Lorentz Force, Ampere's Law, Biot-Savart Law

Karthik Siva

April 9, 2018

Equations

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \tag{1}$$

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 I_{\text{enclosed}} \tag{2}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{d\vec{\ell} \times \hat{\boldsymbol{\imath}}}{|\boldsymbol{\imath}|^2} I = KL = JA \tag{3}$$

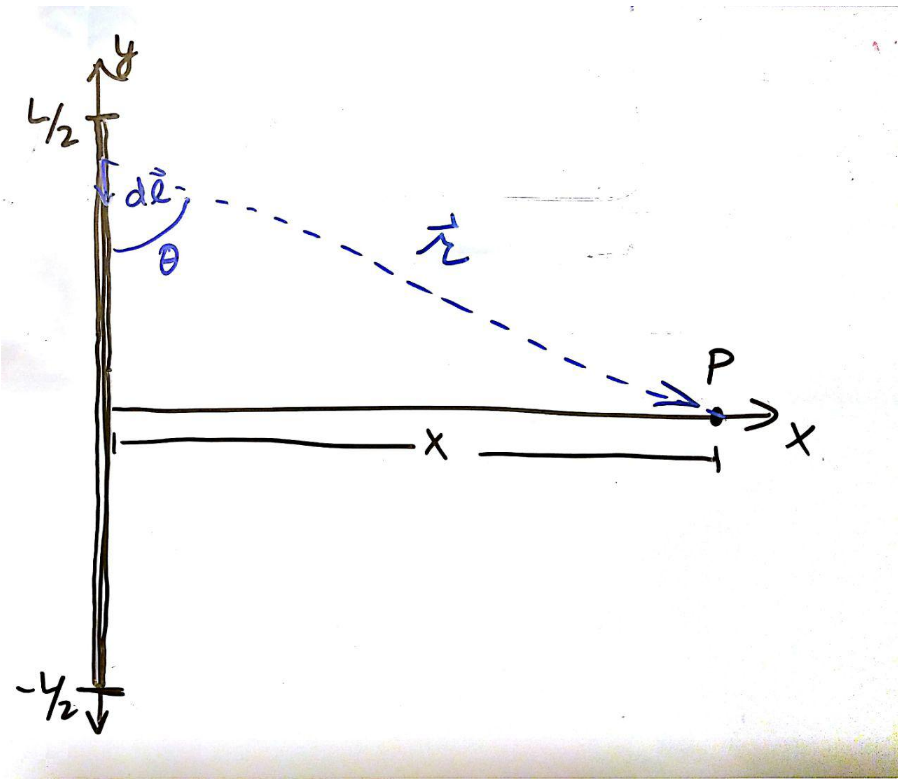
Problem 2

Statement

A wire of length L carries a current I. What is the magnetic field created by this current at point located at a distance x away from the center of the wire?

Solution

First, we want to sketch the problem and sketch a coordinate system.



With the origin of the coordinate system placed at the center of the wire, we are ready to begin. The right-hand rule applied to the vectors $d\vec{\ell}$ and $\hat{\boldsymbol{\imath}}$ suggests that each piece of current will create a little bit of magnetic field out of the board at the point P shown above. This means that all contributions will point along the same direction (out of the board) and we only get a nonzero $\hat{\mathbf{k}}$ component. Next, we should carefully assemble each piece of the integrand in the Biot-Savart law. Since we have reasoned about the resultant direction of the cross-product, all that remains is to find its magnitude:

$$d\vec{\ell} \times \hat{\mathscr{r}} = |d\vec{\ell}||\hat{\mathscr{r}}| \sin\theta \tag{4}$$

where θ is the angle between $d\vec{\ell}$ and $\hat{\mathscr{r}}$. To convert this into an integral over Cartesian coordinates, namely y, we should rewrite

$$|d\vec{\ell}| = -dy \tag{5}$$

$$\sin\theta = \frac{x}{\sqrt{x^2 + y^2}} \tag{6}$$

$$|\vec{\mathscr{r}}|^2 = x^2 + y^2 \tag{7}$$

and since $\hat{\mathscr{r}}$ is a unit vector, $|\hat{\mathscr{r}}| = 1$. Observe the $(-)$ sign on dy. This is a somewhat subtle point: $d\ell$ is not simply dy if we are integrating from the top of the wire to the bottom. To see this, consider the integral of $d\ell$. What should this give? Just the length of the wire, by definition.

$$\int d\ell = L \tag{8}$$

But if I naively set $d\ell = dy$ and integrate from top to bottom,

$$\int_{L/2}^{-L/2} dy = -L \tag{9}$$

So either we should switch the integration limits and keep $d\ell = dy$ or use $d\ell = -dy$. Either way is valid as long as you remember to keep track of this (or just put the correct sign in at the end of the problem). Now we are ready to assemble the parts.

$$\mathbf{B}(x) = \frac{\mu_0 I}{4\pi} \int_{L/2}^{-L/2} \frac{x(-dy)}{(x^2 + y^2)^{3/2}} \hat{\mathbf{k}} \tag{10}$$

This is a tricky integral and a lesson about the Biot-Savart law. Most of the time, even if you can set up the integral, the integral itself is difficult to do. Only in rare cases is a trick unnecessary. Below I use this neat trick called *Mathematica*:

```
In[4]:= Assuming[x > 0 && L > 0, Integrate[(x^2 + y^2)^{-3/2}, {y, L/2, -L/2}]]
Out[4]= {- 2 L / (x^2 Sqrt[L^2 + 4 x^2])}
```

$$\boxed{\mathbf{B}(x) = \frac{\mu_0 I}{2\pi} \frac{L}{x\sqrt{L^2 + 4x^2}} \hat{\mathbf{k}}} \tag{11}$$

