Physics 112 : Lecture 6

Notes for the Fall 2017 Physics 112 Course taught by Professor Holzapfel
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1 Negative Temperature in Paramagnetism

Once again, consider a system of N spin 1/2 particles which live in a magnetic field of strength B. From our multiple previous discussions of this system, we know that the energy is given by:

$$U = -\vec{M} \cdot \vec{B} = -\vec{B} \sum_{i=1}^{N} \vec{m}_i = -B(N_{\uparrow} - N_{\downarrow})m = -2smB$$

the multiplicity is given by:

$$g(N, s) = \frac{N!}{N_{\uparrow}!N_{\downarrow}!} = \frac{N!}{(N/2 + s)!(N/2 - s)!}$$

and the dimensionless entropy is given by:

$$\sigma(N, s) = \ln(g(N, s)) = \ln N! - \ln\Big[(N/2 + s)!\Big] - \ln\Big[(N/2 - s)!\Big]$$

Using the stirling's approximation $\ln(N!) \approx N \ln N - N$ for large N, and using dimensionless constant $x = 2s/N = -U/(NmB)$ we find that:

$$\sigma(N, x) \approx -N\left[\left(\frac{1+x}{2}\right)\ln\left(\frac{1+x}{2}\right) + \left(\frac{1-x}{2}\right)\ln\left(\frac{1-x}{2}\right)\right]$$

Note that we have the simple formula:

$$\frac{\partial \sigma}{\partial x} = -\frac{N}{2}\ln\left(\frac{1+x}{1-x}\right)$$

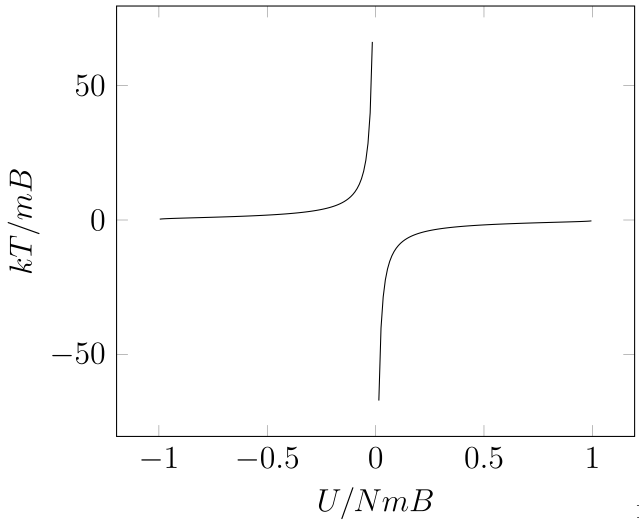
so that we can find:

$$\frac{1}{k_B T} = \frac{1}{\tau} = \frac{\partial \sigma}{\partial U} = \frac{\partial \sigma}{\partial x}\frac{\partial x}{\partial U} = \frac{1}{2mB}\ln\left(\frac{1+x}{1-x}\right)$$

So we have that:

$$\frac{k_B T}{mB} = \frac{2}{\ln\left(\frac{1-U/NmB}{1+U/NmB}\right)}$$

Plotting $k_B T/mB$ as a function of $U/2mB$ we have the graph:



[1]

The important observation is that for this system, it is possible to obtain *negative temperatures*. A system at negative temperature has the property that if you add more energy, the entropy of the system goes *down* (which is abnormal). In the case of paramagnetism, this is because after a certain point adding more energy decreases the number of microstates of the system, since the number of microstates is encoded in the number of particles that are spin up along the magnetic field axis. In other systems (such as an Einstein solid), adding more energy will always lead to an increase in entropy; and you can put an 'infinite' amount of energy into an Einstein solid as well.

For the above situation, if we take the limit as U approaches 0, we find (by manipulation of taylor series) that:

$$\frac{k_B T}{mB} \approx \frac{-NmB}{U}$$

So we see that as U approaches zero from the negative axis, T approaches positive infinity, and as U approaches 0 from the positive axis, then T approaches negative infinity.

If you have a system at negative temperature and a system at positive temperature, then energy will flow from the negative temperature object to the positive temperature object (since energy outflow from a negative temperature object represents an increase in entropy, and energy inflow into a positive temperature object represents an increase in entropy). So we can imagine the spectrum of 'hotness' as starting at 0 kelvin, getting hotter until room temperature, then hotter and hotter until $+\infty$ degrees, so hot that it loops back around to $-\infty$ degrees, even hotter and you get negative room temperature and so on.

