Gauss's Law Problems

Karthik Siva

February 28, 2018

Equations

$$\Phi = \int \mathbf{E} \cdot d\mathbf{A} \tag{1}$$

$$\Phi_{\text{closed}} = \frac{Q_{\text{enclosed}}}{\varepsilon_0} \tag{2}$$

Problem 1

Statement

A charge Q sits at the front, top-left vertex of a cube. What is the total flux through the back, right, and bottom faces of the cube?

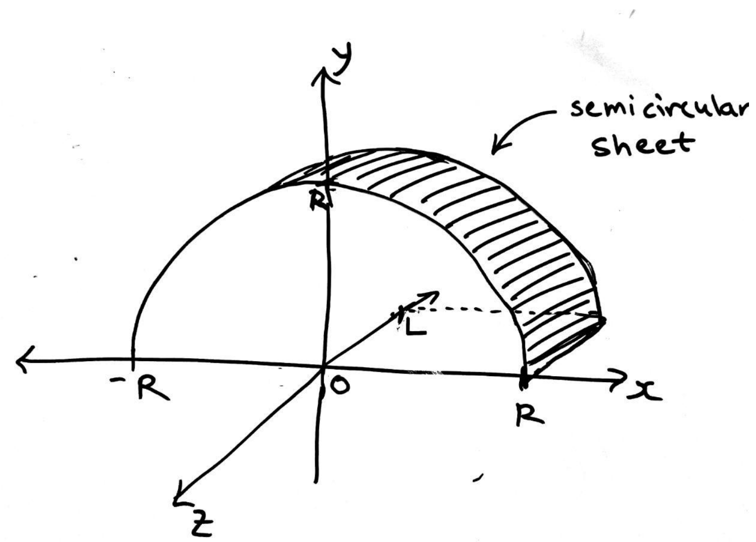
Solution

Imagine that the cube in the problem has side length L. Now imagine that we form a larger cube of side length $2L$ by putting eight of these smaller cubes together. The charge Q is now at the center of the large cube. By Gauss's law, the flux through the entire surface of the large cube is Q/ε_0. Observe that the flux through the surface of each of eight smaller cubes must be the same. Therefore, the flux through the surface of the small cube in question is $Q/(8\varepsilon_0)$.

Problem 2

Statement

A sheet of dimensions $\pi R \times L$ is bent into a semicircle:



Suppose the sheet is centered at $x = 0$ and there is an electric field $\mathbf{E} = cx^2\hat{\mathbf{j}}$ where c is a constant. Although there is a field, there are no source charges near the sheet (as depicted above). Find the magnitude of the flux piercing the sheet.

Solution

There are two ways of solving this. One is to directly integrate $\mathbf{E} \cdot d\mathbf{A}$ over the surface of the sheet and the other is to use Gauss's law to do a different but equivalent integral. To illustrate the practicality of Gauss's law, we will do the second method first. The general approach to solving a problem like this is to find a closed surface which includes the surface over which we are trying to find the flux. So we need to find a closed surface that has, as part of it, the sheet bent into a semicircle. Whenever possible, we want to add surfaces that have no flux going through them. Since the field points in the $\hat{\mathbf{j}}$ direction, we can add surfaces that are parallel to the xy-plane which will not have any flux through them.

To close the surface, we can add a sheet in the xz-plane. The flux through this is not zero, but at least the angle between the electric field and the normal vector of the sheet does not vary. So the closed surface we choose is, in all, the surface of half of a hockey puck. Now,

$$\Phi_{\text{closed}} = \Phi_{\text{top}} + \Phi_{\text{bottom}} + \Phi_{\text{front}} + \Phi_{\text{back}} = \Phi_{\text{top}} + \Phi_{\text{bottom}} \tag{3}$$

Now that we have a closed surface, we can use Gauss's law. Since there is no charge anywhere near the sheet, the half-hockey-puck surface encloses on charge, so

$$\Phi_{\text{closed}} = \Phi_{\text{top}} + \Phi_{\text{bottom}} = 0 \tag{4}$$

So

$$\Phi_{\text{top}} = -\Phi_{\text{bottom}} \tag{5}$$

The flux on the bottom is easy to calculate:

$$\Phi_{\text{bottom}} = \int \mathbf{E} \cdot d\mathbf{A} = \int_{-R}^{R} cx^2 L\hat{\mathbf{j}} \cdot \hat{\mathbf{j}} dx = \frac{2}{3} cLR^3 \tag{6}$$

