Faraday's Law of Induction

Karthik Siva

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Equations

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \tag{1}$$

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 I_{\text{enclosed}} \tag{2}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{d\vec{\ell} \times \hat{\mathscr{r}}}{|\mathscr{r}|^2} \tag{3}$$

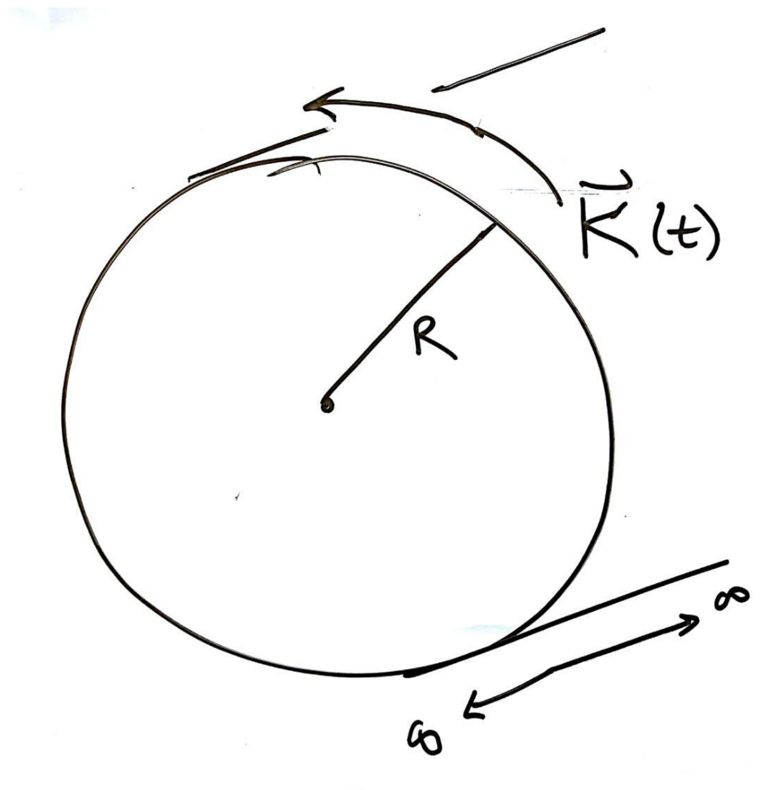
$$I = KL = JA \tag{4}$$

$$\oint \mathbf{E} \cdot d\mathbf{s} = \mathcal{E} = -\frac{d\Phi_B}{dt} \tag{5}$$

Problem 1

Statement

A hollow cylinder of radius R centered on the z-axis carries a uniform surface current density $K(t) = \alpha t\hat{\phi}$ on its surface. The cylinder is infinitely long. Find the magnetic field everywhere created by this current as a function of time, and find the electric field induced by the changing current.



Solution

The hollow cylinder can be thought of as the continuum limit of a solenoid. For a solenoid with n turns per unit length, the magnetic field is zero outside and has magnitude

$$|\mathbf{B}| = \mu_0 n I \tag{6}$$

inside. The direction of $\mathbf{B}$ can be found using any of the right hand rules you know so far, but a convenient one is to wrap your fingers (2-4) in the direction of the flow of current and the direction of your thumb is the direction of the resulting magnetic field. To take the continuum limit, the amount of current flowing through a length L of solenoid is

$$InL \tag{7}$$

To find the current, remember that K is the current per unit length, so

$$InL = KL \tag{8}$$

Making this substituion, the field for $r < R$ is

$$\mathbf{B}(t) = \mu_0 K(t)\hat{\mathbf{z}} = \mu_0 \alpha t \hat{\mathbf{z}} \tag{9}$$

and the field for $r > R$ is

$$\mathbf{B} = 0 \tag{10}$$

Next, we use the induced electric field equation (Eq. 5) to solve for $|E|$. This is pretty much like Ampere's law. For a circular contour of radius r inside the solenoid, the flux enclosed will be

$$\Phi_B(t) = |B(t)|\pi r^2 = \mu_0 K(t)\pi r^2 = \mu_0 \alpha t \pi r^2 \tag{11}$$

