for practice and check with your friends: