# Electric Potential Problems

Karthik Siva

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## Equations

$$\Phi = \int\int \mathbf{E} \cdot d\mathbf{A} \tag{1}$$

$$\Delta V = -\int \mathbf{E} \cdot d\vec{\ell} \tag{2}$$

$$\Delta U = q\Delta V \tag{3}$$

## Problem 1

### Statement

An empty sphere (or spherical shell) of radius $R$ carries total charge $Q$ uniformly distributed over its surface. Suppose that an infinitesimally small hole is drilled in the surface. Along the axis formed by the center of the sphere and the hole, at a distance $R$ outside of the sphere, a point mass of charge $-Q$ and mass $m$ is released from rest. At what speed is the point charge moving when it strikes the inner surface of the sphere opposite the hole? Assume the charges on the sphere do not move and the sphere itself remains stationary.

### Solution

First let us set up a coordinate system. We will set the origin at the center of the spherical shell, the hole at $(R, 0, 0)$, and the initial position of the point mass at $(2R, 0, 0)$. This makes the final position of the point mass $(-R, 0, 0)$.

The basic physics is that the sphere attracts the point charge because they have opposite sign. In the process, the system loses potential energy. By conservation of energy, kinetic energy must be gained. Since the sphere is taken to be stationary, the point charge is the only thing that gains kinetic energy:

$$U_i + \cancelto{0}{KE_i} = U_f + KE_f \tag{4}$$

$$\frac{1}{2}mv^2 = -\Delta U \tag{5}$$

Now we just need to evaluate the change in potential energy between having the point charge at the location $(2R, 0, 0)$ and $(-R, 0, 0)$, which from Eq. 3 means that we need to compute $\Delta V$ between these two points. To do this, we need to know $\mathbf{E}$ everywhere. First let us solve for $\mathbf{E}$ inside the sphere. For the field due to the shell, since the charge distribution is spherically symmetric, we expect $\mathbf{E}$ to only depend on the distance from the center, call it $r$, and to point radially outward if it has nonzero magnitude. To solve for $\mathbf{E}(r)$ for $r < R$, we choose as a Gaussian surface a sphere of radius $r$. Since it is inside the shell, it encloses no charge. Therefore, by Gauss's law

$$\Phi = |\mathbf{E}|(4\pi r^2) = \frac{0}{\varepsilon_0} \tag{6}$$

$$\mathbf{E}(r) = 0 \qquad (r < R) \tag{7}$$

For $r > R$, the Gaussian sphere of radius $r$ encloses all of the charge, $Q$. Therefore,

$$\Phi = |\mathbf{E}|(4\pi r^2) = \frac{Q}{\varepsilon_0} \tag{8}$$

$$\mathbf{E}(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}\hat{\mathbf{r}} \tag{9}$$

So now we are looking for $\Delta V$ between $(2R, 0, 0)$ and $(-R, 0, 0)$.

$$\Delta V = -\int \mathbf{E} \cdot de\vec{l}l \tag{10}$$

where the integral is taken over any path from the starting point to the ending point. But note that if we integrate over the path going straight from $(2R, 0, 0)$ to $(-R, 0, 0)$, the section of the integral that goes over $(R, 0, 0)$ to $(-R, 0, 0)$ will not contribute anything because $\mathbf{E} = 0$ there. Another way to say this is that the interior of the spherical shell is an *equipotential*, so $\Delta V$ between *any* two points inside it is zero. This leaves us with just the integral from $(2R, 0, 0)$ to $(R, 0, 0)$, where $\mathbf{E} \neq 0$.

$$\Delta V = -\int_{2R}^{R} \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}dr - \cancelto{0}{\int_{R}^{-R} 0\, dr} = \frac{Q}{8\pi\varepsilon_0} \tag{11}$$

$$\Delta U = (-Q)\Delta V = \frac{(-Q)Q}{8\pi\varepsilon_0 R} \tag{12}$$

Using Eq. 5,

$$\boxed{v = \sqrt{\frac{1}{4\pi\varepsilon_0}\frac{Q^2}{mR}}} \tag{13}$$

# Problem 2

## Statement

Five point charges, each of charge $Q$, are arranged on the vertices of a plus sign, which has edge length $a$ (so that along each axis, there are 3 charges spaced $a$ apart). How much energy is required to bring the point charges out from infinity into this configuration?

## Solution

The conceptual way to think about this is to picture this as a sequence of steps and evaluate how much the potential energy of the configuration changes each time we add a charge. Then, we add up all of the changes in potential energy and this is the total amount of energy that we have to put into the system. For convenience, I have numbered the charges below:

**Insert figure**

First, we have no charges. Then we introduce charge 1. This takes no energy because there is nothing resisting us. But when we introduce charge 2, we have to work against the fact that charges 1 and 2 do not want to be near each other. How much work is done in this process? We can use Eqs. 2 and 3 to find out:

$$\Delta U_{12} = -Q\int_{\infty}^{a} \frac{Q}{4\pi\varepsilon_0}\frac{1}{r}dr = Q\frac{Q}{4\pi\varepsilon_0 a} \tag{14}$$

