Rate Equation Review

IQE refers to portion of current which is useful → ie portion which falls into QW

$$\begin{cases} \dfrac{dN}{dt} = \dfrac{\eta_i I}{qV} - \dfrac{N}{\tau} - v_g g(N) S \\ \dfrac{dS}{dt} = \Gamma v_g g(N) S - \dfrac{S}{\tau_p} + \Gamma \beta R_{sp} \end{cases}$$

where $R_{sp} = BN^2 = R_{sp}(N)$

$$\frac{N}{\tau} = AN + BN^2 + CN^3$$

Picking apart the equations:

$$\frac{dN}{dt} = \frac{\eta_i I}{qV} - \frac{N}{\tau} - v_g g(N) S$$

- $\frac{dN}{dt}$: Rate of change of carrier concentration
- $\frac{\eta_i I}{qV}$: Amount of charge flowing into "structure"
- $\frac{N}{\tau}$: Charge lost to Recombination (Radiative and Nonradiative)
- $v_g g(N) S$: charge lost due to Stimulated emission

$$\frac{dS}{dt} = \Gamma v_g g(N) S - \frac{S}{\tau_p} + \Gamma \beta R_{sp}$$

- $\frac{dS}{dt}$: rate of change of photon concentration
- $\Gamma v_g g(N) S$: Photons added due to stimulated emission
- $\frac{S}{\tau_p}$: Photons lost due to cavity lifetime
- $\Gamma \beta R_{sp}$: Photons gained due to spontaneous emission

Steady state solution:

$$\begin{cases} \dfrac{dN}{dt} = 0 \Rightarrow 0 = \dfrac{\eta_i I}{qV} - \dfrac{N}{\tau} - v_g g(N) S \Rightarrow N = \tau \dfrac{\eta_i I}{qV} - \tau v_g g(N) S \\ \dfrac{dS}{dt} = 0 \Rightarrow 0 = \Gamma v_g g(N) S - \dfrac{S}{\tau_p} + \Gamma \beta R_{sp}(N) \end{cases}$$

let spontaneous emission be negligable ie $R_{sp} \simeq 0$

$$\Rightarrow g(N) \simeq \frac{1}{\tau_p} \frac{1}{\Gamma v_g} \simeq \frac{n}{\tau_p \Gamma c} = \boxed{\frac{\omega n}{Q \Gamma c} = g(N)}$$

since $\frac{1}{\tau_p} = \frac{\omega}{Q}$

(Side Note: $\tau_p = \frac{1}{\Delta\omega_0}$ and $Q = \frac{\omega_0}{\Delta\omega_0} \Rightarrow \tau_p = \frac{Q}{\omega_0}$)

⚠ Notice that as $t \to \infty$; $g(N) \to g_{th}$ thus the gain becomes clamped

Next substitute $g(N) = \frac{\omega}{Q\Gamma v_g}$ into our 1st expression:

$$\frac{N}{\tau} = \frac{\eta_i I}{qV} - \frac{\omega}{Q\Gamma} S$$

We consider two cases. First, below threshold $S = 0$ (by definition) so,

$$\frac{\eta_i I}{qV} = \frac{N}{\tau} \Rightarrow \boxed{I(N) = \frac{qVN}{\tau \eta_i}} \quad \text{for } S = 0$$

Above threshold, $S > 0$ and $N = N_{th}$:

$$\frac{\omega}{Q\Gamma} S = \frac{\eta_i I}{qV} - \frac{N_{th}}{\tau} = \frac{\eta_i}{qV}\left(I - \frac{qVN_{th}}{\tau \eta_i}\right) = \frac{\eta_i}{qV}(I - I_{th})$$

$$\Rightarrow \boxed{S = \frac{Q\Gamma\eta_i}{\omega q V}(I - I_{th})}$$

Question: what is the output power of the laser

$$\text{Energy} = \hbar\omega S \frac{V}{\Gamma}$$

$\frac{V}{\Gamma}$ = Total volume since V is active volume

$$= \frac{\hbar\omega}{q}\eta_i \frac{1}{g_{th} v_g \Gamma}(I - I_{th})$$

$$\text{Rate Energy lost} = \alpha_m v_g$$

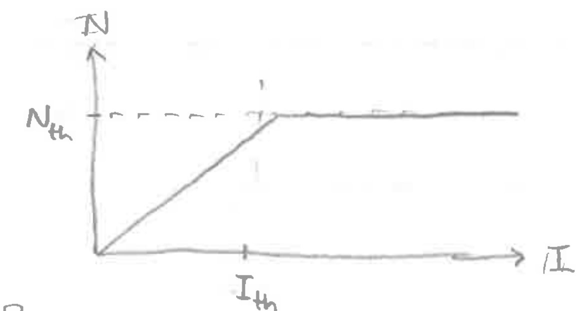
$$= \frac{\alpha_m}{\alpha_i + \alpha_m}(\alpha_i + \alpha_m) v_g$$

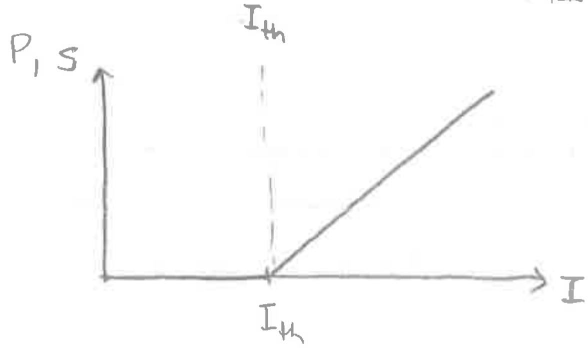
$$= \eta_e \Gamma g_{th} v_g$$

$$\Rightarrow P = \text{Energy} \times \text{Rate lost}$$

$$= \boxed{\frac{\hbar\omega}{q}\eta_e \eta_i (I - I_{th})}$$

Graphically we have:





Differential Analysis

Starting w/ the rate equations:

$$\begin{cases} \dfrac{dN}{dt} = \dfrac{\eta_i I}{qV} - \dfrac{N}{\tau} - v_g g S \\ \dfrac{dS}{dt} = \Gamma v_g g S - \dfrac{S}{\tau_p} - \Gamma \beta R_{sp} \end{cases}$$

We begin by finding the total derivative of the 1st Equation:

$$\frac{d}{dN}\left[\frac{dN}{dt}\right] = \frac{\eta_i}{qV}\frac{dI}{dN} - \frac{d}{dN}\left[AN + BN^2 + CN^3\right] - \frac{d}{dN}(v_g g S)$$

$$= \frac{\eta_i}{qV}\frac{dI}{dN} - \left[A + 2BN + 3CN^2\right] - v_g\left(g\frac{dS}{dN} + S\frac{dg}{dN}\right)$$

$$\Rightarrow \boxed{d\left[\frac{dN}{dt}\right] = \frac{\eta_i}{qV}dI - \frac{dN}{\tau_{\Delta N}} - v_g(g\,dS + S\,dg)} \quad \text{where } \frac{1}{\tau_{\Delta N}} = A + 2BN + 3CN^2$$

Similarly, w/ the 2nd Equation we have:

$$\frac{d}{dN}\left[\frac{dS}{dt}\right] = \Gamma v_g\left(g\frac{dS}{dN} + S\frac{dg}{dN}\right) - \frac{1}{\tau_p}\frac{dS}{dN} - \Gamma\left(R_{sp}\frac{d\beta}{dN} + \beta\frac{dR_{sp}}{dN}\right)$$

$$= \Gamma v_g\left(g\frac{dS}{dN} + S\frac{dg}{dN}\right) - \frac{1}{\tau_p}\frac{dS}{dN} - \Gamma\left(\frac{d\beta}{dN}BN^2 + \beta 2BN\right)$$

$$\Rightarrow \boxed{d\left[\frac{dS}{dt}\right] = \Gamma v_g(g\,dS + S\,dg) - \frac{dS}{\tau_p} - \Gamma\frac{dN}{\tau'_{\Delta N}}} \quad \text{where } \tau'_{\Delta N} = \frac{d\beta}{dN}BN^2 + \beta 2BN$$

Note that the gain is a function of both N & S, ie $g = g(N,S)$ so:

$$dg = \frac{\partial g}{\partial N}dN + \frac{\partial g}{\partial S}dS = \underline{a\,dN - a_p\,dS}$$

We obtain a more detailed picture by including spontaneous emission, i.e. $R_{sp}(N) \neq 0$. Beginning w/ $\frac{dS}{dt} = 0$, we have:

$$0 = \Gamma v_g g(N) S - \frac{S}{\tau_p} + \Gamma \beta R_{sp}(N)$$

$$\Rightarrow \left(-\Gamma v_g g(N) + \frac{1}{\tau_p}\right) S = \Gamma \beta R_{sp}(N)$$

$$\Rightarrow \boxed{S(N) = \frac{\Gamma \beta R_{sp}(N)}{\frac{1}{\tau_p} - \Gamma v_g g(N)}}$$

← Plug in N from 0 to N_{th} to get S!

Likewise, $\frac{dN}{dt} = 0$ yields (w/o additional simplification):

$$\frac{N}{\tau(N)} = \frac{\eta_i I(N)}{qV} - v_g g(N) S(N) \Rightarrow \boxed{I(N) = \frac{qV}{\eta_i}\left[\frac{N}{\tau(N)} + v_g g(N) S(N)\right]}$$

Once again, we could consider $S(N)$ and $I(N)$ below threshold ($S(N) \simeq 0$) and above threshold ($S(N) > 0$).

rewriting our linearized rate equations from above:

$$d\left[\frac{dN}{dt}\right] = \left(-\frac{1}{\tau_{\Delta N}} - v_g a S\right)dN + \left(-v_g g + v_g a_p S\right)dS + \frac{\eta_i}{qV}dI$$

$$d\left[\frac{dS}{dt}\right] = \left(-\frac{\Gamma}{\tau'_{\Delta N}} + \Gamma v_g a S\right)dN + \left(-\Gamma v_g g - \Gamma v_g a_p S\right)dS$$