So the magnitude of Φ_{top} is

$$\boxed{\Phi_{\text{top}} = \frac{2}{3} cLR^3} \tag{7}$$

Now let's check that naively integrating $\mathbf{E} \cdot d\mathbf{A}$ over the top sheet would have given us the same result:

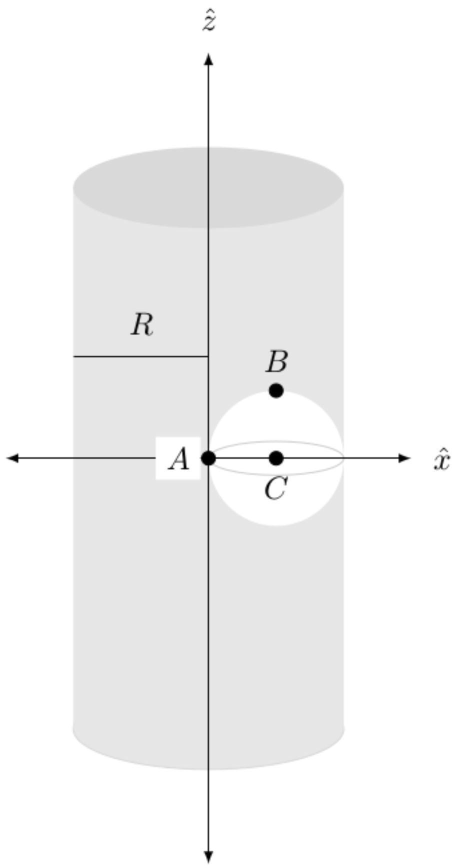
$$\Phi_{\text{bottom}} = \int \mathbf{E} \cdot d\mathbf{A} = \int_0^{\pi} c(R\cos(\theta))^2 LR\sin(\theta) d\theta = cLR^3 \int_0^{\pi} \cos^2(\theta)\sin(\theta) d\theta \tag{8}$$

Observe that $d(\cos^3(\theta))/d\theta = -3\cos^2(\theta)\sin(\theta)$, so

$$\Phi_{\text{bottom}} = cLR^3 \left[-\frac{1}{3}\cos^3(\theta) \right]_0^{\pi} = \frac{2}{3} cLR^3 \tag{9}$$

Problem 3

Statement



The shaded rod above extends infinitely in both directions along the z-axis and carries a volume charge density ρ. Inside the rod, a spherical cavity of radius $R/2$ has been carved out and contains no charge. What are the electric fields at point A, point B, and point C?

Solution

Although this problem lacks any useful symmetry, we can use superposition to solve two problems with high symmetry and superimpose them to get this charge distribution. To create a cavity, we could superimpose a solid sphere with charge density ρ and a solid sphere of charge density $-\rho$ centered at point C. Then, the solid sphere with charge density ρ, together with the rest of the rod, forms a complete rod with no cavities carrying a uniform charge density. We can solve this easily with Gauss's law. First, we observe that by symmetry, the electric field from the rod can only point radially outward, and it has constant magnitude at points that are a fixed distance from the center. This suggests that we should choose our Gaussian (closed) surface to be a cylinder of radius $r < R$ and arbitrary length L (it will drop out). Then,

$$Q_{\text{enclosed}} = \pi r^2 L \rho \tag{10}$$

and since the field does not pierce the caps of the cylinder,

$$\Phi_{\text{closed}} = |E|(2\pi r L) = \frac{\pi r^2 L \rho}{\varepsilon_0} \tag{11}$$

$$\mathbf{E}_{\text{cyl}} = \frac{\rho r}{2\varepsilon_0}\hat{\mathbf{r}} \tag{12}$$