Finally,

$$-\frac{d\Phi_B}{dt} = -\mu_0 \alpha \pi r^2 \tag{12}$$

The left hand side of Eq. 5 gives

$$\oint \mathbf{E} \cdot d\mathbf{s} = |E| 2\pi r \tag{13}$$

Setting these two equal:

$$\boxed{\mathbf{E}(r) = -\frac{\mu_0 \alpha r}{2}\hat{\phi} \qquad (r < R)} \tag{14}$$

For $r > R$, a circular contour of radius r encloses a nonzero magnetic field only for $r < R$, so the total flux is

$$\Phi_B = \mu_0 \alpha t \pi R^2 \tag{15}$$

and

$$-\frac{d\Phi_B}{dt} = -\mu_0 \alpha \pi R^2 \tag{16}$$

The LHS of Faraday's law is unchanged though, as this only depends on the geometry of our imaginary circular contour:

$$\oint \mathbf{E} \cdot d\mathbf{s} = |E| 2\pi r \tag{17}$$

Setting these two equal:

$$\boxed{\mathbf{E}(r) = -\frac{\mu_0 \alpha R^2}{2r}\hat{\phi} \qquad (r > R)} \tag{18}$$

Problem 2

Statement

A square loop of wire of side length L is rotated at constant angular velocity ω in the presence of a constant magnetic field in two scenarios, A and B. The wire has cross sectional area a and resistivity ρ. In both scenarios, the magnetic field is constant and points upward, $\mathbf{B} = B_0\hat{y}$. How much electrical energy is dissipated in the wire as a result of rotating the loop in the field for one cycle?

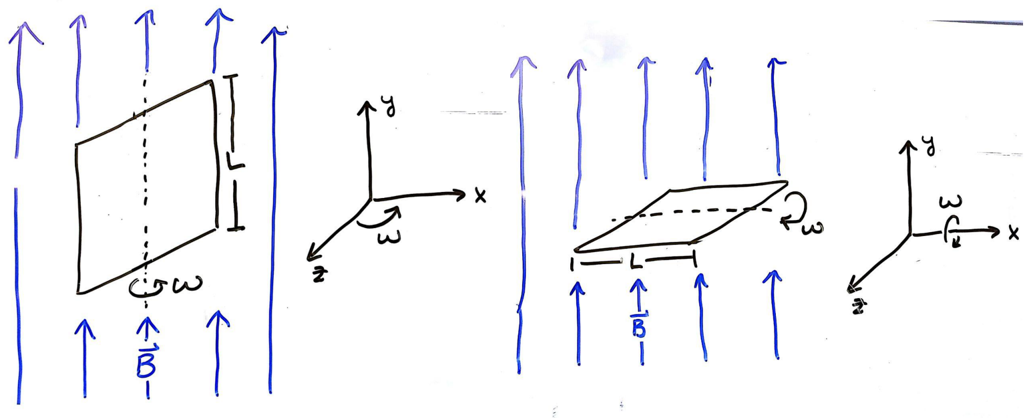


Fig. 1. Scenario A on the left, scenario B on the right.

Solution

The total energy dissipated in the loop over one cycle in each scenario is

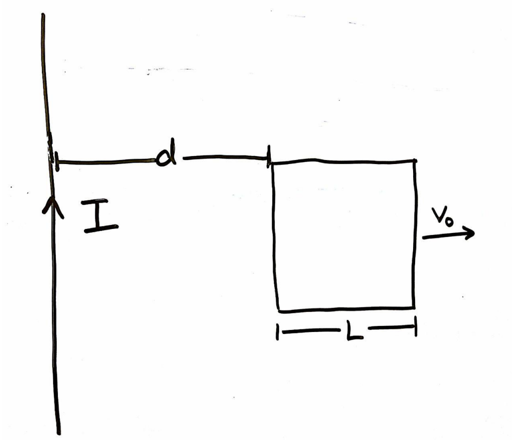
$$\boxed{E_{\text{total}} = 0 \qquad \text{(scenario A)}} \tag{19}$$

$$\boxed{E_{\text{total}} = \frac{B_0^2 L^3 \omega^2 a}{8\rho} \qquad \text{(scenario B)}} \tag{20}$$