Let us check that this makes sense if we take the wire to be infinitely long. When $L \to \infty$, the factor on the right $L/\sqrt{L^2 + 4x^2} \to 1$, which leaves us with

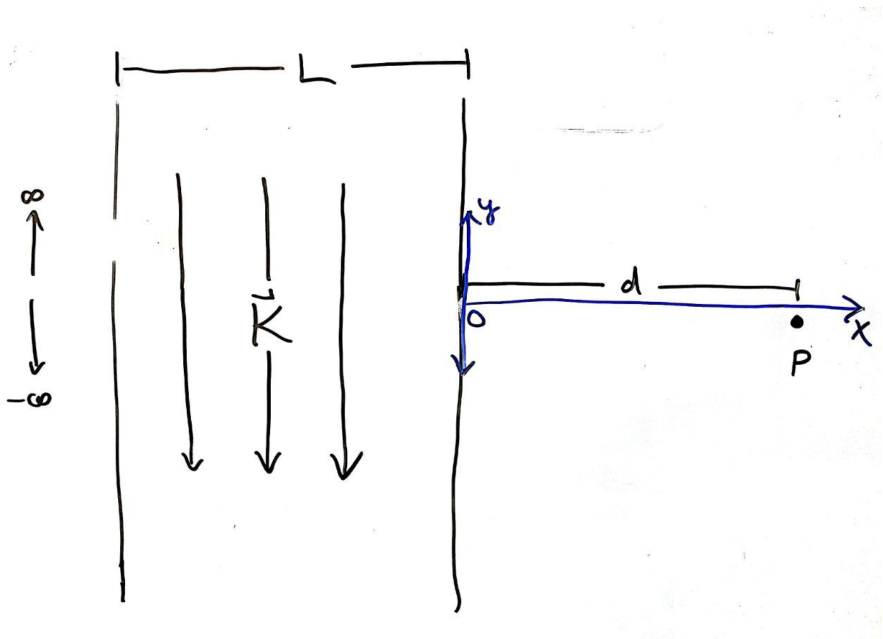
$$\mathbf{B}(x) = \frac{\mu_0 I}{2\pi x} \hat{\mathbf{k}} \tag{12}$$

which is exactly what we expect for the field of an infinitely long wire of current.

Problem 3

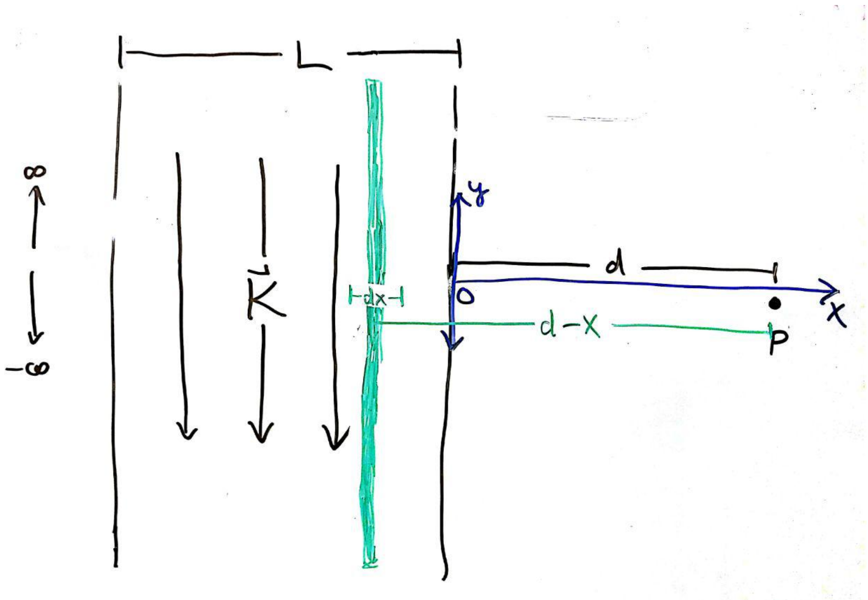
Statement

A thin sheet carrying a current in the $-\hat{\mathbf{j}}$ direction is infinitely long in the y-direction and has width L in the x-direction, as indicated in the diagram below. The current has current density K (current per unit length). What is the magnetic field induced at point P?



Solution

We cannot expect to apply Ampere's law here because there is no translational symmetry along the x-axis. We could potentially use the Biot-Savart law, but it is not necessary. What we can instead do is think of the sheet as being made up of wires of infinitesimal thickness, dx:



For each of these thin wires, we can now use Ampere's law with a circular Amperian loop to determine how magnetic field is created. The calculation of a magnetic field due to an infinitely long wire of current should be familiar, so I will quote the result:

$$\mathbf{B} = \frac{\mu_0 I}{2\pi(d-x)} \tag{13}$$

But the amount of current flowing in our infinitesimally thin wire is an infinitesimal amount of current:

$$dI = K dx \tag{14}$$

and the infinitesimal amount of magnetic field $d\mathbf{B}$ created by an infinitesimal amount of current dI is then:

$$d\mathbf{B} = \frac{\mu_0 K dx}{2\pi(d-x)}\hat{\mathbf{k}} \tag{15}$$

$$\mathbf{B} = \int_{-L}^{0} \frac{\mu_0 K}{2\pi} \frac{dx}{d-x} \hat{\mathbf{k}} = \boxed{\frac{\mu_0 K}{2\pi} \ln\left(1 + \frac{L}{d}\right) \hat{\mathbf{k}}} \tag{16}$$