For the solid sphere of charge density $-\rho$, observe that we only need to evaluate its field on its surface because points A and C are on its surface and point B is at its center, where the field due to the sphere will be zero. So applying Gauss's law again,

$$|E|(4\pi(R/2)^2) = -\frac{4}{3}\pi\left(\frac{R}{2}\right)^3\frac{\rho}{\varepsilon_0} \tag{13}$$

$$\mathbf{E}_{\text{sphere}} = -\frac{\rho R}{6\varepsilon_0}\hat{\mathbf{r}} \tag{14}$$

Finally, we superimpose them to get the fields at each point. At point A, $\mathbf{E}_{\text{cyl}} = 0$. So,

$$\boxed{\mathbf{E}_A = \frac{\rho R}{6\varepsilon_0}\hat{\mathbf{i}}} \tag{15}$$

At point C, $\mathbf{E}_{\text{sphere}} = 0$, so

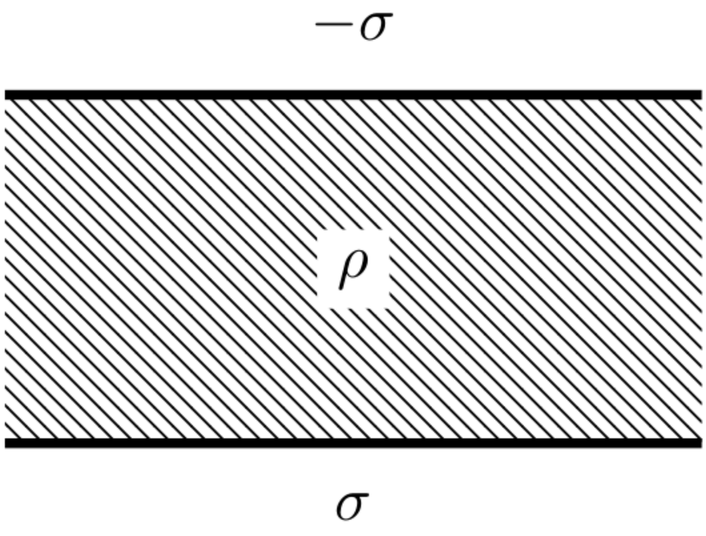
$$\boxed{\mathbf{E}_C = \frac{\rho R}{4\varepsilon_0}\hat{\mathbf{i}}} \tag{16}$$

For point B, neither field is zero, so we have to add them:

$$\boxed{\mathbf{E}_B = \frac{\rho R}{4\varepsilon_0}\hat{\mathbf{i}} + \frac{\rho R}{6\varepsilon_0}\hat{\mathbf{j}}} \tag{17}$$

Problem 3

Statement



In the setup sketched above, two infinite sheets carrying uniform surface charge densities $-\sigma$ and σ are spaced a distance h away from each other. In between them, an infinite slab carrying uniform volume charge density ρ has been inserted. The setup sketched above is a cross section of this sandwich. What is the electric field everywhere?

Solution

We will need to examine this problem for 3 regions: the region above the top sheet, the region in the slab, and the region below the bottom sheet. In each region, we will want to solve for the total electric field by solving for the electric fields of the top sheet, bottom sheet, and slab separately and then adding the 3 fields together (superposition). For convenience, let the z-axis point along the direction perpendicular to the sheets and slab, and let $z = 0$ be the center of the sandwich (halfway through the slab).

Above the top sheet, $z > h/2$

To find the electric field of the top sheet, we observe that by symmetry the electric field must point directly outwards or inwards and cannot have any component parallel to the sheet. Additionally, its magnitude must be constant for all points that are a fixed distance away from the sheet. This suggests that the Gaussian surface we should use is a box or cylinder that intersects the sheet. Then, since the field points straight out from the sheet, there is no flux through the lateral walls of the box/cylinder. For the top and bottom caps of the box/cylinder, the **E**-field will be perpendicular to the Gaussian surface and will have constant magnitude. Now, suppose the box/cylinder has cross sectional area A, then

$$Q_{\text{enclosed}} = -\sigma A \tag{18}$$

and

$$\Phi_{\text{closed}} = 2|\mathbf{E}|A \tag{19}$$

Applying Gauss's law,

$$|\mathbf{E}| = -\frac{\sigma}{2\varepsilon_0} \tag{20}$$

Its direction is $-\hat{\mathbf{k}}$ for $z > h/2$ and $\hat{\mathbf{k}}$ for $z < h/2$. By similar reasoning, the field due to the bottom sheet has magnitude

$$|\mathbf{E}| = \frac{\sigma}{2\varepsilon_0} \tag{21}$$

and direction $\hat{\mathbf{k}}$ for $z > -h/2$ and $-\hat{\mathbf{k}}$ for $z < -h/2$. Take a moment to look at what happens when we superimpose the fields due to the two sheets. Above the top sheet, the fields cancel. Below the bottom sheet, the fields cancel.

