We multiply and divide by $\sec^2 x$ and use the identity $\sec^2 x = 1 + \tan^2 x$:

$$\int \frac{1}{4+5\cos^2 x} dx = \int \frac{\sec^2 x}{5+4\sec^2 x} dx = \int \frac{1+\tan^2 x}{9+4\tan^2 x} dx.$$

Making the substitution $t = \tan x$ (such that $dx = \frac{dt}{1+t^2}$),

$$\int \frac{1+\tan^2 x}{9+4\tan^2 x} dx = \int \frac{1+t^2}{9+4t^2} \cdot \frac{dt}{1+t^2} = \frac{1}{9} \int \frac{dt}{1+\frac{4}{9}t^2}.$$

Substituting $u = \frac{2}{3}t$ (with $dt = \frac{3}{2}du$), we can easily integrate

$$\frac{1}{9} \int \frac{dt}{1+\frac{4}{9}t^2} = \frac{1}{6} \int \frac{du}{1+u^2} = \frac{1}{6} \arctan u + C.$$

Retracing our substitutions leads to

$$\int \frac{1}{4+5\cos^2 x} dx = \frac{1}{6} \arctan\left(\frac{2}{3}\tan x\right) + C.$$