A faster way of doing this is to simply use the absolute potential function for a point charge where the zero of the potential is taken to be at infinity. This may be clear from the explicit calcuation above, but this is

$$V(r) = V(r) - V(\infty) = \frac{Q}{4\pi\varepsilon_0 r} \tag{15}$$

Notice that $QV(a)$ gives the same result as derived slowly above. So now what happens when we bring in charge 3? There are two potentials to deal with now, one from charge 1 and one from charge 2. By superposition, this will give us (using the absolute potential function for each one):

$$\Delta U_{13} = \frac{Q^2}{4\pi\varepsilon_0(2a)} \tag{16}$$

$$\Delta U_{23} = \frac{Q^2}{4\pi\varepsilon_0(a)} \tag{17}$$

Now we bring in charge 4, so we have to deal with potentials from charges 1,2, and 3. You get the idea.

$$\Delta U_{14} = \frac{Q^2}{4\pi\varepsilon_0(\sqrt{2}a)} \tag{18}$$

$$\Delta U_{24} = \frac{Q^2}{4\pi\varepsilon_0(a)} \tag{19}$$

$$\Delta U_{34} = \frac{Q^2}{4\pi\varepsilon_0(\sqrt{2}a)} \tag{20}$$

And for bringing in charge 5, we have four terms:

$$\Delta U_{15} = \frac{Q^2}{4\pi\varepsilon_0(\sqrt{2}a)} \tag{21}$$

$$\Delta U_{25} = \frac{Q^2}{4\pi\varepsilon_0(a)} \tag{22}$$

$$\Delta U_{35} = \frac{Q^2}{4\pi\varepsilon_0(\sqrt{2}a)} \tag{23}$$

$$\Delta U_{45} = \frac{Q^2}{4\pi\varepsilon_0(2a)} \tag{24}$$

Adding up all of these $\Delta U$'s, we get

$$\boxed{\Delta U = \left(5 + 2\sqrt{2}\right)\frac{Q^2}{4\pi\varepsilon_0 a}} \tag{25}$$

You can generalize the way we solve this type of problem. Note that what we ended up doing is calculating the potential energy needed to assemble each *pair* of charges to the given distances of separation and then we added them up. So in general, the way to do this is if you have $N$ point charges, for every distinct pair of charges, calculate how much energy is required to bring them to the distance given in the problem, then add up all of the energies. To check that you have done it right, you should get

$$\frac{N!}{2!(N-2)!} = \frac{1}{2}N(N-1) \text{ terms} \tag{26}$$

in your expansion. An alternative, equivalent way to do it that allows you to not worry about making sure that you only pick distinct pairs of particles is to deliberately double count pairs (for example if you count $\Delta U_{12}$, then you also count $\Delta U_{21}$ in the problem above), and then just divide by 2 at the end. **Potential energy does not obey superposition.** Whereas the electric field due to $N$ point charges is just the sum of fields from each of the $N$ fields, the potential energy required to assemble $N$ charges together requires *quadratically* more energy. In counting the number of terms above, it is clear that you need $O(N^2)$ terms to calculate the potential energy whereas for the electric field you only need $O(N)$ terms.

# Problem 3

## Statement

Two conducting spheres each have total charge $Q$ on them. Sphere 1 has radius $2R$ and sphere 2 has radius $R$. The two spheres are kept an infinite distance apart. At time $t = 0$, the two spheres are connected by an infinitely long, perfectly conducting wire. Charges move through the wire until the equilibrium is achieved and no charge flows. At this point, how much charge is on each sphere?

## Solution

Charges flow due to a potential difference between the endpoints of the wire. This means that when the charge stops flowing, the potential at the surface of sphere 1 must be equal to the potential at the surface of sphere 2. Therefore, we need to look for a way to express the potential at the surface of the spheres as a function of the amount of charge on each one. If we had just sphere 2 sitting in space, then the potential at its surface, relative to infinity, is

$$V_2(R) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R} \tag{27}$$

If we have the sphere 1 also nearby though, the potential at the surface of sphere 2 is no longer simple and will vary across the surface. Luckily, in this problem, sphere 1 is infinitely far away. So the potential at the surface of sphere 2 is unaffected by the potential due to sphere 1, and the potential at the surface of sphere 1 is unaffected by the potential due to sphere 2. Suppose the charges flow and now sphere 2 carries a different charge $Q_2$, and sphere 1 carries a different charge $Q_1$. Since the total amount of charge cnanot change,

$$Q_1 + Q_2 = 2Q \tag{28}$$

The potential at the surface of sphere 2 is now

$$V_2(R) = \frac{1}{4\pi\varepsilon_0}\frac{Q_2}{R} \tag{29}$$

and the potential at the surface of sphere 1 is now

$$V_1(2R) = \frac{1}{4\pi\varepsilon_0}\frac{Q_1}{2R} \tag{30}$$

When the charges stop flowing, $V_1(2R) = V_2(R)$. This gives us

$$\frac{Q_1}{2R} - \frac{Q_2}{R} = 0 \tag{31}$$

We now have two unknowns $(Q_1, Q_2)$ and two equations (Eqs. 28 and 31). Solving this system gives

$$\boxed{Q_1 = \frac{4}{3}Q \text{ and } Q_2 = \frac{2}{3}Q} \tag{32}$$

Let us check that this makes sense. The potential at the surface of sphere 2 was initially higher than that at the surface of sphere 1, because sphere 2 was smaller than sphere 1. This means that (positive) charge would flow from sphere 2 to sphere 1. When the charge stops flowing, sphere 1 should end up with more charge than it started with, and sphere 2 should end up with less charge than it started with. Indeed, the mathematical solution found agrees with the intuition.