Problem 3

Statement

An infinitely long wire runs along the y-axis carrying a current I in the $+\hat{y}$ direction. A square loop of wire with side length L is initially positioned so that its left edge is parallel to the current-carrying wire and is a distance d from it. The square loop of wire has resistance R. If the loop is pulled to the right at a constant velocity $v = v_0$, determine the direction and initial amount of current that flows in the loop. "Initial" meaning when the left edge of the loop is at a distance d from the infinite wire.



Solution

$$I = \frac{\mathcal{E}}{R} = -\frac{1}{R}\frac{d\Phi_B}{dt} = \frac{\mu_0 v_0 I}{2\pi R}\frac{L^2}{d(d+L)} \qquad \text{clockwise} \tag{21}$$

Problem 4 - A Paradox To Show Something is Amiss

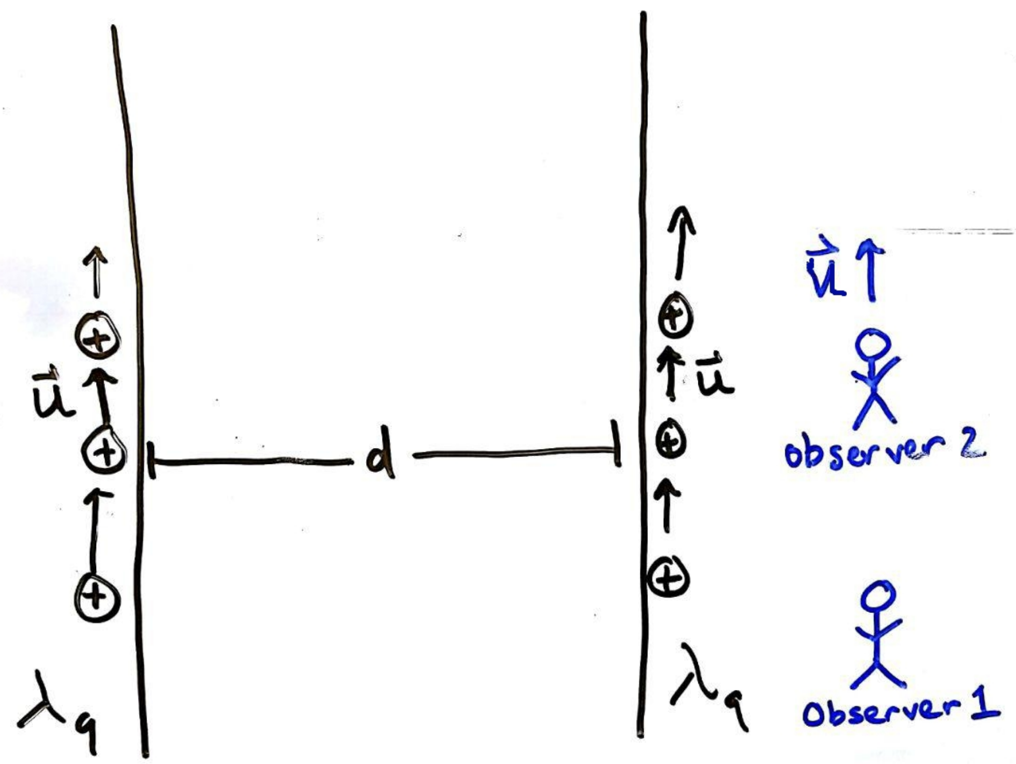
This is not a Faraday's law problem, so if you are trying to practice problems with induced EMF, skip this. This problem is intended to reveal a hint that something is not complete about our understanding of electromagnetism so far.

