# Isobaric Compression Problem

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## Problem

### Statement

I have rectangular prism with square cross-section. The square has side-length $L$. In the prism is a gas of $H_2$ at temperature $T_0$, and above it is a wooden plank of mass $M$ whose cross-section is also $L$x$L$ (so the plank is essentially a piston). Initially the plank is at rest. The plank is then lowered steadily at constant velocity a distance $\Delta h$ so that the pressure is held constant. During this process, the entire system is insulated from the environment. Treating this gas as an ideal gas, explain how this can happen. Calculate the work done on the gas, the change in internal energy, and the change in the RMS speed of the gas molecules. Molecules of $H_2$ have mass $m_{\mathrm{H}_2}$.

### Solution

First, since this is an ideal gas, it satisfies the ideal gas equation:

$$PV = Nk_BT \tag{1}$$

Since $N$, the number of molecules, and $P$ are supposed to be constant,

$$\frac{T}{V} = \text{constant} \tag{2}$$

This means that as we reduce the volume, the temperature must also decrease simultaneously so that we can maintain constant pressure. The strategy here is to relate the RMS speed to the internal energy, and relate the internal energy to the work done. The work is found as follows:

$$W = \int PdV = P\int_{\text{state 1}}^{\text{state 2}} dV = PL^2\Delta h \tag{3}$$

What is the pressure? Since the plank is initially at rest, and is moved at constant velocity downward, the pressures on both sides are equal. So the pressure comes from the weight of the plank, $Mg$:

$$P = \frac{Mg}{L^2} \tag{4}$$

and so

$$W = Mg\Delta h \tag{5}$$

The first law of thermodynamics says

$$U_1 - U_0 = \Delta U = Q - W \tag{6}$$

But since the system is insulated from the environment, $Q = 0$.

$$\Delta U = -Mg\Delta h \tag{7}$$

Now we need to write $U$ in terms of the temperature. This is found by counting the number of degrees of freedom and then using the equipartition theorem. For $H_2$, there is one vibrational degree of freedom, two rotational degrees of freedom, and three translational degrees of freedom. But, at room temperature, the vibrational degrees of freedom are "frozen out" and are inaccessible. So we count only 5 degrees of freedom. By the equipartition theorem,

$$U = 5 \times \frac{1}{2} N k_B T \tag{8}$$

so

$$\Delta T = -\frac{2}{5}\frac{Mg\Delta h}{N k_B} \tag{9}$$

and

$$T_1 = T_0 + \Delta T = T_0 \left(1 - \frac{2}{5}\frac{Mg\Delta h}{N k_B T_0}\right) \tag{10}$$

We also recall that the RMS speed is

$$v_{\text{RMS}} = \sqrt{\frac{3k_B T}{m_{\text{H}_2}}} \tag{11}$$

So

$$v_{\text{RMS,f}} - v_{\text{RMS,i}} = \sqrt{\frac{3k_B}{m_{\text{H}_2}}}(\sqrt{T_1} - \sqrt{T_0}) \tag{12}$$

$$v_{\text{RMS,f}} - v_{\text{RMS,i}} = \sqrt{\frac{3k_B T_0}{m_{\text{H}_2}}}\left(\sqrt{1 - \frac{2}{5}\frac{Mg\Delta h}{N k_B T_0}} - 1\right) \tag{13}$$