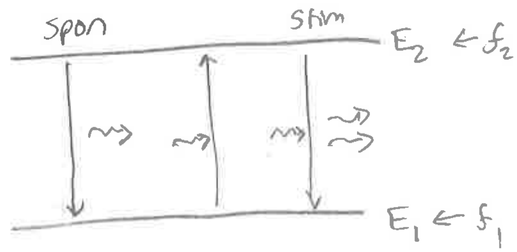
Einstein A-B Coefficients: Spontaneous Emission

We begin by guessing that the rates at which transitions happen between energy levels are governed by occupation probability and photon statistics. We define the spontaneous, stimulated emission, and stimulated absorption rates as:

$$R_{21}^{spon} = A_{21} f_2 (1 - f_1)$$

$$R_{21}^{stim} = B_{21} f_2 (1 - f_1) P(E_{21})$$

$$R_{12} = B_{12} f_1 (1 - f_2) P(E_{21})$$



Where $P(E_{21})$ is the photon occupation probability, defined by:

$$P(E_{21}) = n_{ph} N(E_{21})$$

Here, n_{ph} is the Bose-Einstein distribution

$$n_{ph} = \frac{1}{e^{\hbar\omega/kT} - 1}$$

And $N(E_{21})$ is the photon Density of states:

$$N(E_{21}) = \frac{8\pi n_r^3 E_{21}^2}{h^3 c^2}$$ ⚠ Note E_{21}^2 dependence

which results from the assumption that light follows a linear dispersion:

$$\omega = \frac{kc}{n_r} \quad \text{and} \quad N(E_{21}) = \frac{2}{V} \int \frac{4\pi k^2 dk}{\left(\frac{2\pi}{L}\right)^3} \delta(E_{21} - \hbar\omega_k)$$

Performing a change of variables and integrating yields the expression above

Next, notice that in equilibrium, the absorption rate must equal the combined stimulated and spontaneous emission rate

$$R_{12} = R_{21}^{spon} + R_{21}^{stim} \quad \text{at equilibrium}$$

It follows that:

$$B_{12} f_1 (1-f_2) P(E_{21}) = A_{21} f_2 (1-f_1) + B_{21} f_2 (1-f_1) P(E_{21})$$

Which can be solved to show:

$$P(E_{21}) = \frac{A_{21}}{B_{12} e^{E_{21}/kT} - B_{21}} = N(E_{21}) n_{ph}$$

If we take $A_{21} = A$ and $B_{12} = B_{21} = B$, we find that

$$N(E_{21}) = \frac{A}{B} = \frac{8 \pi n_r^3 E_{21}^2}{h^3 c^3}$$

Using this, we can find an expression for the spontaneous emission rate. Notice that the net absorption rate is:

$$r_{net}^{abs} = R_{12} - R_{21}^{stim} = B[f_2 - f_1] P(E_{21})$$

Which is related to the absorption coefficient by:

$$\alpha(E_{21}) = \frac{r_{abs}^{net}}{P(E_{21}) (c/n_r)} = \frac{n_r}{c} B [f_1 - f_2] = -g(E_{21})$$

It follows that

$$\frac{r_{21}^{spon}}{g(E_{21})} = \frac{A f_2 (1-f_1)}{\frac{n_r}{c} B [f_2 - f_1]} = \frac{c}{n_r} \frac{A}{B} \frac{f_2(1-f_1)}{f_2 - f_1} = \frac{8 \pi n_r^2 E_{21}^2}{h^3 c^2} \frac{f_2 (1-f_1)}{f_2 - f_1}$$

Thus

$$\boxed{r_{21}^{spon} = \frac{8 \pi n_r^2 E_{21}^2}{h^3 c^2} \frac{f_e}{f_g} g(E_{21})}$$