Now, let's continue playing with our equations we found earlier. If we use the dimensionless constants:

$$\alpha = \frac{mB}{k_B T}; \qquad x = \frac{-U}{NmB}$$

we can see that we have the equation:

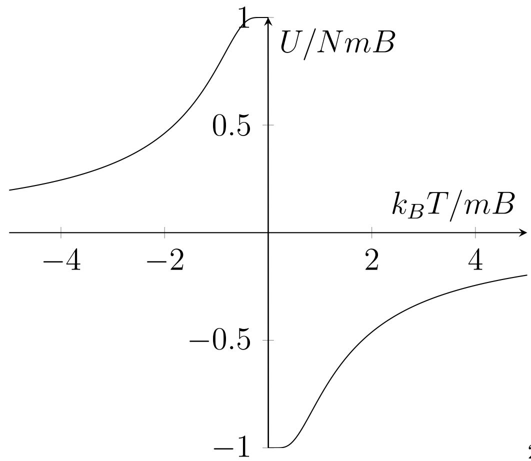
$$2\alpha = \ln\left(\frac{1+x}{1-x}\right)$$

[1] This may well be the first successful latex graph I have drawn

which, solving for x gives us:

$$x = \tanh(\alpha)$$

From this we can make the following plot:



[2]

(This graph should be interpreted as a graph of the energy of the system as a function of it's temperature; the coefficients should be ignored). To get a better understanding of the scales, note that for an electron we have

$$m = \frac{e\hbar}{2m_e} = 5.8 \cdot 10^{-5}\text{eV/T}$$

and in labs, we normally only achieve magnetic fields of strength 1 tesla, at $300K$ (room temperature) we have that $k_B T \approx 1/40\text{eV}$, so all in all we have

$$\frac{mB}{k_B T} << 1$$

In normal situations. So α is very small, so we have the approximation $\tanh(\alpha) \approx \alpha$, so solving for the net magnetisation (from $x = \tanh(\alpha)$) we find that:

$$M \approx \frac{Nm^2 B}{k_B T}$$

which is known as Curie's Law.

2 Heat Capacity

We can define a quantity known as the heat capacity:

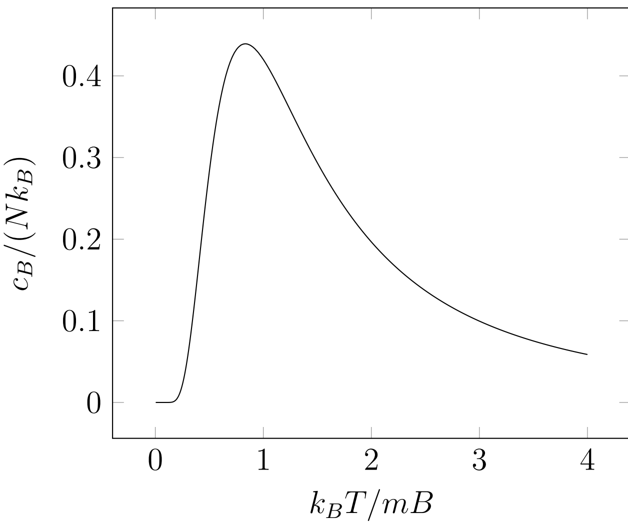
$$c_B = \left(\frac{\partial U}{\partial T}\right)_{N,B}$$

[2]This may well be the second successful latex graph that I have drawn

If we find the heat capacity of the system at hand we find that:

$$c_B = \frac{Nk_B(mB/k_BT)^2}{\cosh^2(mB/k_BT)}$$

We can now plot the following:



[3]

[4]We might be interested in what temperature we would need to have the maximum heat capacity c_B. To this end, we set $\partial c_B/\partial x = 0$, and find that:

$$x = \frac{mB}{k_BT} \approx 1.2$$

is a solution. This corresponds to a solution at:

$$\frac{U}{NmB} \approx 0.83$$

which can be interpreted as 83% unalign-ment of the spins with the external magnetic field.

[3]This may well be the third successful latex graph that I have drawn. It may look funky, but it's generated faithfully from the equation.

[4]