Next, we swap $d[\ ]$ w/ $\frac{d}{dt}$ (⚠ why is this allowed?) and assemble everything as a matrix problem:

$$\frac{d}{dt}\begin{pmatrix} dN \\ dS \end{pmatrix} = \begin{pmatrix} -\gamma_{NN} & \gamma_{NS} \\ \gamma_{SN} & -\gamma_{SS} \end{pmatrix}\begin{pmatrix} dN \\ dS \end{pmatrix} + \frac{\eta_i}{qV}\begin{pmatrix} dI \\ 0 \end{pmatrix}$$

where

$$\gamma_{NN} = \frac{1}{\tau_{\Delta N}} + v_g a S \qquad \gamma_{SN} = -\frac{\Gamma}{\tau'_{\Delta N}} + \Gamma v_g a S$$

$$\gamma_{NS} = -v_g g + v_g a_p S \qquad \gamma_{SS} = \Gamma v_g g + \Gamma v_g a_p S$$

Let us assume harmonic time dependence for everything, ie $dN(t) = N_1 e^{j\omega t}$; $dS(t) = S_1 e^{j\omega t}$; $dI(t) = I_1 e^{j\omega t}$. Substituting yields:

$$j\omega\begin{pmatrix} N_1 \\ S_1 \end{pmatrix} = \begin{pmatrix} -\gamma_{NN} & \gamma_{NS} \\ \gamma_{SN} & -\gamma_{SS} \end{pmatrix}\begin{pmatrix} N_1 \\ S_1 \end{pmatrix} + \frac{\eta_i}{qV}\begin{pmatrix} I_1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} j\omega + \gamma_{NN} & -\gamma_{NS} \\ -\gamma_{SN} & j\omega + \gamma_{SS} \end{pmatrix}\begin{pmatrix} N_1 \\ S_1 \end{pmatrix} = \frac{\eta_i I_1}{qV}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

According to the class notes, the determinant of the matrix on the RHS can give the resonant frequency of the system

It turns out you can use Cramer's rule to solve for relevant quantities:

Solving for N_1:

$$\det(A_1) = \det\begin{pmatrix} \frac{\eta I_1}{qV} & -\gamma_{NS} \\ 0 & j\omega + \gamma_{SS} \end{pmatrix} = \frac{\eta_i I_o}{qV}(j\omega + \gamma_{SS})$$

$$\det(A) = \det\begin{pmatrix} j\omega + \gamma_{NN} & -\gamma_{NS} \\ -\gamma_{SN} & j\omega + \gamma_{SS} \end{pmatrix} = (j\omega + \gamma_{NN})(j\omega + \gamma_{SS}) + \gamma_{NS}\gamma_{SN}$$

$$= (\gamma_{NN}\gamma_{SS} - \omega^2 + j\omega(\gamma_{NN} + \gamma_{SS}) + \gamma_{NS}\gamma_{SN}$$

Thus

$$\boxed{N_1 = N_1(\omega) = \frac{\frac{\eta_i I_1}{qV}(j\omega + \gamma_{SS})}{(\gamma_{NN}\gamma_{SS} + \gamma_{NS}\gamma_{SN}) - \omega^2 + j\omega(\gamma_{NN} + \gamma_{SS})}}$$

Notice that this quantity is maximized when

$$\omega^2 = \gamma_{NN}\gamma_{SS} + \gamma_{NS}\gamma_{SN} \implies \boxed{\omega_R^2 = \gamma_{NN}\gamma_{SS} + \gamma_{NS}\gamma_{SN}}$$

where ω_R is the Resonant frequency of the system.

We can also highlight the dampening coefficient of the system

$$\gamma = \gamma_{NN} + \gamma_{SS}$$

Likewise, we can solve for $S_o = S_o(\omega)$

$$\det(A_2) = \begin{pmatrix} j\omega + \gamma_{NN} & \frac{\eta_i I_1}{qV} \\ -\gamma_{SN} & 0 \end{pmatrix} = + \frac{\eta_i I_1}{qV}\gamma_{SN}$$

Thus

$$\boxed{S_o(\omega) = \frac{\eta_i I_1(\omega)}{qV}\frac{\gamma_{SN}}{\omega_R}\frac{\omega_R}{\omega_R^2 - \omega^2 + j\omega\gamma}}$$

where ω_R is defined above

Correction: We do not assume pure harmonic time dependence. When doing small signal analysis, se assume a DC value plus a small oscillatory part, i.e.

$$N(t) = N_0 + N_1 e^{j\omega t} \quad ; \quad S(t) = S_0 + S_1 e^{j\omega t} \quad ; \quad I(t) = I_0 + I_1 e^{j\omega t}$$

Notice that unless a time derivative is taken, it is generally safe to make the approximation $S(t) \simeq S_0$ since $S_0 \gg S_1$