$$\mathbf{E}_{\text{sheets}} = 0 \qquad (z < -h/2, z > h/2) \tag{22}$$

In between the top and bottom sheets, the fields add to give

$$\mathbf{E}_{\text{sheets}} = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{k}} \qquad (-h/2 < z < h/2) \tag{23}$$

Ok now let us concentrate on the $z > h/2$ for the slab. There are a couple of equivalent ways of thinking about the slab. You could, for example, view it as a bunch of infinite sheets stacked on top of each other and use superposition to solve the problem. After all, it is essentially an infinite sheet with some thickness. So that inspires us to take exactly the same approach as we did with the two sheets:

$$Q_{\text{enclosed}} = \rho A h \tag{24}$$

$$\Phi_{\text{closed}} = 2|\mathbf{E}|A \tag{25}$$

$$\mathbf{E} = \frac{\rho h}{2\varepsilon_0}\hat{\mathbf{k}} \qquad (z > h/2) \tag{26}$$

So for $z > h/2$, we add Eq. 22 and Eq. 26 to get:

$$\boxed{\mathbf{E} = \frac{\rho h}{2\varepsilon_0}\hat{\mathbf{k}}} \tag{27}$$

Below the bottom sheet, $z < -h/2$

We will do the region below the bottom sheet now because the reasoning should be exactly the same as for the top sheet. The only difference is that the field due to the slab points the opposite way (the direction of the field due to a slab or sheet flips when you go through it. Convince yourself why this must be true by symmetry). So Eq. 26 becomes

$$\mathbf{E} = -\frac{\rho h}{2\varepsilon_0}\hat{\mathbf{k}} \qquad (z < -h/2) \tag{28}$$

and so adding Eq. 22 and Eq. 28, we get:

$$\boxed{\mathbf{E} = -\frac{\rho h}{2\varepsilon_0}\hat{\mathbf{k}}} \tag{29}$$

Inside the slab, $-h/2 < z < h/2$

We already found the field in the slab due to the sheets. We now need to find the field in the slab due to the slab. By the same symmetry arguments from before, the field in the slab must point up or down (depending on ρ), and if it points in the $\hat{\mathbf{k}}$ direction for $z > 0$, it must point in the $-\hat{\mathbf{k}}$ direction for $z < 0$ (symmetry of flipping about the xy-plane). This also means that $\mathbf{E}_{\text{slab}} = 0$ in the center of the slab. So let us choose the Gaussian surface to be a cylinder of cross-sectional area A with one cap in the xy-plane (at $z = 0$, where $\mathbf{E}_{\text{slab}} = 0$), and the other cap to be at a height h above the xy-plane. Since the field points straight up/down, there is no flux through the lateral area of the cylinder. This means that

$$Q_{\text{enclosed}} = \rho A z \tag{30}$$

$$\Phi_{\text{closed}} = |\mathbf{E}|A \tag{31}$$

So

$$\mathbf{E}_{\text{slab}} = \frac{\rho z}{\varepsilon_0}\hat{\mathbf{k}} \qquad (z > 0) \tag{32}$$

Similarly, you can reason that

$$\mathbf{E}_{\text{slab}} = -\frac{\rho z}{\varepsilon_0}\hat{\mathbf{k}} \qquad (z < 0) \tag{33}$$

Adding these equations to Eq. 21, we get that

$$\boxed{\mathbf{E}(z) = \begin{cases} \frac{\sigma - \rho z}{\varepsilon_0}\hat{\mathbf{k}} & z \geq 0 \\ -\frac{\sigma - \rho z}{\varepsilon_0}\hat{\mathbf{k}} & z < 0 \end{cases}} \tag{34}$$