Statement

Two infinitely long wires, each carrying net positive linear charge density λ_q run parallel to each other, separated by a distance d. The charges in these wires both move in the same direction along the wires with the same speed u. The wires also have mass per unit length λ_m.

- What is the electric force per unit length exerted by the wires on each other?
- What is the magnetic force per unit length exerted by the wires on each other?
- What is the net acceleration of one of the wires due to these two forces?
- Without rederiving the electric and magnetic forces, determine the following. If you run at speed u (sketched as observer 2 running) in the same direction that the charges are moving, what net acceleration should you see for one of the wires?
- As a runner at speed u, how fast are the charges moving relative to you?
- Now, calculate the electric force per unit length exerted by the wires on each other *as you, the runner, would calculate.*
- Now, calculate the magnetic force per unit length exerted by the wires on each other *as you, the runner, would calculate.*
- Now, calculate the net acceleration of one of the wires due to these two forces *as you, the runner, would calculate.*

Compare the acceleration you expected to get to the acceleration you derived by rederiving the forces. Will a person who is running alongside the charges agree with a person who is standing still about how fast the wires accelerate away from each other? By what multiplicative factor do these answers disagree?



Solution

In the first case, the observer calculates electric force per unit length

$$\frac{\mathbf{F}_E}{L} = \frac{\lambda_q^2}{2\pi\varepsilon_0 d}\hat{\mathbf{x}} \tag{22}$$

and magnetic force per unit length

$$\frac{\mathbf{F}_M}{L} = -\frac{\mu_0\lambda_q^2 u^2}{2\pi d}\hat{\mathbf{x}} \tag{23}$$

The acceleration is therefore

$$\boxed{\mathbf{a} = \frac{F_{\text{net}}}{m} = \frac{F_E - F_M}{\lambda_m L} = \frac{\lambda_q^2}{2\pi\lambda_m d\varepsilon_0}\left(1 - \mu_0\varepsilon_0 u^2\right)\hat{\mathbf{x}}} \tag{24}$$

If we run alongside the charges in the wire at speed u, the charges do not appear to move along the wire relative to us. But the acceleration perpendicular to the direction of wires is should be unchanged. On the other hand, if the speed of the charges is zero for the runner, they see only an electric force

$$\frac{\mathbf{F}'_E}{L} = \frac{\lambda_q^2}{2\pi\varepsilon_0 d}\hat{\mathbf{x}} \tag{25}$$

because there is no current to create a magnetic field. Now,

$$\boxed{\mathbf{a}' = \frac{F_{\text{net}}}{m} = \frac{F'_E - F'_M}{\lambda_m L} = \frac{\lambda_q^2}{2\pi\lambda_m d\varepsilon_0}\hat{\mathbf{x}}} \tag{26}$$

The accelerations $\mathbf{a}$ and $\mathbf{a}'$ disagree by a factor of

$$\frac{|a'|}{|a|} = \frac{1}{1 - \mu_0\varepsilon_0 u^2} = \boxed{\frac{1}{1 - u^2/(1/\sqrt{\mu_0\varepsilon_0})^2}} \tag{27}$$

Evidently $\mathbf{a}' \neq \mathbf{a}$, and a running observer and a standing observer will disagree about how fast the wires accelerate away from each other. At this point in your study of electricity in magnetism, there is no way for us to reconcile this–it is only a hint that this cannot be the whole story. However, this factor in Eq. 27 has a special place in the theory of special relativity. If we make the identification that $\sqrt{1/\mu_0\varepsilon_0}$ must have units of speed, it is not unreasonable that it turns out that this is the speed of light

$$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \tag{28}$$

and the factor in Eq. 27 is the square of a special factor γ that shows up in special relativity:

$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}} \tag{29}$$

If you study E& M or special relativity in a higher level course, you will at some point learn how the two theories are related, and this factor describes how spatial lengths and durations of time transform between the standing observer and the running observer.